Phase Sensitive Faraday rotation

Aysha Aftab, Rabiya Salman and Sabieh Anwar
LUMS School of Science and Engineering

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Can light propagating through a medium be influenced by the application of an external magnetic field? You have observed optical activity in chiral molecules in your freshmen lab. The present experiment extends these concepts to magnetically induced birefringence through the historically important Faraday Effect, which reveals the rich interplay between optics and magnetism.

KEYWORDS

Polarization · Birefringence · Faraday rotation · Verdet constant · Phase-Sensitive Detection · Jones Calculus · Laser · Helmholtz coil · Resonance in RLC series circuit.

APPROXIMATE PERFORMANCE TIME 1 week.

PRE-REQUISITE EXPERIMENT: Basic measurements with the Lock-in amplifier.

1 Objectives

In this experiment, we will,

1. shed some light on the underlying mechanism of magnetically induced birefringence,
2. demonstrate the advantages of phase sensitive detection (PSD),
3. understand the mathematical formalism for polarized light and its manipulation,
4. build or use sources of uniform magnetic fields and measure the field strengths using a commercial magnetometer,

5. calculate numerical integrals,
6. build resonant RLC series circuit and understand the resonance phenomenon,
7. calculate the Verdet constant of terbium gallium garnet (TGG) and of a diamagnetic liquid.

References

[1] Eugene Hetch and A. R. Ganesan, "Optics", 4th edition, Pearson Education, Inc, India, 2008, pp. 318-320.

[2] Daryl. W. Preston and Eric. R. Dietz, "The Art of Experimental Physics", John Wiley and Sons, 1999, pp. 355-362.

[3] Frank L. Pedrotti and Peter Bandettini, "Faraday rotation in the undergraduate advanced laboratory", Am. J. Phys. **58**, 542-544 (1990).

[4] Lock-in amplifier, user Manual, Stanford Research System, SR 510, http://www.thinksrs.com.

[5] Aloke Jain, Jaynat Kumar, Fumin zhou and Lian Li "A simple experiment for determining Verdet constants using alternating current magnetic fields" Am. J. Phys. **67**, 714-717 (1999).

[6] David Jiles, "Introduction to Magnetism and Magnetic Materials", 2nd edition, CHAPMAN and HALL/CRC, 1998, pp 20-25.

[7] JAN SMIT, "Magnetic Properties of Materials", Inter-University Electronic Series, Vol 13, McGRAW-HILL, New York, p 197.

[8] Frank J. Loeffier, "A Faraday rotation experiment for the undergraduate physics laboratory", Am. J. Phys. **51**, 661-663 (1983).

[9] K. Turvey, "Determination of Verdet constant from combined ac and dc measurents, "Am. J. Phys. **64**, 1561-1567 (1993).

[10] Eric W. Weisstein, "Jacobi-Anger Expansion" from MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Jacobi-AngerExpansion.html.

2 Theoretical introduction

Q 1. What is polarization of light? Write down the equation for linear and circular polarization. Also, show that linearly polarized light can be written as a sum of left and right circular light [1]?

2.1 Magneto optical effect in transmission geometry

Michael Faraday observed the relationship between electromagnetism and light in 1845. Faraday's observation gave birth to the field of **magneto optics**: the interaction of optical radiation with magnetic media or the interaction of light with an optically inactive medium placed inside a magnetic field.

2.1.1 Birefringence

Some substances are optically anisotropic *i.e.*, their optical properties are direction dependent. An atom can be viewed as a positive charge surrounded by an electron shell with some binding forces (the dipole oscillator model).

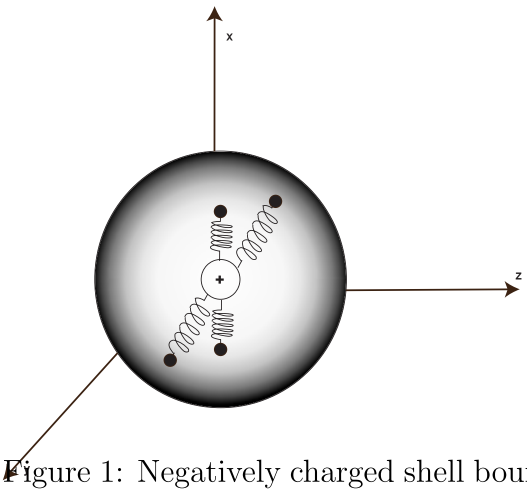

Figure 1: Negatively charged shell bound to positive nucleus by pairs of spring having different stiffness

For an anisotropic substance, the binding forces on the electron are anisotropic implying that the spring constant will be different in different directions: an electron displaced from its equilibrium position along one direction will oscillate with a different frequency than another direction. Since the electric field associated with light drives the electrons of medium at its frequency, these electrons reradiate. The resulting secondary wavelets recombine and light propagates through a medium. The speed of the wave through the medium, is therefore, determined by the difference in natural resonating frequency of electrons and the frequency of the applied electric field. With anisotropy, the whole process becomes direction-dependent. Since the refractive index, $(n = c/v)$ is a function of speed, the anisotropy results in different refractive indices along different directions. This so-called *birefringence* manifests itself the in rotation of the plane of polarization [1].

2.1.2 Faraday rotation

Chiral compounds exhibit rotation of linearly polarized light due to natural birefringence, but this birefringence can also be induced in otherwise optically inactive materials either by applying stress, magnetic or electric field. The Faraday effect is magnetically induced birefringence.

Linearly polarized monochromatic light while transmitting through an optically inactive material, under the influence of an axial magnetic field, is rotated by an angle θ as shown in Figure 2. The angle of rotation θ is given by,

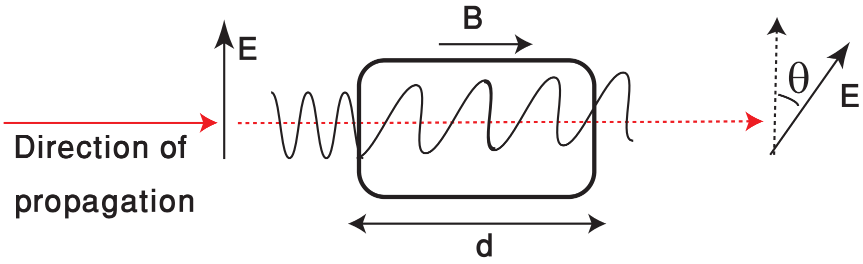

Figure 2: Faraday rotation, The plane of polarization of light is rotated under the action of an axial magnetic field.

$$\theta = VBd, \tag{1}$$

provided the magnetic field remains uniform throughout the length d of sample. For non uniform magnetic field, θ is given by,

$$\theta \;=\; V\int_0^d B(z)dz. \tag{2}$$

The proportionality constant V is a characteristic of the material, called the **Verdet constant** and is a function of the wavelength of light, temperature and refractive index of the material. It is the rotation per unit path length per unit applied magnetic field. In other words, it quantifies the induced birefringence. In this experiment you will measure this induced birefringence.

2.1.3 Larmor precession of the electron cloud in an applied magnetic field

We now try to posit some foundational arguments describing the underlying mechanism of Faraday rotation. Consider an electron, moving in a circle of radius $\mathbf{r}$ in a plane whose normal makes an angle α with an applied magnetic field $\mathbf{B}$. Since an electron is negatively charged its angular momentum $\mathbf{L}$ and magnetic moment $\boldsymbol{\mu_e}$ are opposite to each other. The magnetic field exerts a torque $\boldsymbol{\tau}$ on the magnetic dipole $\boldsymbol{\mu_e}$,

$$\boldsymbol{\tau} = \boldsymbol{\mu_e} \times \mathbf{B} = \mu_e B \sin\alpha.$$

Q 2. Referring to Figure 3 what is the direction of the torque on the magnetic dipole?

According to Newton's second law, an angular impulse $\boldsymbol{\tau}$ produces a change in angular momentum,

$$\boldsymbol{\tau} dt = d\mathbf{L}.$$

Thus, the attached vector **L** rotates in anticlockwise direction. The resulting precession traced out by tip of the vector **L** is shown in Figure 3. The angle of rotation through which angular momentum's projection along the applied field, $\mathbf{L}'$, moves in time dt is,

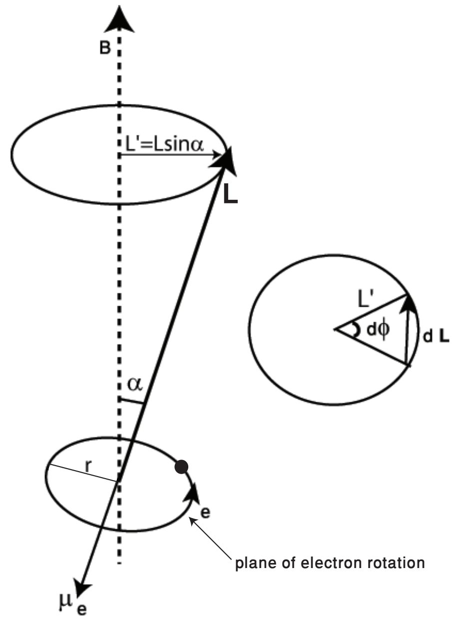

Figure 3: Precession of angular momentum vector about the direction of the applied magnetic field.

$$
\begin{aligned}
d\phi &= dL/L' \\
&= \tau dt/L \sin\alpha
\end{aligned}
$$

and the precessional or the Larmor angular velocity becomes,

$$
\omega_L = \frac{d\phi}{dt} = \frac{\tau}{L \sin\alpha} = \frac{\mu_e B \sin\alpha}{L \sin\alpha} = \frac{\mu_e B}{L}. \tag{3}
$$

The magnetic moment of circular current is given by

$$
\mu_e = iA = i(\pi r^2), \tag{4}
$$

where,

$$
i = \frac{e\omega}{2\pi}, \tag{5}
$$

whereas the angular momentum of electron is given by,

$$
\begin{aligned}
\mathbf{L} &= \mathbf{r} \times \mathbf{p} \\
L &= mvr = mr^2\omega. \qquad (6)
\end{aligned}
$$

Substituting Eqs (4), (5), (6) into (3), we get,

$$
\omega_L = (\frac{e\omega}{2\pi})(\frac{\pi r^2}{mr^2\omega})B \tag{7}
$$

$$
= \frac{eB}{2m}, \tag{8}
$$

showing that the Larmor frequency ω_L is independent of the orientation of the current loop and the overall effect is the rotation of electronic structure about the direction of applied magnetic field [3].

2.1.4 Semi-Classical description of induced birefringence

You must have realized from Q 1 that plane polarized light is a combination of left and right circular (l and r) polarized light. Now, if light of vacuum frequency f is traveling through a medium whose electrons are rotating at the Larmor frequency then the l and r components will rotate the electron clouds with frequencies $f+f_L$ and $f-f_L$. Therefore in the dispersive medium, (refractive index is frequency dependent,) the functional dependence of the respective refractive indices can be written as,

$$n_l = n(f - f_l)$$

and

$$n_r = n(f + f_l),$$

If plane polarized light traverses a distance d then the optical path lengths for l and r light are $n_l d$ and $n_r d$ respectively, so the optical path difference is $(n_r - n_l)d$. The difference of two refractive indices, the induced birefringence is,

$$n_r - n_l = n(f + f_L) - n(f - f_L)$$

Using the Taylor Series,

$$n_r - n_l = (n(f) + \frac{dn}{df} f_L) - (n(f) - \frac{dn}{df} f_L) \tag{9}$$

$$= 2 f_L \frac{dn}{df}. \tag{10}$$

From Equation 8,

$$f_L = \frac{\omega_L}{2\pi} = \frac{eB}{4\pi m},$$

Eq (10) becomes,

$$n_r - n_l = 2(\frac{eB}{4\pi m})(\frac{dn}{df}).$$

Since, phase change of a wave is k $(= 2\pi/\lambda)$ times the physical path traversed by the wave, the phase change for the two components is,

$$\phi_l = (\frac{n_l d}{\lambda})(2\pi) \tag{11}$$

$$\phi_r = (\frac{n_r d}{\lambda})(2\pi). \tag{12}$$

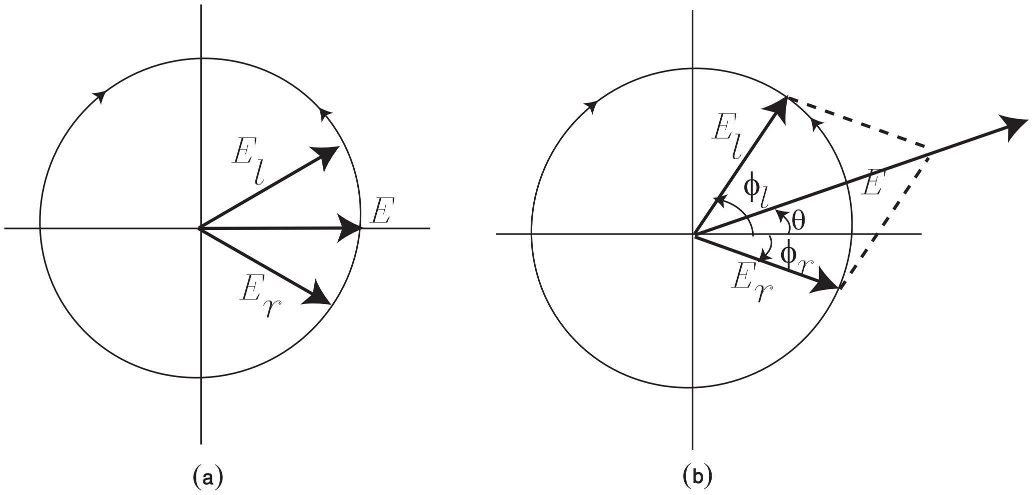

Figure 4: Superposition of left and right circularly polarized light into linearly polarized light. (a) Entering the sample, both the l and r components are moving with same speed and (b) while passing through the sample, these components have travelled with different velocities.

When l and r waves enter the sample, the phase difference is zero, but the phase difference accumulates as light passes through the sample. The vector sum of the two electric fields on emerging from the sample is shown as $\boldsymbol{E}$ with a net rotation θ from its initial value. Since, $\boldsymbol{E}$ is an *equal* superposition of l and r components, we see from Figure (4) that,

$$\begin{aligned} \phi_l - \theta &= \phi_r + \theta \\ \Rightarrow \theta &= \frac{\phi_l - \phi_r}{2}. \end{aligned}$$

Thus, the Faraday rotation angle is,

$$\begin{aligned} \theta &= \frac{1}{2}(\frac{2\pi d}{\lambda})(n_l - nr) \\ &= (\frac{\pi d}{\lambda}\frac{eB}{2\pi m})(\frac{dn}{df}) \\ &= \frac{e}{2m\lambda}(\frac{dn}{df})Bd. \end{aligned} \tag{13}$$

Comparing Eq(1) and (13), the Verdet constant,

$$V = \frac{e}{2m\lambda}(\frac{dn}{df}) \tag{14}$$

which is a function of wavelength and the dispersion [3]. The Faraday rotation is a direct result of $n_l \neq n_r$ arising because of the frequency dependent refractive index.

2.1.5 Description of dispersion based on the Zeeman Effect

The physical reason behind the change in refractive index can also be explained through Zeeman splitting. The splitting of spectral lines of the atom when placed in magnetic field is called the Zeeman effect, after the name of discoverer P. Zeeman. From Eq (3),

$$\mu_e = \frac{L\omega_L}{B} = (\frac{L}{B})(\frac{eB}{2m}) = \frac{e}{2m}L \tag{15}$$

Since, the left and right (l and r) components of light carry an angular momentum of $+\hbar$ and $-\hbar$ respectively, the l component drives electron into left circular motion and the r component drives electrons into right circularly motion, resulting in different magnetic moments. Interaction of the magnetic moment μ_e with the magnetic field $\mathbf{B}$ slightly shifts the energy of atomic level by an amount

$$\Delta E = -\Delta(\mu_e B) = -(\Delta\mu_e)B = -(\frac{e}{m}\hbar)B. \tag{16}$$

Thus, under the application of an axial magnetic field, dispersion curves for left and right circularly polarized light are identical but displaced by the frequency difference between the two Zeeman components,

$$\Delta\omega = \frac{\Delta E}{\hbar} = \frac{-e}{m}B$$

which results in two different refractive indices n_r and n_l and therefore a different speed, at a given ω [2].

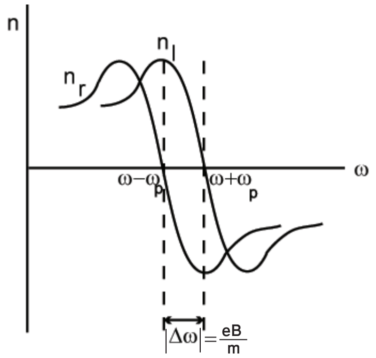

Figure 5: Refractive indices for left and right circularly polarized components of plane wave in the presence of magnetic field. The dispersion curves for the two components are shifted by $\Delta\omega$

2.2 Jones calculus

Jones calculus, invented by the American physicist R. Clark Jones, in 1941, is a useful formalism to understand the state of polarization of perfectly polarized light as well as its transformation by various optical devices. For example, polarized light given by,

$$\mathbf{E}(z,t) = \hat{\imath}E_{ox}\cos(kz-\omega t+\phi_x)+\hat{\jmath}E_{oy}\cos(kz-\omega t+\phi_y) \tag{17}$$

is represented in the Jones formalism as,

$$\tilde{\mathbf{E}}(z,t) = \begin{pmatrix} \tilde{\mathbf{E}}_x(z,t) \\ \tilde{\mathbf{E}}_y(z,t) \end{pmatrix} = \begin{pmatrix} E_{ox}e^{i\phi_x} \\ E_{oy}e^{i\phi_y} \end{pmatrix} e^{i(kz-\omega t)}. \tag{18}$$

The two component column vector completely specifies the amplitude and phase of electric field and hence its state of polarization. This is called the Jones vector. Most of the times, it is not necessary to know the exact phase but the phase difference $\varepsilon = \phi_x - \phi_x$ between the x and y components. Moreover, $e^{i(kz-\omega t)}$ is always understood to be present. Accordingly, Jones vector can also be written as,

$$\tilde{\mathbf{E}}(z,t) = \begin{pmatrix} E_{ox} \\ E_{oy}e^{i\varepsilon} \end{pmatrix} e^{i\phi_x}. \tag{19}$$

Ignoring the term $e^{i\phi_x}$

$$\tilde{\mathbf{E}}(z,t) = \begin{pmatrix} E_{ox} \\ E_{oy}e^{i\varepsilon} \end{pmatrix}. \tag{20}$$

For linearly polarized light $\varepsilon = 0$ or 180^o, therefore the general form of Jones vector for linearly polarized light is,

$$\tilde{\mathbf{E}}(z,t) = \begin{pmatrix} E_{ox} \\ E_{oy} \end{pmatrix}. \tag{21}$$

Jones vectors can be normalized such that the sum of the squares of their components is 1, i.e,

$$E_{ox}E_{ox}^* + E_{oy}E_{oy}^* = 1.$$

This normalized form discards the amplitude information needed for absorption calculations, but simplifies analysis in many other cases. The normalized form of (21) at an angle α w.r.t an arbitrary reference axis is,

$$\tilde{\mathbf{E}}(z,t) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix},$$

where, the angle α is defined such that,

$$\begin{aligned} \cos\alpha &= \frac{E_{ox}}{\sqrt{E_{ox}^2+E_{oy}^2}} \\ &= E_{ox} \\ \sin\alpha &= \frac{E_{oy}}{\sqrt{E_{ox}^2+E_{oy}^2}} \\ &= E_{oy}. \end{aligned}$$

Q 3. Write down the normalized Jones column vector for horizontally, vertically, left and right circularly polarized light?

Suppose that the Jones vector for polarized incident beam $\tilde{\mathbf{E}}_i$ is represented by $\tilde{\mathbf{E}}_t$ after transmission through an optical element then, the optical element can be represented as a 2×2 transformation matrix J, called the Jones matrix, given by

$$\tilde{\mathbf{E}}_t = J\tilde{\mathbf{E}}_i \tag{22}$$

where

$$J = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}. \tag{23}$$

Equation 22 can be written as,

$$\begin{pmatrix} \tilde{\mathbf{E}}_{tx} \\ \tilde{\mathbf{E}}_{ty} \end{pmatrix} = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{E}}_{ix} \\ \tilde{\mathbf{E}}_{iy} \end{pmatrix}. \tag{24}$$

If the beam passes through a series of optical elements represented by the matrices $J_1, J_2, J_3, ..., J_n$, then

$$\tilde{\mathbf{E}}_t = J_n, ..., J_3, J_2, J_1\tilde{\mathbf{E}}_i. \tag{25}$$

The matrices do not commute, so they must be applied in proper order.

Q 4. Show that the transformation matrix J_h for horizontal linear polarizer is

$$J_h = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{26}$$

(HINT: Write expression (25) for maximum and zero transmittance, solve the simultaneous equations to get the coefficients of transformation matrix.)

3 Experimental Technique

3.1 Why PSD in Faraday rotation?

You have already performed an introductory experiment of using the lock-in amplifier, so without discussing the details of the technique and the instrumentation any further, we will only focus on why are we using phase sensitive detection (PSD) in this experiment. Consider a simple optical system used to measure the transmission of light through a medium. Let us suppose a small response obscured by overwhelming noise is to be measured. The output signal in this case will be,

$$V_o = V_{sig} + V_{noise}. \tag{27}$$

The noise and signal amplitudes for such a system as a function of frequency are shown in Figure (7) [4]. The large peaks at 50 Hz and its multiples are due

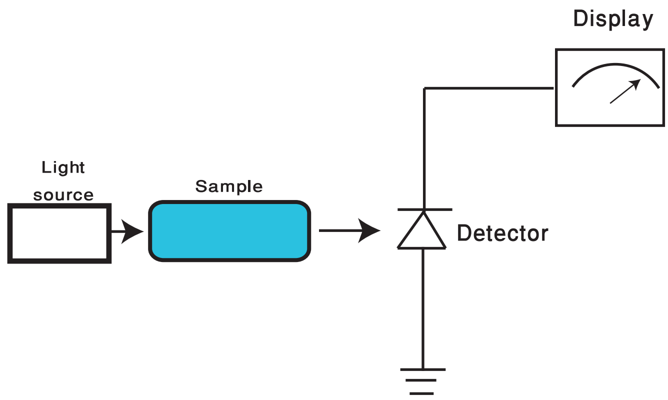

Figure 6: A simple optical system.

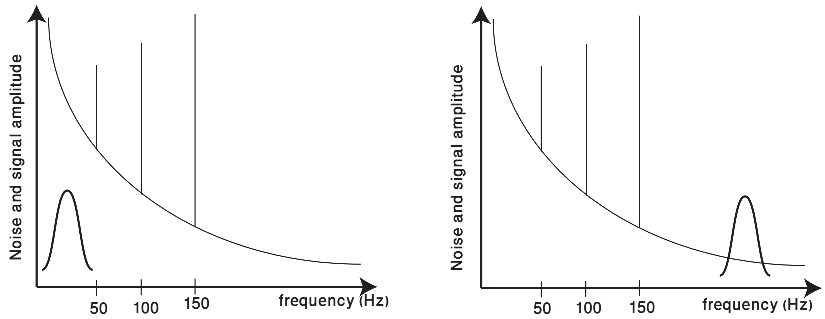

Figure 7: (a) Noise and signal amplitude as a function of frequency. (b) Modulating the signal to a region of low noise.

to electrical interference from the mains power lines. The noise power increases at lower frequencies (remember this is due to $1/f$ noise). Faraday rotation is extremely small in magnitude. If such a small signal buried in noise is to be measured, amplifying the signal will not improve the signal-to-noise ratio, the noise is amplified with the signal. A clever approach is to move the signal to a region of low noise, to higher frequency. For example, in the present experiment, we use an ac magnetic field for inducing Faraday rotation instead of a dc field produced by dc current or a permanent magnet. This technique gives two real advantages.

- The weak signal of interest buried in noise can be extracted successfully through PSD.

- Faraday rotation can be observed at smaller values of magnetic field (e.g., 80 G rms). This circumvents the need for large, expensive, bulky, water-cooled electromagnets for producing large magnetic fields.

Q 5. Can you think of a simple experiment that measures the noise spectrum of laser light detected by a photodetector?

Q 6. What is Malus's law? How does a polarizer work?

3.2 Overview of the experiment

The plane of polarization of linearly polarized monochromatic light traversing through the sample S of length d placed under the influence of an ac magnetic field is rotated. Since the field is oscillatory, the rotation angle is also oscillatory. Another polarizer set at an arbitrary angle relative to input polarizer subsequent to the sample is required to analyze the rotation. The analyzer converts the polarization modulation to an amplitude modulation by the way of Malus's Law. The emerging light beam carrying the information in the form of amplitude variations is incident upon a photodiode whose output appears in the form of current proportional to the light intensity.

Let us suppose incident light polarized along the x-axis is propagating in the z direction. The electric field in terms of Jones vector is,

$$\tilde{\mathbf{E}}_{\mathbf{o}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} A_o \exp i(kz - wt) \tag{28}$$

where A_o corresponds to the amplitude of the electric field.

After passing through the sample S of length d placed in magnetic field, the plane of polarization of light is rotated by an angle θ, so the Jones vector after emerging from the sample is,

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{29}$$

and the corresponding electric field is,

$$\tilde{\mathbf{E}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} A_o \exp i(kz - wt). \tag{30}$$

Q 7. Suppose, the analyzer is set at an angle ϕ w.r.t the polarizer, show that

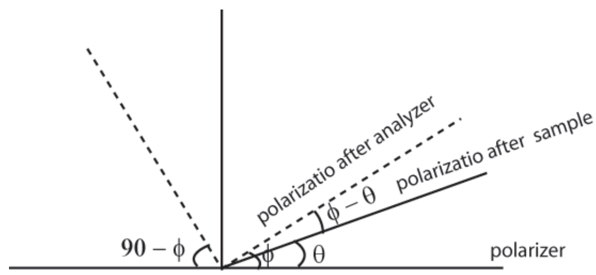

Figure 8: Rotation θ of plane of polarization of light, analyzer is oriented at ϕ relative to the polarizer.

the transformation matrix is,

$$J(\phi) = \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix}. \tag{31}$$

Q 8. Write the Jones vector for the combination of the polarizer, sample and analyzer, placed in the same order.

You can conclude from Q 8 that electric field of the light beam after emerging from the sample followed by the analyzer is,

$$\tilde{\mathbf{E}} = \begin{pmatrix} \cos(\phi-\theta)\cos\phi \\ \cos(\phi-\theta)\sin\phi \end{pmatrix} A_o \exp i(kz - wt). \tag{32}$$

The intensity of light measured by the photodetector is,

$$I = kA_0^2[\cos^2(\phi-\theta)]. \tag{33}$$

Q 9. Derive the expression (33). What are the dimensions of the constant k? In the subsequent discussion, we will normalize $k = 1$. (HINT: Use the concept of the Poynting vector.)

3.2.1 Optimization of the analyzer angle

According to Eq. 33, the rotation of the plane of polarization manifests as a change in intensity at the photodiode. To get maximum change in intensity, the analyzer angle needs to be optimized. Differentiating the intensity w.r.t ϕ, we get,

$$\frac{dI}{d\phi} = A_0^2 2\cos(\phi-\theta)\sin(\phi-\theta) \tag{34}$$

$$= A_0^2 \sin 2(\phi-\theta). \tag{35}$$

Differentiating again,

$$\frac{d^2I}{d^2\phi} = 2A_0^2\cos\big(2(\phi-\theta)\big). \tag{36}$$

Maximum change in intensity is obtained by maximizing $\frac{dI}{d\phi}$ or by setting $\frac{d^2I}{d\phi^2} = 0$,

$$2A_0^2\cos 2(\phi-\theta) = 0$$

since, $A_0 \neq 0$, we have,

$$\cos 2(\phi-\theta) = 0$$
$$(\phi-\theta) = 45^o.$$

Since, the Faraday rotation θ is much smaller than ϕ, maximum ΔI is obtained when the analyzer is set at 45^o relative to polarizer. The measured intensity is,

$$\begin{aligned} I &= \frac{A_0^2}{2}[1 + cos2(\phi-\theta)] \\ &= \frac{A_0^2}{2}[1 + \cos(2\phi)\cos(2\theta) + \sin(2\phi)\sin(2\theta)] \\ &= A_0^2\frac{(1+\sin(2\phi)\sin(2\theta))}{2}. \end{aligned}$$

For $\phi = 45^o$ and $\sin(2\theta) \cong (2\theta)$,

$$I \cong \frac{A_0^2}{2}(1+2\theta). \tag{37}$$

The field is made oscillatory, with an oscillating frequency Ω,

$$B = B_0 \sin(\Omega t),$$

and since the angle of rotation is directly dependent on the magnetic field,

$$\theta = \theta_0 \sin(\Omega t), \tag{38}$$

therefore, Eq. (37) can be written as,

$$I \cong \frac{A_0^2\big(1+2\theta_0\sin(\Omega t)\big)}{2}. \tag{39}$$

3.2.2 Converting light intensities into photocurrents

The photodiode converts the photon intensities into current, thus

$$i = i_{dc} + i'_{ac}$$

where $i_{dc} = \frac{A_0^2}{2}$ and $i'_{ac} = \theta_0 A_0^2 \sin(\Omega t)$. Modulated photocurrent due to Faraday rotation i_{ac} is measured through lock-in amplifier which displays the rms values, therefore the output of the lock-in amplifier is,

$$i_{ac} = \frac{i'_{ac}}{\sqrt{2}} = \frac{\theta_0 A_0^2}{\sqrt{2}}.$$

Taking the ratio of the modulated current (shown by the lock-in amplifier) to the steady current, we obtain,

$$\frac{i_{ac}}{i_{dc}} = \frac{\theta_0 A_0^2}{\sqrt{2}}\frac{2}{A_0^2} \tag{40}$$

$$\Rightarrow \theta_0 = \frac{i_{ac}}{\sqrt{2}\, i_{dc}} \tag{41}$$

and as far as the rms value of the Faraday rotation angle is concerned,

$$\theta_{rms} = \theta = \frac{\theta_0}{\sqrt{2}} = \frac{i_{ac}}{2\, i_{dc}}. \tag{42}$$

The dc component is measured by an oscilloscope in the absence of magnetic field while the ac component is measured by the lock in amplifier in the presence of the magnetic field. For a uniform magnetic field Verdet constant is determined from the experimental values of θ, B and d,

$$\theta = VBd, \tag{43}$$

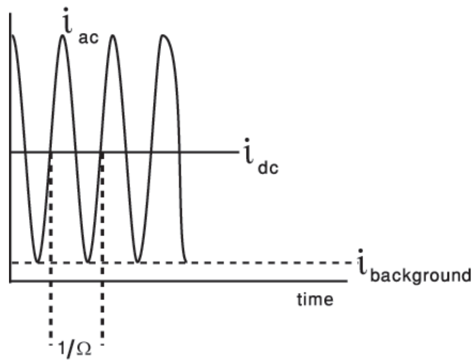

Figure 9: Signal measured by photodiode is made up of two parts, average light intensity, i_{dc} and modulated intensity at the frequency of ac magnetic field, i'_{ac} . The currents are proportional to the intensities.

whereas for non uniform magnetic field, θ, is given by,

$$\theta = V \int_0^d B(z)dz. \tag{44}$$

Q 10. What is the working principle of a photodetector? What does the photodetector measure? The electric field or the intensity?

Q 11. Can the photodiode measurement be affected by stray magnetic field? (HINT: the Hall Effect)

3.2.3 Schematic of the experiment

Figure 10 shows the schematic diagram of the experimental setup for the observation of Faraday rotation.

The setup comprises these components:

(a) Light Source

(b) Mechanism for producing and measuring an oscillating magnetic field

(c) Detection devices

3.2.4 Light source

Light from a lamp can be used after collimating it by a lens and passing through a color filter to make it monochromatic, however, since, LASER is a source of highly directional and monochromatic light and is easily available, it is convenient to use

Table 1: List of Equipment used in the experiment.

Component		**Supplier**
Light source	laser 633 nm, 2 mW	Thorlabs (HRR-020)
	laser 405 nm, 40 mW	B&W TEK (405-40E)
Linear Polarizer	extinction ratio=1000 : 1	Thorlabs (LPVIS050)
Magnetic field production	Signal generator, $10V_{pp}$	GW-Instek (SFG-1013)
	Audio amplifier, 150 W	CERWIN VEGA
	Power supply, 10 A, 12 V	Panasonic electronics
	Helmholtz coil, 120 G rms	Homemade
Detection element	Photodiode	Newport (818-SL)
Measuring instruments	Lock in amplifier	Stanford Research System (SR-510)
	Oscilloscope	GW-Instek
	Gaussmeter with axial and transverse probes	LakeShore (410)
	Clamp meter	Kyoritsu (KEW SNAP 2017)
	LCR meter	QuadTech, Inc.
Accessories	Optical breadboard 90 × 60 × 6 cm	Thorlabs (PBI51506)
	Optical rail, 60 cm long	Thorlabs (RLA600/M)
	Rail carrier 2.5 cm	Thorlabs (RC1)
	SS Post, 5 cm long	Homemade
	Post holder, 7.6 cm long	Thorlabs (PH3/M)
	Rotation mount, 2 degree resolution	Thorlabs (RSPO5/M)
	Laser post, 20 cm long	Thorlabs (P200/M)
	V shaped laser housing	Thorlabs (C1502/M)
	Glass cell, 6 cm long	Homemade
	Crescent shaped cell holder	Homemade
	Teflon crystal holder	Homemade
	Laser safety glasses OD=4, OD=7	Thorlabs (LG4, LG10)
	M6 and M4 screws	Thorlabs
	TGG crystal d=1 cm	Castech Inc.

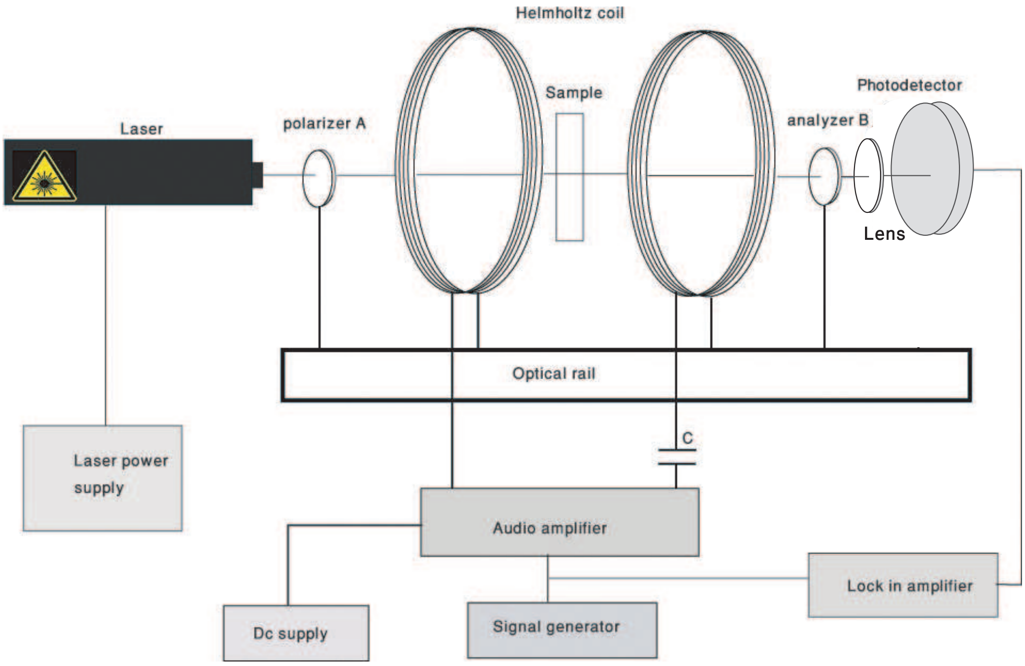

Figure 10: Schematic of experimental setup for Faraday rotation.

it as the light source. He Ne laser of wavelength 633 nm or an electrically pumped diode laser of wavelength 405 nm will be used in the experiment.

Q 12. What is the basic principle of a laser? How does a He Ne laser work?

3.2.5 Mechanism for producing and detecting the magnetic field

In principle, both ac and dc magnetic field can be used in this experiment. Dc sources include permanent magnets or solenoids having steady current in their windings. Since, Faraday rotation is small in magnitude, of the order of microradians, so a large dc magnetic field, of several kilo gauss will be required to achieve a sizeable rotation, which in turn requires large and bulky dc magnets or a large dc power supply to produce required field [8]. However, using an ac magnetic field, the rotation becomes oscillatory and can be tracked by PSD. For example, in this experiment, you will be provided with a Helmholtz coil capable of generating a field of approximately 120 G rms.

The Helmholtz coil

A pair of Helmholtz coils is used to produce a uniform magnetic field over a large volume of space. It consists of two identical coils such that separation d of the coils is equal to their common radius a.

Let us consider a single loop of conductor of radius a, carrying a current i. Using the Biot-Savart rule, magnetic induction at the point P, at a distance r is,

$$d\mathbf{B} = \frac{\mu_o}{4\pi r^2} i\, d\mathbf{l} \times \mathbf{u}, \tag{45}$$

where, $\mathbf{u}$ is the unit vector connecting the conducting element with the point at

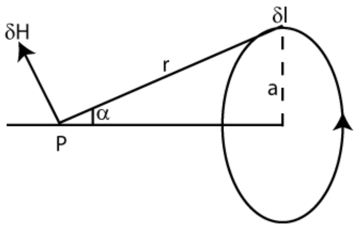

Figure 11: Magnetic field at point P due to single circular coil carrying a current i.

which the field is to be determined, μ_o is permeability of free space $= 4\pi \times 10^{-7}$ Hm^{-1}. This geometry is shown in Figure 11. Substituting,

$$r = \frac{a}{\sin \alpha} \tag{46}$$

into Eq (45), we get,

$$d\mathbf{B} = \frac{\mu_o}{4\pi a^2}(\sin^2 \alpha) i \, d\mathbf{l} \times \mathbf{u}. \tag{47}$$

The axial component of magnetic induction is

$$d\mathbf{B}_{axial} = d\mathbf{B} \sin \alpha.$$

Therefore,

$$d\mathbf{B}_{axial} = \frac{\mu_o}{4\pi a^2}(\sin^3 \alpha) i \, d\mathbf{l} \times \mathbf{u} \tag{48}$$

Since, $d\mathbf{l}$ is perpendicular to $\mathbf{u}$, and integrating round the coil $\int dl = 2\pi a$, we obtain the total axial field,

$$\begin{aligned} B_{axial} &= \frac{\mu_o i}{4\pi a^2}(\sin^3 \alpha) 2\pi a \\ &= \frac{\mu_o i}{2a} \sin^3 \alpha \\ &= \frac{\mu_o i}{2a} \frac{a^3}{(a^2 + z^2)^{3/2}} \\ &= \frac{\mu_o i}{2} \frac{a^2}{(a^2 + z^2)^{3/2}}. \end{aligned} \tag{49}$$

For N number of turns, Eq. (49) becomes,

$$\begin{aligned} B_{axial} &= N \frac{\mu_o i}{2} \frac{a^2}{(a^2 + z^2)^{3/2}} \\ &= \frac{\mu_o N i}{2a}. \end{aligned} \tag{50}$$

For the Helmholtz pair, if one coil is placed at $z = 0$ and the other at $z = a$, and if current flows through both the coils in same direction (referred to as superposition condition, Figure (12)), by symmetry the radial component of magnetic field along the axis must be zero. Hence, the magnetic field on the common axis of the coils becomes [6],

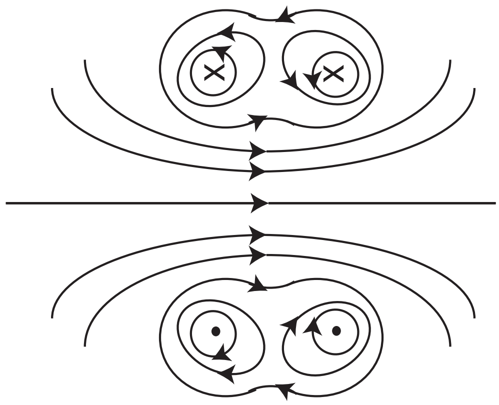

Figure 12: Pair of Helmholtz coil with separation equal to the common radius and carrying the current in same direction.

$$B = \frac{\mu_o N i}{2a}[(1+\frac{z^2}{a^2})^{-3/2} + (1+\frac{(a-z)^2}{a^2})^{-3/2}]. \tag{51}$$

Q 13. Show that at the point on the axis midway between the coils $z = a/2$, the field is ,

$$B = (\frac{4}{5})^{3/2}(\frac{\mu_o N i}{a}). \tag{52}$$

Q 14. Using the binomial expansion $(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} + \ldots$, show that Eq. (51) can also be written as,

$$B = \frac{\mu_o N i}{2a}(1 + c_4 z^4 + c_6 z^6 + ...), \tag{53}$$

where, $c_4 = 15/8a^4$ and $c_6 = -105/48a^6$.

Q 15. What do you conclude from equation (53) about the uniformity of the magnetic field? How does the field in the middle of the coils compare with the field in the center of a single circular loop of the same radius?

In our experiment, the Helmholtz coil is constructed from 18 gauge copper wire (diameter 1.2 mm). Each multilayer coil consists of 18 turns in 18 layers, the coil's outer and inner diameters are 10.2 cm and 6.5 cm respectively. The length of each coil is 2.7 cm and radius is 4.5 cm. Inductance of the coils, determined using the LCR meter, is found to be 7 mH with a resistance of 1.5 Ω for each coil, so the total inductance of the Helmholtz pair is 15 mH and the total resistance is 3 Ω . The Helmholtz coil pair constitutes a series RLC circuit. At resonating frequency

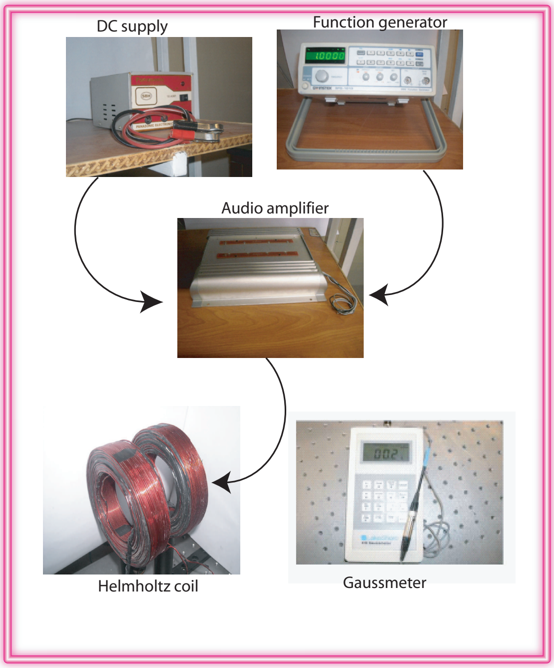

Figure 13: Instruments for creating and detecting the oscillating magnetic field.

ω_r, the inductive reactance X_L is equal to the capacitive reactance X_L and total impedance is purely resistive. The resonating frequency is,

$$\omega_r = \sqrt{\frac{1}{LC}}$$

or

$$f_r = \frac{1}{2\pi\sqrt{(LC)}}.$$

Q 16. Calculate the resonating frequency when a capacitor of 0.97 μF is connected in series with the coil? Why is the Helmholtz coil made resonating?

4 The Experiment

1. Assemble the setup according to Figure (10). Turn on the audio amplifier and the function generator. Amplify an approximately 1 V, 70 Hz sinusoidal signal through audio amplifier. Apply this amplified output to the Helmholtz coil.

2. Increase the frequency of the ac signal applied to the coil. Tabulate the frequency against current passing through the Helmholtz coil (Table 2) and plot the frequency response. The current is measured with the help of a clamp meter or an ammeter.

Frequency (Hz)	Current (A rms)
100	
.	
.	
.	
2500	

Table 2: Mapping the frequency response of the Helmholtz coil.

3. Set the function generator at the resonating frequency. Increase the current by increasing the gain of the amplifier (Table 3). Measure the magnetic field using the Gaussmeter in ac mode (LakeShore, Model 410) equipped with transverse probe at the midpoint between the two coil. Plot a graph between current and magnetic field. Do you observe a linear relationship as predicted by equation (52)?

Current (A rms)	Magnetic field (G rms)
1	
.	
.	
.	
1.6	

Table 3: Linear relationship between the current (A) and the magnetic field (Gauss).

Next you will determine the Verdet constant for Terbium Gallium Garnet(TGG) crystal and a diamagnetic liquid such as carbon disulphide CS_2. **Never touch the lateral surfaces of the TGG crystal.**

4. Turn on the provided laser. The HeNetakes about 30 mins to warm up and reach a stable value. **Never look directly at the laser light. Always wear safety goggles when operating the laser.** For the time being, keep the laser power switched on but close the shutter of the laser head.

5. The TGG crystal is 1 cm long but CS_2 is filled in a 6 cm long glass cell. We cannot expect the magnetic field in between the coils of Helmholtz pair to be uniform over this large a distance. So you need to map the magnetic field profile and perform numerical integration of the magnetic field as suggested in Equation 44.

   The next few steps will help you calibrate the magnetic field.

6. Fix a scale with the edge of the sample holder. Place the glass cell over the crescent shaped sample holder. Open the laserhead shutter. Adjust the height of the laser and sample holder to pass the beam through the center of the coils. you might need to adjust the heights of optical components by translating the stainless steel posts. Close the shutter again. The height of

the sample holder will now be kept fixed. Mark the end points of the sample on the scale.

7. Remove the sample. Fix the axial probe on another holder on either side of the Helmholtz coil and turn on the Gauss meter. Select the 200 G range and ac mode. Switch on the audio amplifier and tune the function generator to the resonating frequency, 1.22 kHz. Current is now passing through the Helmholtz coil. Select some value of current and measure the corresponding magnetic field at the midpoint between the Helmholtz coils (A field of 90 G rms is a reasonably good value). Move the probe away from the center of coils on both sides. Check that magnetic field is not reaching the polarizers and photodiode. If required, adjust the distances by moving the rail carriers along the length of the optical rail. Map the magnetic field profile by moving the probe along the length of the sample with a step size of 0.5 cm, for different values of current (similar to the observation tabulated in (4)). Remove the probe and turn off the magnetic field. Tabulate the variation in magnetic field along the length of the sample as in Table 4.

8. Estimate the numerically integrated magnetic field over the length of the sample d,

$$\Gamma = \Sigma_k B^k(z)\, \Delta z^k. \tag{54}$$

Table 4: Numerical integration of the magnetic field.

i (A rms)	.1	.2	.3	.4	.5	.6	.7	.8	.9	1	1.1	1.2	1.3	1.4
z (cm)	B(G rms)													
.	.													
.	.													
.	.													
Γ(G cm)														

You will now measure i_{dc}, the dc component of the detected signal, in the absence of magnetic field, with the sample in place. Perform optical alignment if required.

9. Connect the photodiode to the oscilloscope, select the dc input mode. Remove the background reading from ambient light either by placing a black tube or by making the background level at the datum. Open the laserhead shutter.

10. Rotate the analyzer angle ϕ and find out the maximum and minimum intensity. Set the analyzer at angle of 45^o approximately w.r.t polarizer. Note

down the value of voltage at the oscilloscope and divide it by 1 MΩ, the input impedance of oscilloscope, to get i_{dc}.

Next, you will determine i_{ac}, the (rms value of the) ac component of the photocurrent ($\propto$ intensity), using the lock-in amplifier.

11. Activate the magnetic field. Provide the reference signal from the oscillator to the lock-in amplifier. Select the reference frequency mode, f Hz in the reference channel of lock-in amplifier. Select the current input mode I, from the input section. Make the offset equal to zero. Connect the photodiode to the input BNC connector. Make sure that no error indication (unlock or overload) occurs. Turn on the band pass and line frequency filter. The effect of each key press on the lock-in amplifier is indicated by a nearby LED. Select a suitable sensitivity (usually 200 nA) and time constant (3 ms) for the pre filter. Check that no offset is introduced by the lock-in amplifier at the selected sensitivity scale.

12. Adjust the phase located in reference section to make the output equal to zero. Then introduce a phase shift of 90^o, bringing the reference and input signals in-phase.

13. Rotate the analyzer angle ϕ in steps of 10^o and tabulate i_{ac}, for any fixed value of the magnetic field. You will observe that the maximum rotation occurs when analyzer is at an angle of approximately 45^o relative to polarizer.

    Q 17. What does the reading on the lock-in amplifier physically represent?

14. Fix the analyzer at 45^o relative to the polarizer. Increase the magnetic field, from an initial value of 10 Gauss, in steps of 5 or 10 Gauss by increasing the current. The transverse probe of Gaussmeter can be fixed to observe the magnetic field in the center of Helmholtz coil.

15. Tabulate the values for i_{ac} for each value of current (and hence the corresponding magnetic field) passing through the coil.

    Q 18. Use the above results to calculate the Verdet constant of your sample?

    Q 19. Clearly quantify your uncertainties. What are the major sources of error?

    Q 20. Can you measure i_{dc} with the help of the lock-in amplifier?

16. Remove the HeNe laser from setup, place AlGaN diode laser. Turn the diode laser on, its warm up time is 15 minutes and its output is linearly polarized, therefore, remove the polarizer A, use analyzer B only and repeat all the steps.

4.1 Applications of Faraday rotation

4.1.1 Optical isolators

An optical isolator acts as a photon valve, passing radiation in one direction and blocking in the other. An isolator is shown in Figure 14. Polarizer A is used to make the beam horizontally polarized which is then passed through a 45° Faraday rotator C, followed by another linear polarizer(analyzer, B) at 45° relative to A. If any of the light is reflected or backscattered from analyzer, it undergoes an extra rotation of 45° by C and thus is blocked by A. In a LASER, if any of the emitted light returns into the active medium through an unwanted reflection, it can generate instabilities in the emission. Optical isolators are used to prevent the unwanted reflection in lasers [7].

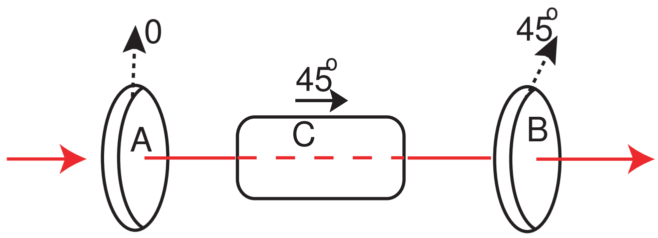

Figure 14: Optical isolator, backscattered radiation undergoes an additional 45° rotation by C , thus is blocked by polarizer A.

4.1.2 Domain Observation

Light will have different characteristics after reflection or transmission by regions having different orientations of magnetic moments. Let a sample be made up of three domains, the magnetization of each domain is shown in Figure 15. Plane polarized light of wavelength λ, passing through domain A, is rotated through some angle θ_1, while interacting with C, is rotated $-\theta_1$. If the analyzer is at $-\theta_1$, A may be dark, C bright and D of intermediate shade. For analyzer at 90°, D will be dark, A and C will be equally bright. For analyzer set at θ_1, A will be bright, C dark and D of intermediate shade, i.e, polarization direction may be turned one way or the other, depending on magnetization, thus resulting in different intensities, it is therefore possible to image magnetic domains [7].

4.1.3 Circulator

Optical circulators are used in fiber optics, to separate light traveling in opposite directions. Figure 16 shows one such circulator. It is made up of two Foster Seely Prisms and a 45° rotator placed between the prisms. In these prisms the rejected polarized light is internally reflected, so that it exits perpendicular to the axis of prism. Horizontally polarized light entering along **a** passes straight through the

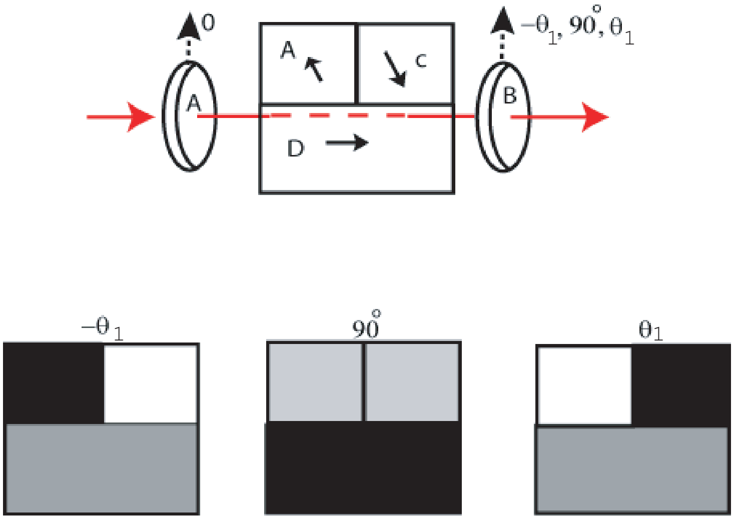

Figure 15: The domain imaging through magneto optic rotation.

prism, is rotated to 45° by the rotator using Faraday rotation and emerges from second prism at **b**. However, any light reflected back to the circulator entering at **b** with polarization azimuth 45° undergoes a 45° rotation through the rod, thus polarized at 90° and exits from port **c**. Similarly, light entering at **c** emerges at **d** and entering at **d** exits at **a** [7]. A circulator has at least three ports. The

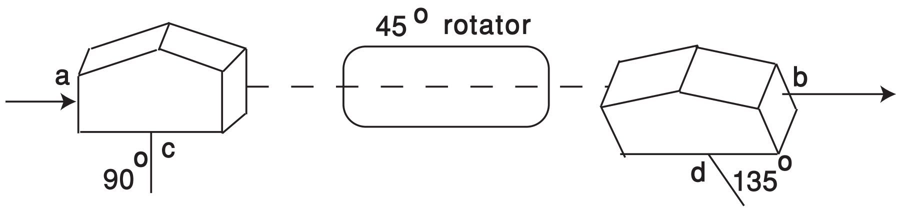

Figure 16: 45° rotator placed between two Foster-Seely prisms constitutes a circulator

light beam if entering from one port after passing through circulator exits from the second. Another light beam entering from the second port or light reflected from second port exits from third port and so on.

5 (OPTIONAL) Measurement of the the Verdet constant using higher harmonic components

The light rotated by the Faraday medium incident on the photodetector from analyzer, contains fundamental as well as components at higher frequencies. The rms values u_1 and u_2, at f and $2f$, respectively of these current components are measured, where f is the frequency of ac signal passing through Helmholtz coil. The ratios u_1/U_o and u_2/U_o can also be used to determine the Verdet constant, where, U_o is the steady output from the photodiode under zero magnetic field and analyzer set for maximum transmittance [9]. The power transmitted through a Faraday rotator is,

$$\begin{aligned} I &= \frac{A_o^2}{2}[1+\cos 2(\phi-\theta)] \qquad (55) \\ &= \frac{A_o^2}{2}[1+\cos 2(\phi-\theta_o\cos(\Omega t))] \\ &= \frac{A_o^2}{2}[1+\cos 2\phi\cos(2\theta_o\cos\Omega t)+\sin 2\phi\sin(2\theta_o\cos\Omega t)]. \end{aligned}$$

Using the Jacobi-Anger expansion, we obtain[10],

$$\begin{aligned} \cos(2\theta_o\cos\Omega t) &= J_o(2\theta_o)+2\sum_{m=1}^{\infty}(-1)^m J_{2m}(2\theta_o)\cos(2m\Omega t) \\ \sin(2\theta_o\cos\Omega t) &= 2\sum_{m=1}^{\infty}(-1)^m J_{2m+1}(2\theta_o)\cos((2m+1)\Omega t) \end{aligned}$$

where the Bessel function is,

$$J_\alpha(x) = \sum_{q=0}^{\infty}\frac{(-1)^q}{q!\gamma(q+\alpha+1)}(\frac{x}{2})^{2q+\alpha}$$

and γ is the factorial function, given by,[1]

$$\gamma(n)=(n-1)!$$

Therefore, Eq (56) becomes,

$$\begin{aligned} I = \frac{A_o^2}{2}\Bigg[&1+\cos(2\phi)\left(J_o(2\theta_o)+2\sum_{m=1}^{\infty}(-1)^m J_{2m}(2\theta_o)\cos(2m\Omega t)\right) \\ &+\sin(2\phi)\left(2\sum_{m=0}^{\infty}(-1)^m J_{2m+1}(2\theta_o)\cos(2m+1)\Omega t\right)\Bigg]. \qquad (56) \end{aligned}$$

Let the amplitude of coefficient of the terms containing Ωt and $2\Omega t$ be represented by s_1 and s_2 respectively. Then,

$$\begin{aligned} s_1 &= \frac{A_o^2}{2}2(-1)^0 J_1(2\theta_o)|\sin(2\phi)| \\ &= A_o^2\sum_{q=0}^{\infty}\frac{(-1)^q}{q!\gamma(q+1+1)}\left[\frac{2\theta_o}{2}\right]^{2q+1}|\sin(2\phi)| \\ &= {A_o}^2\left[\frac{(-1)^0}{0!\gamma(2)}\theta_o+\frac{-1}{1!\gamma(3)}\theta_o^3+\frac{(-1)^2}{2!\gamma(4)}\theta_o^5+...\right]|\sin(2\phi)| \qquad (57) \\ &= {A_o}^2\left[\frac{1}{1!}\theta_o+\frac{-1}{1!2!}\theta_o^3+\frac{1}{2!3!}\theta_o^5...\right]|\sin(2\phi)| \\ &= {A_o}^2\theta_o\left[1+\frac{-1}{2}\theta_o^2+\frac{1}{12}\theta_o^4...\right]|\sin(2\phi)| \qquad (58) \end{aligned}$$

[1] Γ is the conventional symbol to generalize the factorial function. Since, we are using Γ for numerically integrated magnetic field, therefore, we have used γ to denote the general form of factorial function.

$$
\begin{aligned}
s_2 &= \frac{A_o^2}{2} 2J_2(2\theta_o)|\cos(2\phi)| \\
&= A_o^2 \sum_{q=0}^{\infty} \frac{(-1)^q}{q!\gamma(q+2+1)} \left[\frac{2\theta_o}{2}\right]^{2q+2} |\cos(2\phi)| \\
&= A_o^2 \left[\frac{1}{\gamma(3)}\theta_o^2 + \frac{-1}{\gamma(4)}\theta_o^4 + \frac{(-1)^2}{2!\gamma(5)}\theta_o^6 + ...\right] |\cos(2\phi)| \\
&= A_o^2 \left[\frac{1}{2!}\theta_o^2 - \frac{1}{3!}\theta_o^4 + \frac{1}{2!4!}\theta_o^6 ...\right] |\cos(2\phi)| \\
&= \frac{A_o^2}{2}\theta_o^2 \left[1 - \frac{1}{3}\theta_o^2 + \frac{1}{24}\theta_o^4 ...\right] |\cos(2\phi)| \qquad (59)
\end{aligned}
$$

Since,

$$
\theta_o = VB_o d \qquad (60)
$$

Substituting equation (60) in (58)

$$
\begin{aligned}
s_1 &= {A_o}^2 VB_o d \left[1 + \frac{-1}{2}(VB_o d)^2 + \frac{1}{12}(VB_o d)^4 ...\right] |\sin(2\phi)| \\
&= U_o VB_o d \left[1 + \frac{-1}{2}(VB_o d)^2 + \frac{1}{12}(VB_o d)^4 ...\right] |\sin(2\phi)|. \qquad (61)
\end{aligned}
$$

where, U_o is the steady power on photodetector when polarizers are set for maximum transmittance (in the absence of applied magnetic field).

Substituting equation (60) into (59), we obtain,

$$
\begin{aligned}
s_2 &= \frac{A_o^2}{2}(VB_o d)^2 \left[1 - \frac{1}{3}(VB_o d)^2 + \frac{1}{24}(VB_o d)^4 + ...\right] |\cos(2\phi)| \\
&= \frac{U_o}{2}(VB_o d)^2 \left[1 - \frac{1}{3}(VB_o d)^2 + \frac{1}{24}(VB_o d)^4 + ...\right] |\cos(2\phi)|. \qquad (62)
\end{aligned}
$$

The f and $2f$ components are determined through lock-in amplifier which displays rms values, so from equation (61), the rms value of the first harmonic component of output current (ignoring higher order terms) is,

$$
\begin{aligned}
u_1 &\approx \frac{U_o VB_o d}{\sqrt{2}} |\sin(2\phi)| \\
&= U_o VBd |\sin(2\phi)| \qquad (63)
\end{aligned}
$$

where $B = B_o/\sqrt{2}$, B represents the rms value of the field measured by the Gauss-meter. Similarly, from (62) the rms value of the second harmonic component of output current is,

$$
\begin{aligned}
u_2 &\approx \frac{U_o}{2\sqrt{2}}(VB_o d)^2 |\cos(2\phi)| \qquad (64) \\
&= \frac{U_o}{\sqrt{2}}(VBd)^2 |\cos(2\phi)|. \qquad (65)
\end{aligned}
$$

Both equations (63) and (65) can be used to determine Verdet constant.

In short, we have three three different means of measuring the Farday rotation,

Method 1. The gradient of the plot of u_1 or i_{ac} against $\mathbf{B}$ for $\phi = 45^o$ results in the Verdet constant. This is, in fact, the method you have used in previous section. Since, $U_o = 2i_{dc}$ and $u_1 = i_{ac}$, Equation (63) is actually Eq. (43) in disguise.

Method 2. Determine the gradient of the least squares-fit line to a plot of u_1/B against $|\sin 2\phi|$ for fixed U_o. Equate the gradient to $VBdU_o$ and find the Verdet constant [9].

Method 3. Determine the gradient of a plot of u_2 against $\mathbf{B}^2$ when $\phi = 90^o$. equate this to to $V^2d^2U_o/\sqrt{2}$ and find the Verdet constant.

Q 21. Find the Verdet constant for TGG at 405 nm using methods 2 and 3.