Linear Algebra

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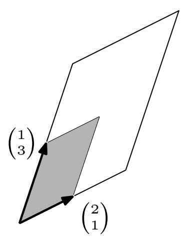
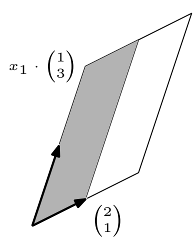
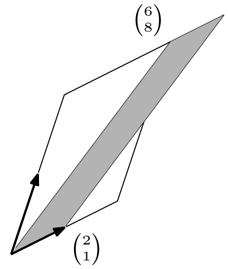


$$\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$



$$\begin{vmatrix} x \cdot 1 & 2 \\ x \cdot 3 & 1 \end{vmatrix}$$



$$\begin{vmatrix} 6 & 2 \\ 8 & 1 \end{vmatrix}$$

Notation

$\mathbb{R}$	real numbers
$\mathbb{N}$	natural numbers: $\{0, 1, 2, \ldots\}$
$\mathbb{C}$	complex numbers
$\{\ldots \mid \ldots\}$	set of ... such that ...
$\langle \ldots \rangle$	sequence; like a set but order matters
V, W, U	vector spaces
$\vec{v}, \vec{w}$	vectors
$\vec{0}$, $\vec{0}_V$	zero vector, zero vector of V
B, D	bases
$\mathcal{E}_n = \langle \vec{e}_1, \ldots, \vec{e}_n \rangle$	standard basis for $\mathbb{R}^n$
$\vec{\beta}, \vec{\delta}$	basis vectors
$\operatorname{Rep}_B(\vec{v})$	matrix representing the vector
$\mathcal{P}_n$	set of n-th degree polynomials
$\mathcal{M}_{n \times m}$	set of $n \times m$ matrices
$[S]$	span of the set S
$M \oplus N$	direct sum of subspaces
$V \cong W$	isomorphic spaces
h, g	homomorphisms, linear maps
H, G	matrices
t, s	transformations; maps from a space to itself
T, S	square matrices
$\operatorname{Rep}_{B,D}(h)$	matrix representing the map h
$h_{i,j}$	matrix entry from row i, column j
$\lvert T \rvert$	determinant of the matrix T
$\mathscr{R}(h), \mathscr{N}(h)$	rangespace and nullspace of the map h
$\mathscr{R}_\infty(h), \mathscr{N}_\infty(h)$	generalized rangespace and nullspace

Lower case Greek alphabet

name	character	name	character	name	character
alpha	α	iota	ι	rho	ρ
beta	β	kappa	κ	sigma	σ
gamma	γ	lambda	λ	tau	τ
delta	δ	mu	μ	upsilon	υ
epsilon	ϵ	nu	ν	phi	ϕ
zeta	ζ	xi	ξ	chi	χ
eta	η	omicron	o	psi	ψ
theta	θ	pi	π	omega	ω

Cover. This is Cramer's Rule for the system $x + 2y = 6$, $3x + y = 8$. The size of the first box is the determinant shown (the absolute value of the size is the area). The size of the second box is x times that, and equals the size of the final box. Hence, x is the final determinant divided by the first determinant.

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* *Note:* starred subsections are optional.

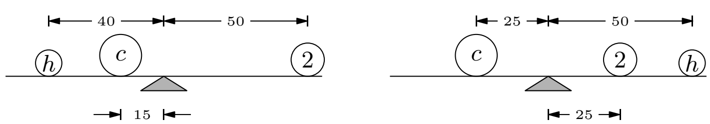
Chapter One

Linear Systems

I Solving Linear Systems

Systems of linear equations are common in science and mathematics. These two examples from high school science [Onan] give a sense of how they arise.

The first example is from Physics. Suppose that we are given three objects, one with a mass known to be 2 kg, and are asked to find the unknown masses. Suppose further that experimentation with a meter stick produces these two balances.



Since the sum of moments on the left of each balance equals the sum of moments on the right (the moment of an object is its mass times its distance from the balance point), the two balances give this system of two equations.

$$\begin{aligned} 40h + 15c &= 100 \\ 25c &= 50 + 50h \end{aligned}$$

The second example of a linear system is from Chemistry. We can mix, under controlled conditions, toluene C_7H_8 and nitric acid HNO_3 to produce trinitrotoluene $C_7H_5O_6N_3$ along with the byproduct water (conditions have to be controlled very well, indeed — trinitrotoluene is better known as TNT). In what proportion should those components be mixed? The number of atoms of each element present before the reaction

$$x\,C_7H_8 \;+\; y\,HNO_3 \quad\longrightarrow\quad z\,C_7H_5O_6N_3 \;+\; w\,H_2O$$

must equal the number present afterward. Applying that principle to the elements C, H, N, and O in turn gives this system.

$$\begin{aligned} 7x &= 7z \\ 8x + 1y &= 5z + 2w \\ 1y &= 3z \\ 3y &= 6z + 1w \end{aligned}$$

To finish each of these examples requires solving a system of equations. In each, the equations involve only the first power of the variables. This chapter shows how to solve any such system.

I.1 Gauss' Method

1.1 Definition A *linear equation* in variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = d$$

where the numbers $a_1, \ldots, a_n \in \mathbb{R}$ are the equation's *coefficients* and $d \in \mathbb{R}$ is the *constant*. An n-tuple $(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$ is a *solution* of, or *satisfies*, that equation if substituting the numbers $s_1, \ldots, s_n$ for the variables gives a true statement: $a_1s_1 + a_2s_2 + \ldots + a_ns_n = d$.

A *system of linear equations*

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= d_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= d_2 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= d_m \end{aligned}$$

has the solution $(s_1, s_2, \ldots, s_n)$ if that n-tuple is a solution of all of the equations in the system.

1.2 Example The ordered pair $(-1, 5)$ is a solution of this system.

$$\begin{aligned} 3x_1 + 2x_2 &= 7 \\ -x_1 + x_2 &= 6 \end{aligned}$$

In contrast, $(5, -1)$ is not a solution.

Finding the set of all solutions is *solving* the system. No guesswork or good fortune is needed to solve a linear system. There is an algorithm that always works. The next example introduces that algorithm, called *Gauss' method*. It transforms the system, step by step, into one with a form that is easily solved.

1.3 Example To solve this system

$$\begin{aligned} 3x_3 &= 9 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ \tfrac{1}{3}x_1 + 2x_2 \qquad &= 3 \end{aligned}$$

we repeatedly transform it until it is in a form that is easy to solve.

$$\xrightarrow{\text{swap row 1 with row 3}} \begin{aligned} \tfrac{1}{3}x_1 + 2x_2 \qquad &= 3 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ 3x_3 &= 9 \end{aligned}$$

$$\xrightarrow{\text{multiply row 1 by 3}} \begin{aligned} x_1 + 6x_2 \qquad &= 9 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ 3x_3 &= 9 \end{aligned}$$

$$\xrightarrow{\text{add } -1 \text{ times row 1 to row 2}} \begin{aligned} x_1 + 6x_2 \qquad &= 9 \\ -x_2 - 2x_3 &= -7 \\ 3x_3 &= 9 \end{aligned}$$

The third step is the only nontrivial one. We've mentally multiplied both sides of the first row by -1, mentally added that to the old second row, and written the result in as the new second row.

Now we can find the value of each variable. The bottom equation shows that $x_3 = 3$. Substituting 3 for x_3 in the middle equation shows that $x_2 = 1$. Substituting those two into the top equation gives that $x_1 = 3$ and so the system has a unique solution: the solution set is $\{(3, 1, 3)\}$.

Most of this subsection and the next one consists of examples of solving linear systems by Gauss' method. We will use it throughout this book. It is fast and easy. But, before we get to those examples, we will first show that this method is also safe in that it never loses solutions or picks up extraneous solutions.

1.4 Theorem (Gauss' method) If a linear system is changed to another by one of these operations

(1) an equation is swapped with another

(2) an equation has both sides multiplied by a nonzero constant

(3) an equation is replaced by the sum of itself and a multiple of another

then the two systems have the same set of solutions.

Each of those three operations has a restriction. Multiplying a row by 0 is not allowed because obviously that can change the solution set of the system. Similarly, adding a multiple of a row to itself is not allowed because adding -1 times the row to itself has the effect of multiplying the row by 0. Finally, swapping a row with itself is disallowed to make some results in the fourth chapter easier to state and remember (and besides, self-swapping doesn't accomplish anything).

Proof. We will cover the equation swap operation here and save the other two cases for Exercise 29.

Consider this swap of row i with row j.

$$\begin{array}{rcl} a_{1,1}x_1 + a_{1,2}x_2 + \cdots a_{1,n}x_n &=& d_1 \\ &\vdots& \\ a_{i,1}x_1 + a_{i,2}x_2 + \cdots a_{i,n}x_n &=& d_i \\ &\vdots& \\ a_{j,1}x_1 + a_{j,2}x_2 + \cdots a_{j,n}x_n &=& d_j \\ &\vdots& \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots a_{m,n}x_n &=& d_m \end{array} \longrightarrow \begin{array}{rcl} a_{1,1}x_1 + a_{1,2}x_2 + \cdots a_{1,n}x_n &=& d_1 \\ &\vdots& \\ a_{j,1}x_1 + a_{j,2}x_2 + \cdots a_{j,n}x_n &=& d_j \\ &\vdots& \\ a_{i,1}x_1 + a_{i,2}x_2 + \cdots a_{i,n}x_n &=& d_i \\ &\vdots& \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots a_{m,n}x_n &=& d_m \end{array}$$

The n-tuple $(s_1, \ldots, s_n)$ satisfies the system before the swap if and only if substituting the values, the s's, for the variables, the x's, gives true statements: $a_{1,1}s_1 + a_{1,2}s_2 + \cdots + a_{1,n}s_n = d_1$ and ... $a_{i,1}s_1 + a_{i,2}s_2 + \cdots + a_{i,n}s_n = d_i$ and ... $a_{j,1}s_1 + a_{j,2}s_2 + \cdots + a_{j,n}s_n = d_j$ and ... $a_{m,1}s_1 + a_{m,2}s_2 + \cdots + a_{m,n}s_n = d_m$.

In a requirement consisting of statements and-ed together we can rearrange the order of the statements, so that this requirement is met if and only if $a_{1,1}s_1 + a_{1,2}s_2 + \cdots + a_{1,n}s_n = d_1$ and ... $a_{j,1}s_1 + a_{j,2}s_2 + \cdots + a_{j,n}s_n = d_j$ and ... $a_{i,1}s_1 + a_{i,2}s_2 + \cdots + a_{i,n}s_n = d_i$ and ... $a_{m,1}s_1 + a_{m,2}s_2 + \cdots + a_{m,n}s_n = d_m$. This is exactly the requirement that $(s_1, \ldots, s_n)$ solves the system after the row swap. QED

1.5 Definition The three operations from Theorem 1.4 are the *elementary reduction operations*, or *row operations*, or *Gaussian operations*. They are *swapping*, *multiplying by a scalar* or *rescaling*, and *pivoting*.

When writing out the calculations, we will abbreviate 'row i' by 'ρ_i'. For instance, we will denote a pivot operation by $k\rho_i + \rho_j$, with the row that is changed written second. We will also, to save writing, often list pivot steps together when they use the same ρ_i.

1.6 Example A typical use of Gauss' method is to solve this system.

$$\begin{aligned} x + y &= 0 \\ 2x - y + 3z &= 3 \\ x - 2y - z &= 3 \end{aligned}$$

The first transformation of the system involves using the first row to eliminate the x in the second row and the x in the third. To get rid of the second row's $2x$, we multiply the entire first row by -2, add that to the second row, and write the result in as the new second row. To get rid of the third row's x, we multiply the first row by -1, add that to the third row, and write the result in as the new third row.

$$\xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{aligned} x + y &= 0 \\ -3y + 3z &= 3 \\ -3y - z &= 3 \end{aligned}$$

(Note that the two ρ_1 steps $-2\rho_1+\rho_2$ and $-\rho_1+\rho_3$ are written as one operation.) In this second system, the last two equations involve only two unknowns. To finish we transform the second system into a third system, where the last equation involves only one unknown. This transformation uses the second row to eliminate y from the third row.

$$\xrightarrow{-\rho_2+\rho_3}\quad \begin{aligned} x + \quad y \qquad\quad &= 0 \\ -3y + \quad 3z &= 3 \\ -4z &= 0 \end{aligned}$$

Now we are set up for the solution. The third row shows that $z=0$. Substitute that back into the second row to get $y=-1$, and then substitute back into the first row to get $x=1$.

1.7 Example For the Physics problem from the start of this chapter, Gauss' method gives this.

$$\begin{aligned} 40h + 15c &= 100 \\ -50h + 25c &= \;\;50 \end{aligned} \quad\xrightarrow{5/4\rho_1+\rho_2}\quad \begin{aligned} 40h + \qquad 15c &= 100 \\ (175/4)c &= 175 \end{aligned}$$

So $c=4$, and back-substitution gives that $h=1$. (The Chemistry problem is solved later.)

1.8 Example The reduction

$$\begin{aligned} x + \;\; y + \;\; z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \quad\xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2}\quad \begin{aligned} x + \;\; y + \;\; z &= \quad 9 \\ 2y - 5z &= -17 \\ 3y - 8z &= -27 \end{aligned}$$

$$\xrightarrow{-(3/2)\rho_2+\rho_3}\quad \begin{aligned} x + \;\; y + \qquad\quad z &= \quad 9 \\ 2y - \qquad 5z &= -17 \\ -(1/2)z &= -(3/2) \end{aligned}$$

shows that $z=3$, $y=-1$, and $x=7$.

As these examples illustrate, Gauss' method uses the elementary reduction operations to set up back-substitution.

1.9 Definition In each row, the first variable with a nonzero coefficient is the row's *leading variable*. A system is in *echelon form* if each leading variable is to the right of the leading variable in the row above it (except for the leading variable in the first row).

1.10 Example The only operation needed in the examples above is pivoting. Here is a linear system that requires the operation of swapping equations. After the first pivot

$$\begin{aligned} x - \;\; y \qquad\qquad\quad &= 0 \\ 2x - 2y + \;\; z + 2w &= 4 \\ y \qquad + \;\; w &= 0 \\ 2z + \;\; w &= 5 \end{aligned} \quad\xrightarrow{-2\rho_1+\rho_2}\quad \begin{aligned} x - y \qquad\qquad &= 0 \\ z + 2w &= 4 \\ y \qquad + \;\; w &= 0 \\ 2z + \;\; w &= 5 \end{aligned}$$

the second equation has no leading y. To get one, we look lower down in the system for a row that has a leading y and swap it in.

$$\xrightarrow{\rho_2\leftrightarrow\rho_3}\;\begin{array}{rcrcrcrcr} x&-&y&&&&&=&0\\ &&y&&&+&w&=&0\\ &&&&z&+&2w&=&4\\ &&&&2z&+&w&=&5 \end{array}$$

(Had there been more than one row below the second with a leading y then we could have swapped in any one.) The rest of Gauss' method goes as before.

$$\xrightarrow{-2\rho_3+\rho_4}\;\begin{array}{rcrcrcrcr} x&-&y&&&&&=&0\\ &&y&&&+&w&=&0\\ &&&&z&+&2w&=&4\\ &&&&&&-3w&=&-3 \end{array}$$

Back-substitution gives $w = 1$, $z = 2$, $y = -1$, and $x = -1$.

Strictly speaking, the operation of rescaling rows is not needed to solve linear systems. We have included it because we will use it later in this chapter as part of a variation on Gauss' method, the Gauss-Jordan method.

All of the systems seen so far have the same number of equations as unknowns. All of them have a solution, and for all of them there is only one solution. We finish this subsection by seeing for contrast some other things that can happen.

1.11 Example Linear systems need not have the same number of equations as unknowns. This system

$$\begin{array}{rcrcr} x&+&3y&=&1\\ 2x&+&y&=&-3\\ 2x&+&2y&=&-2 \end{array}$$

has more equations than variables. Gauss' method helps us understand this system also, since this

$$\xrightarrow[-2\rho_1+\rho_3]{-2\rho_1+\rho_2}\;\begin{array}{rcrcr} x&+&3y&=&1\\ &&-5y&=&-5\\ &&-4y&=&-4 \end{array}$$

shows that one of the equations is redundant. Echelon form

$$\xrightarrow{-(4/5)\rho_2+\rho_3}\;\begin{array}{rcrcr} x&+&3y&=&1\\ &&-5y&=&-5\\ &&0&=&0 \end{array}$$

gives $y = 1$ and $x = -2$. The '$0 = 0$' is derived from the redundancy.

That example's system has more equations than variables. Gauss' method is also useful on systems with more variables than equations. Many examples are in the next subsection.

Another way that linear systems can differ from the examples shown earlier is that some linear systems do not have a unique solution. This can happen in two ways.

The first is that it can fail to have any solution at all.

1.12 Example Contrast the system in the last example with this one.

$$\begin{array}{rcl} x+3y&=&1 \\ 2x+y&=&-3 \\ 2x+2y&=&0 \end{array} \xrightarrow[-2\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{array}{rcl} x+3y&=&1 \\ -5y&=&-5 \\ -4y&=&-2 \end{array}$$

Here the system is inconsistent: no pair of numbers satisfies all of the equations simultaneously. Echelon form makes this inconsistency obvious.

$$\xrightarrow{-(4/5)\rho_2+\rho_3} \begin{array}{rcl} x+3y&=&1 \\ -5y&=&-5 \\ 0&=&2 \end{array}$$

The solution set is empty.

1.13 Example The prior system has more equations than unknowns, but that is not what causes the inconsistency — Example 1.11 has more equations than unknowns and yet is consistent. Nor is having more equations than unknowns necessary for inconsistency, as is illustrated by this inconsistent system with the same number of equations as unknowns.

$$\begin{array}{rcl} x+2y&=&8 \\ 2x+4y&=&8 \end{array} \xrightarrow{-2\rho_1+\rho_2} \begin{array}{rcl} x+2y&=&8 \\ 0&=&-8 \end{array}$$

The other way that a linear system can fail to have a unique solution is to have many solutions.

1.14 Example In this system

$$\begin{array}{rcl} x+y&=&4 \\ 2x+2y&=&8 \end{array}$$

any pair of numbers satisfying the first equation automatically satisfies the second. The solution set $\{(x,y) \mid x+y=4\}$ is infinite; some of its members are $(0,4)$, $(-1,5)$, and $(2.5,1.5)$. The result of applying Gauss' method here contrasts with the prior example because we do not get a contradictory equation.

$$\xrightarrow{-2\rho_1+\rho_2} \begin{array}{rcl} x+y&=&4 \\ 0&=&0 \end{array}$$

Don't be fooled by the '$0=0$' equation in that example. It is not the signal that a system has many solutions.

1.15 Example The absence of a '$0 = 0$' does not keep a system from having many different solutions. This system is in echelon form

$$\begin{aligned} x + y + z &= 0 \\ y + z &= 0 \end{aligned}$$

has no '$0 = 0$', and yet has infinitely many solutions. (For instance, each of these is a solution: $(0, 1, -1)$, $(0, 1/2, -1/2)$, $(0, 0, 0)$, and $(0, -\pi, \pi)$. There are infinitely many solutions because any triple whose first component is 0 and whose second component is the negative of the third is a solution.)

Nor does the presence of a '$0 = 0$' mean that the system must have many solutions. Example 1.11 shows that. So does this system, which does not have many solutions — in fact it has none — despite that when it is brought to echelon form it has a '$0 = 0$' row.

$$\begin{array}{rcl} \begin{aligned} 2x \qquad - 2z &= 6 \\ y + z &= 1 \\ 2x + y - z &= 7 \\ 3y + 3z &= 0 \end{aligned} & \xrightarrow{-\rho_1+\rho_3} & \begin{aligned} 2x \qquad - 2z &= 6 \\ y + z &= 1 \\ y + z &= 1 \\ 3y + 3z &= 0 \end{aligned} \\ & \xrightarrow[-3\rho_2+\rho_4]{-\rho_2+\rho_3} & \begin{aligned} 2x \qquad - 2z &= 6 \\ y + z &= 1 \\ 0 &= 0 \\ 0 &= -3 \end{aligned} \end{array}$$

We will finish this subsection with a summary of what we've seen so far about Gauss' method.

Gauss' method uses the three row operations to set a system up for back substitution. If any step shows a contradictory equation then we can stop with the conclusion that the system has no solutions. If we reach echelon form without a contradictory equation, and each variable is a leading variable in its row, then the system has a unique solution and we find it by back substitution. Finally, if we reach echelon form without a contradictory equation, and there is not a unique solution (at least one variable is not a leading variable) then the system has many solutions.

The next subsection deals with the third case — we will see how to describe the solution set of a system with many solutions.

Exercises

✓ **1.16** Use Gauss' method to find the unique solution for each system.

(a) $\begin{aligned} 2x + 3y &= 13 \\ x - y &= -1 \end{aligned}$ **(b)** $\begin{aligned} x \qquad - z &= 0 \\ 3x + y \qquad &= 1 \\ -x + y + z &= 4 \end{aligned}$

✓ **1.17** Use Gauss' method to solve each system or conclude 'many solutions' or 'no solutions'.

(a) $\begin{aligned} 2x + 2y &= 5 \\ x - 4y &= 0 \end{aligned}$ **(b)** $\begin{aligned} -x + y &= 1 \\ x + y &= 2 \end{aligned}$ **(c)** $\begin{aligned} x - 3y + z &= 1 \\ x + y + 2z &= 14 \end{aligned}$

(d) $\begin{aligned} -x - y &= 1 \\ -3x - 3y &= 2 \end{aligned}$ **(e)** $\begin{aligned} 4y + z &= 20 \\ 2x - 2y + z &= 0 \\ x + z &= 5 \\ x + y - z &= 10 \end{aligned}$ **(f)** $\begin{aligned} 2x + z + w &= 5 \\ y - w &= -1 \\ 3x - z - w &= 0 \\ 4x + y + 2z + w &= 9 \end{aligned}$

✓ **1.18** There are methods for solving linear systems other than Gauss' method. One often taught in high school is to solve one of the equations for a variable, then substitute the resulting expression into other equations. That step is repeated until there is an equation with only one variable. From that, the first number in the solution is derived, and then back-substitution can be done. This method both takes longer than Gauss' method, since it involves more arithmetic operations and is more likely to lead to errors. To illustrate how it can lead to wrong conclusions, we will use the system

$$\begin{aligned} x + 3y &= 1 \\ 2x + y &= -3 \\ 2x + 2y &= 0 \end{aligned}$$

from Example 1.12.

(a) Solve the first equation for x and substitute that expression into the second equation. Find the resulting y.

(b) Again solve the first equation for x, but this time substitute that expression into the third equation. Find this y.

What extra step must a user of this method take to avoid erroneously concluding a system has a solution?

✓ **1.19** For which values of k are there no solutions, many solutions, or a unique solution to this system?

$$\begin{aligned} x - y &= 1 \\ 3x - 3y &= k \end{aligned}$$

✓ **1.20** This system is not linear, in some sense,

$$\begin{aligned} 2\sin\alpha - \cos\beta + 3\tan\gamma &= 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 10 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= 9 \end{aligned}$$

and yet we can nonetheless apply Gauss' method. Do so. Does the system have a solution?

✓ **1.21** What conditions must the constants, the b's, satisfy so that each of these systems has a solution? *Hint.* Apply Gauss' method and see what happens to the right side. [Anton]

(a) $\begin{aligned} x - 3y &= b_1 \\ 3x + y &= b_2 \\ x + 7y &= b_3 \\ 2x + 4y &= b_4 \end{aligned}$ **(b)** $\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 + 8x_3 &= b_3 \end{aligned}$

1.22 True or false: a system with more unknowns than equations has at least one solution. (As always, to say 'true' you must prove it, while to say 'false' you must produce a counterexample.)

1.23 Must any Chemistry problem like the one that starts this subsection — a balance the reaction problem — have infinitely many solutions?

✓ **1.24** Find the coefficients a, b, and c so that the graph of $f(x) = ax^2+bx+c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

1.25 Gauss' method works by combining the equations in a system to make new equations.

(a) Can the equation $3x-2y=5$ be derived, by a sequence of Gaussian reduction steps, from the equations in this system?

$$\begin{aligned} x+y&=1\\ 4x-y&=6 \end{aligned}$$

(b) Can the equation $5x-3y=2$ be derived, by a sequence of Gaussian reduction steps, from the equations in this system?

$$\begin{aligned} 2x+2y&=5\\ 3x+\ \ y&=4 \end{aligned}$$

(c) Can the equation $6x-9y+5z=-2$ be derived, by a sequence of Gaussian reduction steps, from the equations in the system?

$$\begin{aligned} 2x+\ \ y-z&=4\\ 6x-3y+z&=5 \end{aligned}$$

1.26 Prove that, where $a, b, \ldots, e$ are real numbers and $a \neq 0$, if

$$ax+by=c$$

has the same solution set as

$$ax+dy=e$$

then they are the same equation. What if $a=0$?

✓ **1.27** Show that if $ad-bc \neq 0$ then

$$\begin{aligned} ax+by&=j\\ cx+dy&=k \end{aligned}$$

has a unique solution.

✓ **1.28** In the system

$$\begin{aligned} ax+by&=c\\ dx+ey&=f \end{aligned}$$

each of the equations describes a line in the xy-plane. By geometrical reasoning, show that there are three possibilities: there is a unique solution, there is no solution, and there are infinitely many solutions.

1.29 Finish the proof of Theorem 1.4.

1.30 Is there a two-unknowns linear system whose solution set is all of $\mathbb{R}^2$?

✓ **1.31** Are any of the operations used in Gauss' method redundant? That is, can any of the operations be synthesized from the others?

1.32 Prove that each operation of Gauss' method is reversible. That is, show that if two systems are related by a row operation $S_1 \rightarrow S_2$ then there is a row operation to go back $S_2 \rightarrow S_1$.

? **1.33** A box holding pennies, nickels and dimes contains thirteen coins with a total value of 83 cents. How many coins of each type are in the box? [Anton]

? **1.34** Four positive integers are given. Select any three of the integers, find their arithmetic average, and add this result to the fourth integer. Thus the numbers 29, 23, 21, and 17 are obtained. One of the original integers is:

(a) 19 **(b)** 21 **(c)** 23 **(d)** 29 **(e)** 17
[Con. Prob. 1955]

? ✓ **1.35** Laugh at this: AHAHA + TEHE = TEHAW. It resulted from substituting a code letter for each digit of a simple example in addition, and it is required to identify the letters and prove the solution unique. [Am. Math. Mon., Jan. 1935]

? **1.36** The Wohascum County Board of Commissioners, which has 20 members, recently had to elect a President. There were three candidates (A, B, and C); on each ballot the three candidates were to be listed in order of preference, with no abstentions. It was found that 11 members, a majority, preferred A over B (thus the other 9 preferred B over A). Similarly, it was found that 12 members preferred C over A. Given these results, it was suggested that B should withdraw, to enable a runoff election between A and C. However, B protested, and it was then found that 14 members preferred B over C! The Board has not yet recovered from the resulting confusion. Given that every possible order of A, B, C appeared on at least one ballot, how many members voted for B as their first choice? [Wohascum no. 2]

? **1.37** "This system of n linear equations with n unknowns," said the Great Mathematician, "has a curious property."

"Good heavens!" said the Poor Nut, "What is it?"

"Note," said the Great Mathematician, "that the constants are in arithmetic progression."

"It's all so clear when you explain it!" said the Poor Nut. "Do you mean like $6x + 9y = 12$ and $15x + 18y = 21$?"

"Quite so," said the Great Mathematician, pulling out his bassoon. "Indeed, the system has a unique solution. Can you find it?"

"Good heavens!" cried the Poor Nut, "I am baffled."

Are you? [Am. Math. Mon., Jan. 1963]

I.2 Describing the Solution Set

A linear system with a unique solution has a solution set with one element. A linear system with no solution has a solution set that is empty. In these cases the solution set is easy to describe. Solution sets are a challenge to describe only when they contain many elements.

2.1 Example This system has many solutions because in echelon form

$$\begin{array}{rcl} 2x + z &=& 3 \\ x - y - z &=& 1 \\ 3x - y &=& 4 \end{array} \xrightarrow[-(3/2)\rho_1+\rho_3]{-(1/2)\rho_1+\rho_2} \begin{array}{rcl} 2x + z &=& 3 \\ -y - (3/2)z &=& -1/2 \\ -y - (3/2)z &=& -1/2 \end{array}$$

$$\xrightarrow{-\rho_2+\rho_3} \begin{array}{rcl} 2x + z &=& 3 \\ -y - (3/2)z &=& -1/2 \\ 0 &=& 0 \end{array}$$

not all of the variables are leading variables. The Gauss' method theorem showed that a triple satisfies the first system if and only if it satisfies the third. Thus, the solution set $\{(x, y, z) \mid 2x + z = 3 \text{ and } x - y - z = 1 \text{ and } 3x - y = 4\}$

can also be described as $\{(x, y, z) \mid 2x + z = 3 \text{ and } -y - 3z/2 = -1/2\}$. However, this second description is not much of an improvement. It has two equations instead of three, but it still involves some hard-to-understand interaction among the variables.

To get a description that is free of any such interaction, we take the variable that does not lead any equation, z, and use it to describe the variables that do lead, x and y. The second equation gives $y = (1/2) - (3/2)z$ and the first equation gives $x = (3/2) - (1/2)z$. Thus, the solution set can be described as $\{(x, y, z) = ((3/2) - (1/2)z, (1/2) - (3/2)z, z) \mid z \in \mathbb{R}\}$. For instance, $(1/2, -5/2, 2)$ is a solution because taking $z = 2$ gives a first component of $1/2$ and a second component of $-5/2$.

The advantage of this description over the ones above is that the only variable appearing, z, is unrestricted — it can be any real number.

2.2 Definition The non-leading variables in an echelon-form linear system are *free variables.*

In the echelon form system derived in the above example, x and y are leading variables and z is free.

2.3 Example A linear system can end with more than one variable free. This row reduction

$$\begin{array}{rcl}
\begin{aligned}
x + y + z - w &= 1 \\
y - z + w &= -1 \\
3x + 6z - 6w &= 6 \\
-y + z - w &= 1
\end{aligned}
& \xrightarrow{-3\rho_1+\rho_3} &
\begin{aligned}
x + y + z - w &= 1 \\
y - z + w &= -1 \\
-3y + 3z - 3w &= 3 \\
-y + z - w &= 1
\end{aligned} \\
& \xrightarrow[\rho_2+\rho_4]{3\rho_2+\rho_3} &
\begin{aligned}
x + y + z - w &= 1 \\
y - z + w &= -1 \\
0 &= 0 \\
0 &= 0
\end{aligned}
\end{array}$$

ends with x and y leading, and with both z and w free. To get the description that we prefer we will start at the bottom. We first express y in terms of the free variables z and w with $y = -1 + z - w$. Next, moving up to the top equation, substituting for y in the first equation $x + (-1 + z - w) + z - w = 1$ and solving for x yields $x = 2 - 2z + 2w$. Thus, the solution set is $\{2 - 2z + 2w, -1 + z - w, z, w) \mid z, w \in \mathbb{R}\}$.

We prefer this description because the only variables that appear, z and w, are unrestricted. This makes the job of deciding which four-tuples are system solutions into an easy one. For instance, taking $z = 1$ and $w = 2$ gives the solution $(4, -2, 1, 2)$. In contrast, $(3, -2, 1, 2)$ is not a solution, since the first component of any solution must be 2 minus twice the third component plus twice the fourth.

2.4 Example After this reduction

$$\begin{array}{rcl} 2x-2y \qquad\qquad\;\; =0 & & 2x-2y \qquad\quad =0 \\ z+3w=2 & \xrightarrow[-(1/2)\rho_1+\rho_4]{-(3/2)\rho_1+\rho_3} & z+3w=2 \\ 3x-3y \qquad\qquad\;\; =0 & & 0=0 \\ x-\;\; y+2z+6w=4 & & 2z+6w=4 \\ & & 2x-2y \qquad\quad =0 \\ & \xrightarrow{-2\rho_2+\rho_4} & z+3w=2 \\ & & 0=0 \\ & & 0=0 \end{array}$$

x and z lead, y and w are free. The solution set is $\{(y,y,2-3w,w) \bigm| y,w\in\mathbb{R}\}$. For instance, $(1,1,2,0)$ satisfies the system — take $y=1$ and $w=0$. The four-tuple $(1,0,5,4)$ is not a solution since its first coordinate does not equal its second.

We refer to a variable used to describe a family of solutions as a *parameter* and we say that the set above is *paramatrized* with y and w. (The terms 'parameter' and 'free variable' do not mean the same thing. Above, y and w are free because in the echelon form system they do not lead any row. They are parameters because they are used in the solution set description. We could have instead paramatrized with y and z by rewriting the second equation as $w=2/3-(1/3)z$. In that case, the free variables are still y and w, but the parameters are y and z. Notice that we could not have paramatrized with x and y, so there is sometimes a restriction on the choice of parameters. The terms 'parameter' and 'free' are related because, as we shall show later in this chapter, the solution set of a system can always be paramatrized with the free variables. Consequenlty, we shall paramatrize all of our descriptions in this way.)

2.5 Example This is another system with infinitely many solutions.

$$\begin{array}{rcl} x+2y \qquad\qquad\; =1 & & x+\;\;2y \qquad\qquad =1 \\ 2x \qquad +z \qquad =2 & \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2} & -4y+z \qquad =0 \\ 3x+2y+z-w=4 & & -4y+z-w=1 \\ & & x+\;\;2y \qquad\qquad =1 \\ & \xrightarrow{-\rho_2+\rho_3} & -4y+z \qquad =0 \\ & & -w=1 \end{array}$$

The leading variables are x, y, and w. The variable z is free. (Notice here that, although there are infinitely many solutions, the value of one of the variables is fixed — $w=-1$.) Write w in terms of z with $w=-1+0z$. Then $y=(1/4)z$. To express x in terms of z, substitute for y into the first equation to get $x=1-(1/2)z$. The solution set is $\{(1-(1/2)z,(1/4)z,z,-1) \bigm| z\in\mathbb{R}\}$.

We finish this subsection by developing the notation for linear systems and their solution sets that we shall use in the rest of this book.

2.6 Definition An $m\times n$ *matrix* is a rectangular array of numbers with m *rows* and n *columns*. Each number in the matrix is an *entry*, Matrices are usually named by upper case roman letters, e.g. A. Each entry is denoted by the corresponding lower-case letter, e.g. $a_{i,j}$ is the number in row i and column j of the array. For instance,

$$A = \begin{pmatrix} 1 & 2.2 & 5 \\ 3 & 4 & -7 \end{pmatrix}$$

has two rows and three columns, and so is a 2×3 matrix. (Read that "two-by-three"; the number of rows is always stated first.) The entry in the second row and first column is $a_{2,1} = 3$. Note that the order of the subscripts matters: $a_{1,2} \neq a_{2,1}$ since $a_{1,2} = 2.2$. (The parentheses around the array are a typographic device so that when two matrices are side by side we can tell where one ends and the other starts.)

2.7 Example We can abbreviate this linear system

$$\begin{aligned} x_1 + 2x_2 \phantom{{}- x_3} &= 4 \\ x_2 - x_3 &= 0 \\ x_1 + 2x_3 &= 4 \end{aligned}$$

with this matrix.

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right)$$

The vertical bar just reminds a reader of the difference between the coefficients on the systems's left hand side and the constants on the right. When a bar is used to divide a matrix into parts, we call it an *augmented* matrix. In this notation, Gauss' method goes this way.

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right) \xrightarrow{-\rho_1+\rho_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right) \xrightarrow{2\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The second row stands for $y - z = 0$ and the first row stands for $x + 2y = 4$ so the solution set is $\{(4-2z, z, z) \mid z \in \mathbb{R}\}$. One advantage of the new notation is that the clerical load of Gauss' method — the copying of variables, the writing of $+$'s and $=$'s, etc. — is lighter.

We will also use the array notation to clarify the descriptions of solution sets. A description like $\{(2-2z+2w, -1+z-w, z, w) \mid z, w \in \mathbb{R}\}$ from Example 2.3 is hard to read. We will rewrite it to group all the constants together, all the coefficients of z together, and all the coefficients of w together. We will write them vertically, in one-column wide matrices.

$$\{\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot z + \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \cdot w \mid z, w \in \mathbb{R}\}$$

For instance, the top line says that $x = 2 - 2z + 2w$. The next section gives a geometric interpretation that will help us picture the solution sets when they are written in this way.

2.8 Definition A *vector* (or *column vector*) is a matrix with a single column. A matrix with a single row is a *row vector*. The entries of a vector are its *components*.

Vectors are an exception to the convention of representing matrices with capital roman letters. We use lower-case roman or greek letters overlined with an arrow: $\vec{a}$, $\vec{b}$, ... or $\vec{\alpha}$, $\vec{\beta}$, ... (boldface is also common: $\boldsymbol{a}$ or $\boldsymbol{\alpha}$). For instance, this is a column vector with a third component of 7.

$$\vec{v} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$$

2.9 Definition The linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$ with unknowns $x_1, \ldots, x_n$ is *satisfied* by

$$\vec{s} = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$$

if $a_1s_1 + a_2s_2 + \cdots + a_ns_n = d$. A vector satisfies a linear system if it satisfies each equation in the system.

The style of description of solution sets that we use involves adding the vectors, and also multiplying them by real numbers, such as the z and w. We need to define these operations.

2.10 Definition The *vector sum* of $\vec{u}$ and $\vec{v}$ is this.

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix}$$

In general, two matrices with the same number of rows and the same number of columns add in this way, entry-by-entry.

2.11 Definition The *scalar multiplication* of the real number r and the vector $\vec{v}$ is this.

$$r \cdot \vec{v} = r \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} rv_1 \\ \vdots \\ rv_n \end{pmatrix}$$

In general, any matrix is multiplied by a real number in this entry-by-entry way.

Scalar multiplication can be written in either order: $r \cdot \vec{v}$ or $\vec{v} \cdot r$, or without the '$\cdot$' symbol: $r\vec{v}$. (Do not refer to scalar multiplication as 'scalar product' because that name is used for a different operation.)

2.12 Example

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2+3 \\ 3-1 \\ 1+4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix} \qquad 7 \cdot \begin{pmatrix} 1 \\ 4 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 28 \\ -7 \\ -21 \end{pmatrix}$$

Notice that the definitions of vector addition and scalar multiplication agree where they overlap, for instance, $\vec{v} + \vec{v} = 2\vec{v}$.

With the notation defined, we can now solve systems in the way that we will use throughout this book.

2.13 Example This system

$$\begin{aligned} 2x + y \phantom{{}-z} - w \phantom{{}+u} &= 4 \\ y \phantom{{}-z} + w + u &= 4 \\ x \phantom{{}+y} - z + 2w \phantom{{}+u} &= 0 \end{aligned}$$

reduces in this way.

$$\left(\begin{array}{ccccc|c} 2 & 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 4 \\ 1 & 0 & -1 & 2 & 0 & 0 \end{array}\right) \xrightarrow{-(1/2)\rho_1+\rho_3} \left(\begin{array}{ccccc|c} 2 & 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 4 \\ 0 & -1/2 & -1 & 5/2 & 0 & -2 \end{array}\right)$$

$$\xrightarrow{(1/2)\rho_2+\rho_3} \left(\begin{array}{ccccc|c} 2 & 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 4 \\ 0 & 0 & -1 & 3 & 1/2 & 0 \end{array}\right)$$

The solution set is $\{(w + (1/2)u, 4 - w - u, 3w + (1/2)u, w, u) \mid w, u \in \mathbb{R}\}$. We write that in vector form.

$$\{\begin{pmatrix} x \\ y \\ z \\ w \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} w + \begin{pmatrix} 1/2 \\ -1 \\ 1/2 \\ 0 \\ 1 \end{pmatrix} u \mid w, u \in \mathbb{R}\}$$

Note again how well vector notation sets off the coefficients of each parameter. For instance, the third row of the vector form shows plainly that if u is held fixed then z increases three times as fast as w.

That format also shows plainly that there are infinitely many solutions. For example, we can fix u as 0, let w range over the real numbers, and consider the first component x. We get infinitely many first components and hence infinitely many solutions.

Another thing shown plainly is that setting both w and u to zero gives that this

$$\begin{pmatrix} x \\ y \\ z \\ w \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is a particular solution of the linear system.

2.14 Example In the same way, this system

$$\begin{aligned} x - y + z &= 1 \\ 3x + z &= 3 \\ 5x - 2y + 3z &= 5 \end{aligned}$$

reduces

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 3 & 0 & 1 & 3 \\ 5 & -2 & 3 & 5 \end{array}\right) \xrightarrow[-5\rho_1+\rho_3]{-3\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & 0 \\ 0 & 3 & -2 & 0 \end{array}\right) \xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

to a one-parameter solution set.

$$\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/3 \\ 2/3 \\ 1 \end{pmatrix} z \bigm| z \in \mathbb{R}\}$$

Before the exercises, we pause to point out some things that we have yet to do.

The first two subsections have been on the mechanics of Gauss' method. Except for one result, Theorem 1.4 — without which developing the method doesn't make sense since it says that the method gives the right answers — we have not stopped to consider any of the interesting questions that arise.

For example, can we always describe solution sets as above, with a particular solution vector added to an unrestricted linear combination of some other vectors? The solution sets we described with unrestricted parameters were easily seen to have infinitely many solutions so an answer to this question could tell us something about the size of solution sets. An answer to that question could also help us picture the solution sets, in $\mathbb{R}^2$, or in $\mathbb{R}^3$, etc.

Many questions arise from the observation that Gauss' method can be done in more than one way (for instance, when swapping rows, we may have a choice of which row to swap with). Theorem 1.4 says that we must get the same solution set no matter how we proceed, but if we do Gauss' method in two different ways must we get the same number of free variables both times, so that any two solution set descriptions have the same number of parameters? Must those be the same variables (e.g., is it impossible to solve a problem one way and get y and w free or solve it another way and get y and z free)?

In the rest of this chapter we answer these questions. The answer to each is 'yes'. The first question is answered in the last subsection of this section. In the second section we give a geometric description of solution sets. In the final section of this chapter we tackle the last set of questions. Consequently, by the end of the first chapter we will not only have a solid grounding in the practice of Gauss' method, we will also have a solid grounding in the theory. We will be sure of what can and cannot happen in a reduction.

Exercises

✓ **2.15** Find the indicated entry of the matrix, if it is defined.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & 4 \end{pmatrix}$$

(a) $a_{2,1}$ **(b)** $a_{1,2}$ **(c)** $a_{2,2}$ **(d)** $a_{3,1}$

✓ **2.16** Give the size of each matrix.

(a) $\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 5 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 3 & -1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 5 & 10 \\ 10 & 5 \end{pmatrix}$

✓ **2.17** Do the indicated vector operation, if it is defined.

(a) $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ **(b)** $5\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ **(d)** $7\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 9\begin{pmatrix} 3 \\ 5 \end{pmatrix}$

(e) $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ **(f)** $6\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - 4\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

✓ **2.18** Solve each system using matrix notation. Express the solution using vectors.

(a) $\begin{aligned} 3x + 6y &= 18 \\ x + 2y &= 6 \end{aligned}$ **(b)** $\begin{aligned} x + y &= 1 \\ x - y &= -1 \end{aligned}$ **(c)** $\begin{aligned} x_1 \quad\;\; + x_3 &= 4 \\ x_1 - x_2 + 2x_3 &= 5 \\ 4x_1 - x_2 + 5x_3 &= 17 \end{aligned}$

(d) $\begin{aligned} 2a + b - c &= 2 \\ 2a \quad + c &= 3 \\ a - b \quad &= 0 \end{aligned}$ **(e)** $\begin{aligned} x + 2y - z \quad &= 3 \\ 2x + y \quad + w &= 4 \\ x - y + z + w &= 1 \end{aligned}$ **(f)** $\begin{aligned} x \quad + z + w &= 4 \\ 2x + y \quad - w &= 2 \\ 3x + y + z \quad &= 7 \end{aligned}$

✓ **2.19** Solve each system using matrix notation. Give each solution set in vector notation.

(a) $\begin{aligned} 2x + y - z &= 1 \\ 4x - y \quad &= 3 \end{aligned}$ **(b)** $\begin{aligned} x \quad - z \quad &= 1 \\ y + 2z - w &= 3 \\ x + 2y + 3z - w &= 7 \end{aligned}$ **(c)** $\begin{aligned} x - y + z \quad &= 0 \\ y \quad + w &= 0 \\ 3x - 2y + 3z + w &= 0 \\ -y \quad - w &= 0 \end{aligned}$

(d) $\begin{aligned} a + 2b + 3c + d - e &= 1 \\ 3a - b + c + d + e &= 3 \end{aligned}$

✓ **2.20** The vector is in the set. What value of the parameters produces that vector?

(a) $\begin{pmatrix} 5 \\ -5 \end{pmatrix}$, $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix} k \mid k \in \mathbb{R}\}$

(b) $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} i + \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} j \mid i, j \in \mathbb{R}\}$

(c) $\begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix}$, $\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} m + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} n \bigm| m, n \in \mathbb{R}\}$

2.21 Decide if the vector is in the set.

(a) $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\{\begin{pmatrix} -6 \\ 2 \end{pmatrix} k \bigm| k \in \mathbb{R}\}$

(b) $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$, $\{\begin{pmatrix} 5 \\ -4 \end{pmatrix} j \bigm| j \in \mathbb{R}\}$

(c) $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\{\begin{pmatrix} 0 \\ 3 \\ -7 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} r \bigm| r \in \mathbb{R}\}$

(d) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} j + \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} k \bigm| j, k \in \mathbb{R}\}$

2.22 Paramatrize the solution set of this one-equation system.

$$x_1 + x_2 + \cdots + x_n = 0$$

✓ **2.23** **(a)** Apply Gauss' method to the left-hand side to solve

$$\begin{array}{rcrcrcrcl} x & + & 2y & & & - & w & = & a \\ 2x & & & + & z & & & = & b \\ x & + & y & & & + & 2w & = & c \end{array}$$

for x, y, z, and w, in terms of the constants a, b, and c.

(b) Use your answer from the prior part to solve this.

$$\begin{array}{rcrcrcrcr} x & + & 2y & & & - & w & = & 3 \\ 2x & & & + & z & & & = & 1 \\ x & + & y & & & + & 2w & = & -2 \end{array}$$

✓ **2.24** Why is the comma needed in the notation '$a_{i,j}$' for matrix entries?

✓ **2.25** Give the $4{\times}4$ matrix whose i,j-th entry is

(a) $i + j$; **(b)** -1 to the $i + j$ power.

2.26 For any matrix A, the *transpose* of A, written A^{trans}, is the matrix whose columns are the rows of A. Find the transpose of each of these.

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 5 & 10 \\ 10 & 5 \end{pmatrix}$ **(d)** $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

✓ **2.27** **(a)** Describe all functions $f(x) = ax^2 + bx + c$ such that $f(1) = 2$ and $f(-1) = 6$.

(b) Describe all functions $f(x) = ax^2 + bx + c$ such that $f(1) = 2$.

2.28 Show that any set of five points from the plane $\mathbb{R}^2$ lie on a common conic section, that is, they all satisfy some equation of the form $ax^2 + by^2 + cxy + dx + ey + f = 0$ where some of $a, \ldots, f$ are nonzero.

2.29 Make up a four equations/four unknowns system having

(a) a one-parameter solution set;

(b) a two-parameter solution set;

(c) a three-parameter solution set.

? **2.30** **(a)** Solve the system of equations.

$$\begin{array}{rcrcl} ax & + & y & = & a^2 \\ x & + & ay & = & 1 \end{array}$$

For what values of a does the system fail to have solutions, and for what values of a are there infinitely many solutions?

(b) Answer the above question for the system.

$$\begin{aligned} ax + y &= a^3 \\ x + ay &= \phantom{a^{3}}1 \end{aligned}$$

[USSR Olympiad no. 174]

? 2.31 In air a gold-surfaced sphere weighs 7588 grams. It is known that it may contain one or more of the metals aluminum, copper, silver, or lead. When weighed successively under standard conditions in water, benzene, alcohol, and glycerine its respective weights are 6588, 6688, 6778, and 6328 grams. How much, if any, of the forenamed metals does it contain if the specific gravities of the designated substances are taken to be as follows?

Aluminum	2.7	Alcohol	0.81
Copper	8.9	Benzene	0.90
Gold	19.3	Glycerine	1.26
Lead	11.3	Water	1.00
Silver	10.8		

[Math. Mag., Sept. 1952]

I.3 General = Particular + Homogeneous

The prior subsection has many descriptions of solution sets. They all fit a pattern. They have a vector that is a particular solution of the system added to an unrestricted combination of some other vectors. The solution set from Example 2.13 illustrates.

$$\{\underbrace{\begin{pmatrix}0\\4\\0\\0\\0\end{pmatrix}}_{\substack{\text{particular}\\\text{solution}}}+\underbrace{w\begin{pmatrix}1\\-1\\3\\1\\0\end{pmatrix}+u\begin{pmatrix}1/2\\-1\\1/2\\0\\1\end{pmatrix}}_{\substack{\text{unrestricted}\\\text{combination}}}\,\big|\,w,u\in\mathbb{R}\}$$

The combination is unrestricted in that w and u can be any real numbers — there is no condition like "such that $2w-u=0$" that would restrict which pairs w, u can be used to form combinations.

That example shows an infinite solution set conforming to the pattern. We can think of the other two kinds of solution sets as also fitting the same pattern. A one-element solution set fits in that it has a particular solution, and the unrestricted combination part is a trivial sum (that is, instead of being a combination of two vectors, as above, or a combination of one vector, it is a combination of no vectors). A zero-element solution set fits the pattern since there is no particular solution, and so the set of sums of that form is empty.

We will show that the examples from the prior subsection are representative, in that the description pattern discussed above holds for every solution set.

3.1 Theorem For any linear system there are vectors $\vec{\beta}_1, \ldots, \vec{\beta}_k$ such that the solution set can be described as

$$\{\vec{p} + c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$$

where $\vec{p}$ is any particular solution, and where the system has k free variables.

This description has two parts, the particular solution $\vec{p}$ and also the unrestricted linear combination of the $\vec{\beta}$'s. We shall prove the theorem in two corresponding parts, with two lemmas.

We will focus first on the unrestricted combination part. To do that, we consider systems that have the vector of zeroes as one of the particular solutions, so that $\vec{p} + c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k$ can be shortened to $c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k$.

3.2 Definition A linear equation is *homogeneous* if it has a constant of zero, that is, if it can be put in the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$.

(These are 'homogeneous' because all of the terms involve the same power of their variable — the first power — including a '$0x_0$' that we can imagine is on the right side.)

3.3 Example With any linear system like

$$\begin{aligned} 3x + 4y &= 3 \\ 2x - y &= 1 \end{aligned}$$

we associate a system of homogeneous equations by setting the right side to zeros.

$$\begin{aligned} 3x + 4y &= 0 \\ 2x - y &= 0 \end{aligned}$$

Our interest in the homogeneous system associated with a linear system can be understood by comparing the reduction of the system

$$\begin{aligned} 3x + 4y &= 3 \\ 2x - y &= 1 \end{aligned} \;\xrightarrow{-(2/3)\rho_1+\rho_2}\; \begin{aligned} 3x + \quad 4y &= 3 \\ -(11/3)y &= -1 \end{aligned}$$

with the reduction of the associated homogeneous system.

$$\begin{aligned} 3x + 4y &= 0 \\ 2x - y &= 0 \end{aligned} \;\xrightarrow{-(2/3)\rho_1+\rho_2}\; \begin{aligned} 3x + \quad 4y &= 0 \\ -(11/3)y &= 0 \end{aligned}$$

Obviously the two reductions go in the same way. We can study how linear systems are reduced by instead studying how the associated homogeneous systems are reduced.

Studying the associated homogeneous system has a great advantage over studying the original system. Nonhomogeneous systems can be inconsistent. But a homogeneous system must be consistent since there is always at least one solution, the vector of zeros.

3.4 Definition A column or row vector of all zeros is a *zero vector*, denoted $\vec{0}$.

There are many different zero vectors, e.g., the one-tall zero vector, the two-tall zero vector, etc. Nonetheless, people often refer to "the" zero vector, expecting that the size of the one being discussed will be clear from the context.

3.5 Example Some homogeneous systems have the zero vector as their only solution.

$$\begin{array}{rcl} 3x+2y+z&=&0\\ 6x+4y&=&0\\ y+z&=&0 \end{array} \xrightarrow{-2\rho_1+\rho_2} \begin{array}{rcl} 3x+2y+z&=&0\\ -2z&=&0\\ y+z&=&0 \end{array} \xrightarrow{\rho_2\leftrightarrow\rho_3} \begin{array}{rcl} 3x+2y+z&=&0\\ y+z&=&0\\ -2z&=&0 \end{array}$$

3.6 Example Some homogeneous systems have many solutions. One example is the Chemistry problem from the first page of this book.

$$\begin{array}{rcl} 7x-7z&=&0\\ 8x+y-5z-2k&=&0\\ y-3z&=&0\\ 3y-6z-k&=&0 \end{array} \xrightarrow{-(8/7)\rho_1+\rho_2} \begin{array}{rcl} 7x-7z&=&0\\ y+3z-2w&=&0\\ y-3z&=&0\\ 3y-6z-w&=&0 \end{array}$$

$$\xrightarrow[-3\rho_2+\rho_4]{-\rho_2+\rho_3} \begin{array}{rcl} 7x-7z&=&0\\ y+3z-2w&=&0\\ -6z+2w&=&0\\ -15z+5w&=&0 \end{array}$$

$$\xrightarrow{-(5/2)\rho_3+\rho_4} \begin{array}{rcl} 7x-7z&=&0\\ y+3z-2w&=&0\\ -6z+2w&=&0\\ 0&=&0 \end{array}$$

The solution set:

$$\{\begin{pmatrix}1/3\\1\\1/3\\1\end{pmatrix}w \mid w\in\mathbb{R}\}$$

has many vectors besides the zero vector (if we interpret w as a number of molecules then solutions make sense only when w is a nonnegative multiple of 3).

We now have the terminology to prove the two parts of Theorem 3.1. The first lemma deals with unrestricted combinations.

3.7 Lemma For any homogeneous linear system there exist vectors $\vec{\beta}_1$, ..., $\vec{\beta}_k$ such that the solution set of the system is

$$\{c_1\vec{\beta}_1+\cdots+c_k\vec{\beta}_k \mid c_1,\ldots,c_k\in\mathbb{R}\}$$

where k is the number of free variables in an echelon form version of the system.

Before the proof, we will recall the back substitution calculations that were done in the prior subsection. Imagine that we have brought a system to this echelon form.

$$\begin{aligned} x + \quad 2y - z + 2w &= 0 \\ -3y + z \qquad &= 0 \\ -w &= 0 \end{aligned}$$

We next perform back-substitution to express each variable in terms of the free variable z. Working from the bottom up, we get first that w is $0 \cdot z$, next that y is $(1/3) \cdot z$, and then substituting those two into the top equation $x + 2((1/3)z) - z + 2(0) = 0$ gives $x = (1/3) \cdot z$. So, back substitution gives a paramatrization of the solution set by starting at the bottom equation and using the free variables as the parameters to work row-by-row to the top. The proof below follows this pattern.

Comment: That is, this proof just does a verification of the bookkeeping in back substitution to show that we haven't overlooked any obscure cases where this procedure fails, say, by leading to a division by zero. So this argument, while quite detailed, doesn't give us any new insights. Nevertheless, we have written it out for two reasons. The first reason is that we need the result — the computational procedure that we employ must be verified to work as promised. The second reason is that the row-by-row nature of back substitution leads to a proof that uses the technique of mathematical induction.* This is an important, and non-obvious, proof technique that we shall use a number of times in this book. Doing an induction argument here gives us a chance to see one in a setting where the proof material is easy to follow, and so the technique can be studied. Readers who are unfamiliar with induction arguments should be sure to master this one and the ones later in this chapter before going on to the second chapter.

Proof. First use Gauss' method to reduce the homogeneous system to echelon form. We will show that each leading variable can be expressed in terms of free variables. That will finish the argument because then we can use those free variables as the parameters. That is, the $\vec{\beta}$'s are the vectors of coefficients of the free variables (as in Example 3.6, where the solution is $x = (1/3)w$, $y = w$, $z = (1/3)w$, and $w = w$).

We will proceed by mathematical induction, which has two steps. The base step of the argument will be to focus on the bottom-most non-'$0 = 0$' equation and write its leading variable in terms of the free variables. The inductive step of the argument will be to argue that if we can express the leading variables from the bottom t rows in terms of free variables, then we can express the leading variable of the next row up — the $t + 1$-th row up from the bottom — in terms of free variables. With those two steps, the theorem will be proved because by the base step it is true for the bottom equation, and by the inductive step the fact that it is true for the bottom equation shows that it is true for the next one up, and then another application of the inductive step implies it is true for third equation up, etc.

* More information on mathematical induction is in the appendix.

For the base step, consider the bottom-most non-'$0 = 0$' equation (the case where all the equations are '$0 = 0$' is trivial). We call that the m-th row:

$$a_{m,\ell_m} x_{\ell_m} + a_{m,\ell_m+1} x_{\ell_m+1} + \cdots + a_{m,n} x_n = 0$$

where $a_{m,\ell_m} \neq 0$. (The notation here has 'ℓ' stand for 'leading', so a_{m,ℓ_m} means "the coefficient, from the row m of the variable leading row m".) Either there are variables in this equation other than the leading one x_{ℓ_m} or else there are not. If there are other variables x_{ℓ_m+1}, etc., then they must be free variables because this is the bottom non-'$0 = 0$' row. Move them to the right and divide by a_{m,ℓ_m}

$$x_{\ell_m} = (-a_{m,\ell_m+1}/a_{m,\ell_m}) x_{\ell_m+1} + \cdots + (-a_{m,n}/a_{m,\ell_m}) x_n$$

to expresses this leading variable in terms of free variables. If there are no free variables in this equation then $x_{\ell_m} = 0$ (see the "tricky point" noted following this proof).

For the inductive step, we assume that for the m-th equation, and for the $(m-1)$-th equation, ..., and for the $(m-t)$-th equation, we can express the leading variable in terms of free variables (where $0 \leq t < m$). To prove that the same is true for the next equation up, the $(m-(t+1))$-th equation, we take each variable that leads in a lower-down equation $x_{\ell_m}, \ldots, x_{\ell_{m-t}}$ and substitute its expression in terms of free variables. The result has the form

$$a_{m-(t+1),\ell_{m-(t+1)}} x_{\ell_{m-(t+1)}} + \text{sums of multiples of free variables} = 0$$

where $a_{m-(t+1),\ell_{m-(t+1)}} \neq 0$. We move the free variables to the right-hand side and divide by $a_{m-(t+1),\ell_{m-(t+1)}}$, to end with $x_{\ell_{m-(t+1)}}$ expressed in terms of free variables.

Because we have shown both the base step and the inductive step, by the principle of mathematical induction the proposition is true. QED

We say that the set $\{c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$ is *generated by* or *spanned by* the set of vectors $\{\vec{\beta}_1, \ldots, \vec{\beta}_k\}$. There is a tricky point to this definition. If a homogeneous system has a unique solution, the zero vector, then we say the solution set is generated by the empty set of vectors. This fits with the pattern of the other solution sets: in the proof above the solution set is derived by taking the c's to be the free variables and if there is a unique solution then there are no free variables.

This proof incidentally shows, as discussed after Example 2.4, that solution sets can always be paramatrized using the free variables.

The next lemma finishes the proof of Theorem 3.1 by considering the particular solution part of the solution set's description.

3.8 Lemma For a linear system, where $\vec{p}$ is any particular solution, the solution set equals this set.

$$\{\vec{p} + \vec{h} \mid \vec{h} \text{ satisfies the associated homogeneous system}\}$$

PROOF. We will show mutual set inclusion, that any solution to the system is in the above set and that anything in the set is a solution to the system.*

For set inclusion the first way, that if a vector solves the system then it is in the set described above, assume that $\vec{s}$ solves the system. Then $\vec{s}-\vec{p}$ solves the associated homogeneous system since for each equation index i between 1 and n,

$$\begin{aligned} a_{i,1}(s_1-p_1)+\cdots+a_{i,n}(s_n-p_n) &= (a_{i,1}s_1+\cdots+a_{i,n}s_n) \\ &\quad -(a_{i,1}p_1+\cdots+a_{i,n}p_n) \\ &= d_i-d_i \\ &= 0 \end{aligned}$$

where p_j and s_j are the j-th components of $\vec{p}$ and $\vec{s}$. We can write $\vec{s}-\vec{p}$ as $\vec{h}$, where $\vec{h}$ solves the associated homogeneous system, to express $\vec{s}$ in the required $\vec{p}+\vec{h}$ form.

For set inclusion the other way, take a vector of the form $\vec{p}+\vec{h}$, where $\vec{p}$ solves the system and $\vec{h}$ solves the associated homogeneous system, and note that it solves the given system: for any equation index i,

$$\begin{aligned} a_{i,1}(p_1+h_1)+\cdots+a_{i,n}(p_n+h_n) &= (a_{i,1}p_1+\cdots+a_{i,n}p_n) \\ &\quad +(a_{i,1}h_1+\cdots+a_{i,n}h_n) \\ &= d_i+0 \\ &= d_i \end{aligned}$$

where h_j is the j-th component of $\vec{h}$. QED

The two lemmas above together establish Theorem 3.1. We remember that theorem with the slogan "General = Particular + Homogeneous".

3.9 Example This system illustrates Theorem 3.1.

$$\begin{aligned} x+2y-z&=1 \\ 2x+4y&=2 \\ y-3z&=0 \end{aligned}$$

Gauss' method

$$\xrightarrow{-2\rho_1+\rho_2}\ \begin{aligned} x+2y-z&=1 \\ 2z&=0 \\ y-3z&=0 \end{aligned} \ \xrightarrow{\rho_2\leftrightarrow\rho_3}\ \begin{aligned} x+2y-z&=1 \\ y-3z&=0 \\ 2z&=0 \end{aligned}$$

shows that the general solution is a singleton set.

$$\{\begin{pmatrix}1\\0\\0\end{pmatrix}\}$$

* More information on equality of sets is in the appendix.

That single vector is, of course, a particular solution. The associated homogeneous system reduces via the same row operations

$$\begin{array}{rcl} x+2y-z&=&0\\ 2x+4y\phantom{{}-3z}&=&0\\ y-3z&=&0 \end{array} \xrightarrow{-2\rho_1+\rho_2} \xrightarrow{\rho_2\leftrightarrow\rho_3} \begin{array}{rcl} x+2y-z&=&0\\ y-3z&=&0\\ 2z&=&0 \end{array}$$

to also give a singleton set.

$$\{\begin{pmatrix}0\\0\\0\end{pmatrix}\}$$

As the theorem states, and as discussed at the start of this subsection, in this single-solution case the general solution results from taking the particular solution and adding to it the unique solution of the associated homogeneous system.

3.10 Example Also discussed there is that the case where the general solution set is empty fits the 'General = Particular+Homogeneous' pattern. This system illustrates. Gauss' method

$$\begin{array}{rcl} x\phantom{{}-y}+z+w&=&-1\\ 2x-y\phantom{{}+3z}+w&=&3\\ x+y+3z+2w&=&1 \end{array} \xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{array}{rcl} x\phantom{{}-y}+z+w&=&-1\\ -y-2z-w&=&5\\ y+2z+w&=&2 \end{array}$$

shows that it has no solutions. The associated homogeneous system, of course, has a solution.

$$\begin{array}{rcl} x\phantom{{}-y}+z+w&=&0\\ 2x-y\phantom{{}+3z}+w&=&0\\ x+y+3z+2w&=&0 \end{array} \xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \xrightarrow{\rho_2+\rho_3} \begin{array}{rcl} x\phantom{{}-y}+z+w&=&0\\ -y-2z-w&=&0\\ 0&=&0 \end{array}$$

In fact, the solution set of the homogeneous system is infinite.

$$\{\begin{pmatrix}-1\\-2\\1\\0\end{pmatrix}z+\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}w \mid z,w\in\mathbb{R}\}$$

However, because no particular solution of the original system exists, the general solution set is empty — there are no vectors of the form $\vec{p}+\vec{h}$ because there are no $\vec{p}$'s.

3.11 Corollary Solution sets of linear systems are either empty, have one element, or have infinitely many elements.

Proof. We've seen examples of all three happening so we need only prove that those are the only possibilities.

First, notice a homogeneous system with at least one non-$\vec{0}$ solution $\vec{v}$ has infinitely many solutions because the set of multiples $s\vec{v}$ is infinite — if $s\neq 1$ then $s\vec{v}-\vec{v}=(s-1)\vec{v}$ is easily seen to be non-$\vec{0}$, and so $s\vec{v}\neq\vec{v}$.

Now, apply Lemma 3.8 to conclude that a solution set

$$\{\vec{p}+\vec{h} \mid \vec{h} \text{ solves the associated homogeneous system}\}$$

is either empty (if there is no particular solution $\vec{p}$), or has one element (if there is a $\vec{p}$ and the homogeneous system has the unique solution $\vec{0}$), or is infinite (if there is a $\vec{p}$ and the homogeneous system has a non-$\vec{0}$ solution, and thus by the prior paragraph has infinitely many solutions). QED

This table summarizes the factors affecting the size of a general solution.

		number of solutions of the associated homogeneous system	
		one	*infinitely many*
particular solution exists?	*yes*	unique solution	infinitely many solutions
	no	no solutions	no solutions

The factor on the top of the table is the simpler one. When we perform Gauss' method on a linear system, ignoring the constants on the right side and so paying attention only to the coefficients on the left-hand side, we either end with every variable leading some row or else we find that some variable does not lead a row, that is, that some variable is free. (Of course, "ignoring the constants on the right" is formalized by considering the associated homogeneous system. We are simply putting aside for the moment the possibility of a contradictory equation.)

A nice insight into the factor on the top of this table at work comes from considering the case of a system having the same number of equations as variables. This system will have a solution, and the solution will be unique, if and only if it reduces to an echelon form system where every variable leads its row, which will happen if and only if the associated homogeneous system has a unique solution. Thus, the question of uniqueness of solution is especially interesting when the system has the same number of equations as variables.

3.12 Definition A square matrix is *nonsingular* if it is the matrix of coefficients of a homogeneous system with a unique solution. It is *singular* otherwise, that is, if it is the matrix of coefficients of a homogeneous system with infinitely many solutions.

3.13 Example The systems from Example 3.3, Example 3.5, and Example 3.9 each have an associated homogeneous system with a unique solution. Thus these matrices are nonsingular.

$$\begin{pmatrix} 3 & 4 \\ 2 & -1 \end{pmatrix} \qquad \begin{pmatrix} 3 & 2 & 1 \\ 6 & -4 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ 0 & 1 & -3 \end{pmatrix}$$

The Chemistry problem from Example 3.6 is a homogeneous system with more than one solution so its matrix is singular.

$$\begin{pmatrix} 7 & 0 & -7 & 0 \\ 8 & 1 & -5 & -2 \\ 0 & 1 & -3 & 0 \\ 0 & 3 & -6 & -1 \end{pmatrix}$$

3.14 Example The first of these matrices is nonsingular while the second is singular

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

because the first of these homogeneous systems has a unique solution while the second has infinitely many solutions.

$$\begin{aligned} x + 2y &= 0 \\ 3x + 4y &= 0 \end{aligned} \qquad \begin{aligned} x + 2y &= 0 \\ 3x + 6y &= 0 \end{aligned}$$

We have made the distinction in the definition because a system (with the same number of equations as variables) behaves in one of two ways, depending on whether its matrix of coefficients is nonsingular or singular. A system where the matrix of coefficients is nonsingular has a unique solution for any constants on the right side: for instance, Gauss' method shows that this system

$$\begin{aligned} x + 2y &= a \\ 3x + 4y &= b \end{aligned}$$

has the unique solution $x = b - 2a$ and $y = (3a - b)/2$. On the other hand, a system where the matrix of coefficients is singular never has a unique solutions — it has either no solutions or else has infinitely many, as with these.

$$\begin{aligned} x + 2y &= 1 \\ 3x + 6y &= 2 \end{aligned} \qquad \begin{aligned} x + 2y &= 1 \\ 3x + 6y &= 3 \end{aligned}$$

Thus, 'singular' can be thought of as connoting "troublesome", or at least "not ideal".

The above table has two factors. We have already considered the factor along the top: we can tell which column a given linear system goes in solely by considering the system's left-hand side — the constants on the right-hand side play no role in this factor. The table's other factor, determining whether a particular solution exists, is tougher. Consider these two

$$\begin{aligned} 3x + 2y &= 5 \\ 3x + 2y &= 5 \end{aligned} \qquad \begin{aligned} 3x + 2y &= 5 \\ 3x + 2y &= 4 \end{aligned}$$

with the same left sides but different right sides. Obviously, the first has a solution while the second does not, so here the constants on the right side decide if the system has a solution. We could conjecture that the left side of a linear system determines the number of solutions while the right side determines if solutions exist, but that guess is not correct. Compare these two systems

$$\begin{aligned}3x+2y&=5\\4x+2y&=4\end{aligned}\qquad\begin{aligned}3x+2y&=5\\3x+2y&=4\end{aligned}$$

with the same right sides but different left sides. The first has a solution but the second does not. Thus the constants on the right side of a system don't decide alone whether a solution exists; rather, it depends on some interaction between the left and right sides.

For some intuition about that interaction, consider this system with one of the coefficients left as the parameter c.

$$\begin{aligned}x+2y+3z&=1\\x+y+z&=1\\cx+3y+4z&=0\end{aligned}$$

If $c=2$ this system has no solution because the left-hand side has the third row as a sum of the first two, while the right-hand does not. If $c\neq 2$ this system has a unique solution (try it with $c=1$). For a system to have a solution, if one row of the matrix of coefficients on the left is a linear combination of other rows, then on the right the constant from that row must be the same combination of constants from the same rows.

More intuition about the interaction comes from studying linear combinations. That will be our focus in the second chapter, after we finish the study of Gauss' method itself in the rest of this chapter.

Exercises

✓ **3.15** Solve each system. Express the solution set using vectors. Identify the particular solution and the solution set of the homogeneous system.

(a) $\begin{aligned}3x+6y&=18\\x+2y&=6\end{aligned}$ **(b)** $\begin{aligned}x+y&=1\\x-y&=-1\end{aligned}$ **(c)** $\begin{aligned}x_1+x_3&=4\\x_1-x_2+2x_3&=5\\4x_1-x_2+5x_3&=17\end{aligned}$

(d) $\begin{aligned}2a+b-c&=2\\2a+c&=3\\a-b&=0\end{aligned}$ **(e)** $\begin{aligned}x+2y-z&=3\\2x+y+w&=4\\x-y+z+w&=1\end{aligned}$ **(f)** $\begin{aligned}x+z+w&=4\\2x+y-w&=2\\3x+y+z&=7\end{aligned}$

3.16 Solve each system, giving the solution set in vector notation. Identify the particular solution and the solution of the homogeneous system.

(a) $\begin{aligned}2x+y-z&=1\\4x-y&=3\end{aligned}$ **(b)** $\begin{aligned}x-z&=1\\y+2z-w&=3\\x+2y+3z-w&=7\end{aligned}$ **(c)** $\begin{aligned}x-y+z&=0\\y+w&=0\\3x-2y+3z+w&=0\\-y-w&=0\end{aligned}$

(d) $\begin{aligned}a+2b+3c+d-e&=1\\3a-b+c+d+e&=3\end{aligned}$

✓ **3.17** For the system

$$\begin{aligned}2x-y-w&=3\\y+z+2w&=2\\x-2y-z&=-1\end{aligned}$$

which of these can be used as the particular solution part of some general solution?

(a) $\begin{pmatrix} 0 \\ -3 \\ 5 \\ 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ **(c)** $\begin{pmatrix} -1 \\ -4 \\ 8 \\ -1 \end{pmatrix}$

✓ **3.18** Lemma 3.8 says that any particular solution may be used for $\vec{p}$. Find, if possible, a general solution to this system

$$\begin{aligned} x - y \quad\;\; + w &= 4 \\ 2x + 3y - z \quad\;\; &= 0 \\ y + z + w &= 4 \end{aligned}$$

that uses the given vector as its particular solution.

(a) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} -5 \\ 1 \\ -7 \\ 10 \end{pmatrix}$ **(c)** $\begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

3.19 One of these is nonsingular while the other is singular. Which is which?

(a) $\begin{pmatrix} 1 & 3 \\ 4 & -12 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$

✓ **3.20** Singular or nonsingular?

(a) $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix}$ (Careful!)

(d) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 3 & 4 & 7 \end{pmatrix}$ **(e)** $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 5 \\ -1 & 1 & 4 \end{pmatrix}$

✓ **3.21** Is the given vector in the set generated by the given set?

(a) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \{\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}\}$

(b) $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\}$

(c) $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \{\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}\}$

(d) $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \\ 2 \end{pmatrix}\}$

3.22 Prove that any linear system with a nonsingular matrix of coefficients has a solution, and that the solution is unique.

3.23 To tell the whole truth, there is another tricky point to the proof of Lemma 3.7. What happens if there are no non-'$0 = 0$' equations? (There aren't any more tricky points after this one.)

✓ **3.24** Prove that if $\vec{s}$ and $\vec{t}$ satisfy a homogeneous system then so do these vectors.

(a) $\vec{s}+\vec{t}$ **(b)** $3\vec{s}$ **(c)** $k\vec{s}+m\vec{t}$ for $k, m \in \mathbb{R}$

What's wrong with: "These three show that if a homogeneous system has one solution then it has many solutions — any multiple of a solution is another solution, and any sum of solutions is a solution also — so there are no homogeneous systems with exactly one solution."?

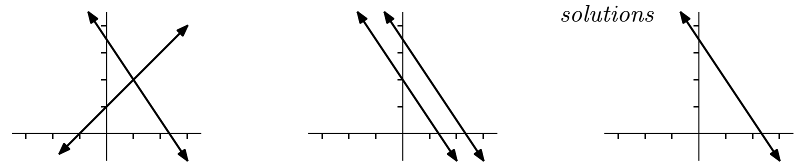
3.25 Prove that if a system with only rational coefficients and constants has a solution then it has at least one all-rational solution. Must it have infinitely many?

II Linear Geometry of n-Space

For readers who have seen the elements of vectors before, in calculus or physics, this section is an optional review. However, later work will refer to this material so it is not optional if it is not a review.

In the first section, we had to do a bit of work to show that there are only three types of solution sets — singleton, empty, and infinite. But in the special case of systems with two equations and two unknowns this is easy to see. Draw each two-unknowns equation as a line in the plane and then the two lines could have a unique intersection, be parallel, or be the same line.

Unique solution *No solutions* *Infinitely many solutions*



$$\begin{array}{rcr} 3x+2y&=&7\\ x-y&=&-1\end{array} \qquad \begin{array}{rcr} 3x+2y&=&7\\ 3x+2y&=&4\end{array} \qquad \begin{array}{rcr} 3x+2y&=&7\\ 6x+4y&=&14\end{array}$$

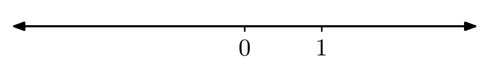
These pictures don't prove the results from the prior section, which apply to any number of linear equations and any number of unknowns, but nonetheless they do help us to understand those results. This section develops the ideas that we need to express our results from the prior section, and from some future sections, geometrically. In particular, while the two-dimensional case is familiar, to extend to systems with more than two unknowns we shall need some higher-dimensional geometry.

II.1 Vectors in Space

"Higher-dimensional geometry" sounds exotic. It is exotic — interesting and eye-opening. But it isn't distant or unreachable.

We begin by defining one-dimensional space to be the set $\mathbb{R}^1$. To see that definition is reasonable, draw a one-dimensional space

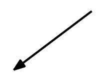
and make the usual correspondence with $\mathbb{R}$: pick a point to label 0 and another to label 1.



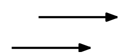
Now, with a scale and a direction, finding the point corresponding to, say $+2.17$, is easy — start at 0 and head in the direction of 1 (i.e., the positive direction), but don't stop there, go 2.17 times as far.

The basic idea here, combining magnitude with direction, is the key to extending to higher dimensions.

An object comprised of a magnitude and a direction is a *vector* (we will use the same word as in the previous section because we shall show below how to describe such an object with a column vector). We can draw a vector as having some length, and pointing somewhere.

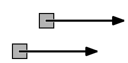


There is a subtlety here — these vectors

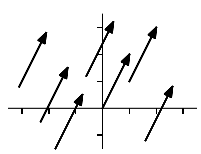


are equal, even though they start in different places, because they have equal lengths and equal directions. Again: those vectors are not just alike, they are equal.

How can things that are in different places be equal? Think of a vector as representing a displacement ('vector' is Latin for "carrier" or "traveler"). These squares undergo the same displacement, despite that those displacements start in different places.



Sometimes, to emphasize this property vectors have of not being anchored, they are referred to as *free* vectors. Thus, these free vectors are equal as each is a displacement of one over and two up.



More generally, vectors in the plane are the same if and only if they have the same change in first components and the same change in second components: the vector extending from (a_1, a_2) to (b_1, b_2) equals the vector from (c_1, c_2) to (d_1, d_2) if and only if $b_1 - a_1 = d_1 - c_1$ and $b_2 - a_2 = d_2 - c_2$.

An expression like 'the vector that, were it to start at (a_1, a_2), would extend to (b_1, b_2)' is awkward. We instead describe such a vector as

$$\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

so that, for instance, the 'one over and two up' arrows shown above picture this vector.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

We often draw the arrow as starting at the origin, and we then say it is in the *canonical position* (or *natural position*). When the vector

$$\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

is in its canonical position then it extends to the endpoint $(b_1 - a_1, b_2 - a_2)$.

We typically just refer to "the point

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}\text{''}$$

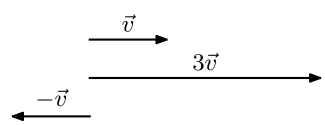
rather than "the endpoint of the canonical position of" that vector. Thus, we will call both of these sets $\mathbb{R}^n$.

$$\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\} \qquad \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1, x_2 \in \mathbb{R}\}$$

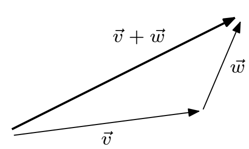
In the prior section we defined vectors and vector operations with an algebraic motivation;

$$r \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \end{pmatrix} \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$$

we can now interpret those operations geometrically. For instance, if $\vec{v}$ represents a displacement then $3\vec{v}$ represents a displacement in the same direction but three times as far, and $-1\vec{v}$ represents a displacement of the same distance as $\vec{v}$ but in the opposite direction.

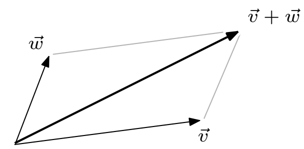


And, where $\vec{v}$ and $\vec{w}$ represent displacements, $\vec{v} + \vec{w}$ represents those displacements combined.



The long arrow is the combined displacement in this sense: if, in one minute, a ship's motion gives it the displacement relative to the earth of $\vec{v}$ and a passenger's motion gives a displacement relative to the ship's deck of $\vec{w}$, then $\vec{v} + \vec{w}$ is the displacement of the passenger relative to the earth.

Another way to understand the vector sum is with the *parallelogram rule*. Draw the parallelogram formed by the vectors $\vec{v}_1, \vec{v}_2$ and then the sum $\vec{v}_1 + \vec{v}_2$ extends along the diagonal to the far corner.



The above drawings show how vectors and vector operations behave in $\mathbb{R}^2$. We can extend to $\mathbb{R}^3$, or to even higher-dimensional spaces where we have no pictures, with the obvious generalization: the free vector that, if it starts at $(a_1, \ldots, a_n)$, ends at $(b_1, \ldots, b_n)$, is represented by this column

$$\begin{pmatrix} b_1 - a_1 \\ \vdots \\ b_n - a_n \end{pmatrix}$$

(vectors are equal if they have the same representation), we aren't too careful to distinguish between a point and the vector whose canonical representation ends at that point,

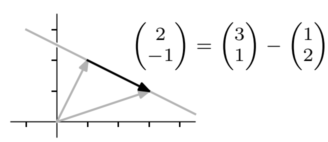
$$\mathbb{R}^n = \{ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mid v_1, \ldots, v_n \in \mathbb{R} \}$$

and addition and scalar multiplication are component-wise.

Having considered points, we now turn to the lines. In $\mathbb{R}^2$, the line through $(1, 2)$ and $(3, 1)$ is comprised of (the endpoints of) the vectors in this set

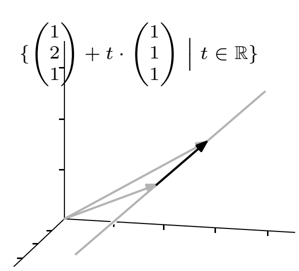
$$\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} \mid t \in \mathbb{R} \}$$

That description expresses this picture.



The vector associated with the parameter t has its whole body in the line — it is a *direction vector* for the line. Note that points on the line to the left of $x = 1$ are described using negative values of t.

In $\mathbb{R}^3$, the line through $(1, 2, 1)$ and $(2, 3, 2)$ is the set of (endpoints of) vectors of this form



and lines in even higher-dimensional spaces work in the same way.

If a line uses one parameter, so that there is freedom to move back and forth in one dimension, then a plane must involve two. For example, the plane through the points $(1,0,5)$, $(2,1,-3)$, and $(-2,4,0.5)$ consists of (endpoints of) the vectors in

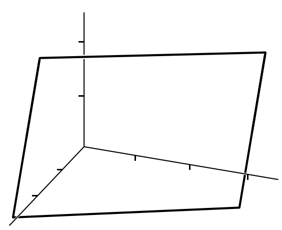
$$\{\begin{pmatrix}1\\0\\5\end{pmatrix}+t\cdot\begin{pmatrix}1\\1\\-8\end{pmatrix}+s\cdot\begin{pmatrix}-3\\4\\-4.5\end{pmatrix}\bigm| t,s\in\mathbb{R}\}$$

(the column vectors associated with the parameters

$$\begin{pmatrix}1\\1\\-8\end{pmatrix}=\begin{pmatrix}2\\1\\-3\end{pmatrix}-\begin{pmatrix}1\\0\\5\end{pmatrix}\qquad\begin{pmatrix}-3\\4\\-4.5\end{pmatrix}=\begin{pmatrix}-2\\4\\0.5\end{pmatrix}-\begin{pmatrix}1\\0\\5\end{pmatrix}$$

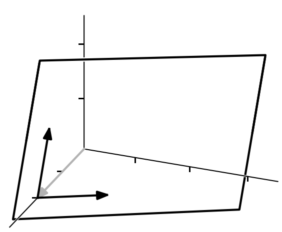
are two vectors whose whole bodies lie in the plane). As with the line, note that some points in this plane are described with negative t's or negative s's or both.

A description of planes that is often encountered in algebra and calculus uses a single equation as the condition that describes the relationship among the first, second, and third coordinates of points in a plane.



$$P=\{\begin{pmatrix}x\\y\\z\end{pmatrix}\bigm| 2x+y+z=4\}$$

The translation from such a description to the vector description that we favor in this book is to think of the condition as a one-equation linear system and paramatrize $x=(1/2)(4-y-z)$.



$$P=\{\begin{pmatrix}2\\0\\0\end{pmatrix}+\begin{pmatrix}-0.5\\1\\0\end{pmatrix}y+\begin{pmatrix}-0.5\\0\\1\end{pmatrix}z\bigm| y,z\in\mathbb{R}\}$$

Generalizing from lines and planes, we define a *k-dimensional linear surface* (or *k-flat*) in $\mathbb{R}^n$ to be $\{\vec{p}+t_1\vec{v}_1+t_2\vec{v}_2+\cdots+t_k\vec{v}_k \bigm| t_1,\ldots,t_k\in\mathbb{R}\}$ where $\vec{v}_1,\ldots,\vec{v}_k\in\mathbb{R}^n$. For example, in $\mathbb{R}^4$,

$$\{\begin{pmatrix}2\\\pi\\3\\-0.5\end{pmatrix}+t\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\bigm| t\in\mathbb{R}\}$$

is a line,

$$\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}+t\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}+s\begin{pmatrix}2\\0\\1\\0\end{pmatrix}\bigm| t,s\in\mathbb{R}\}$$

is a plane, and

$$\{\begin{pmatrix}3\\1\\-2\\0.5\end{pmatrix}+r\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}+s\begin{pmatrix}1\\0\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\0\\1\\0\end{pmatrix}\bigm| r,s,t\in\mathbb{R}\}$$

is a three-dimensional linear surface. Again, the intuition is that a line permits motion in one direction, a plane permits motion in combinations of two directions, etc.

A linear surface description can be misleading about the dimension — this

$$L=\{\begin{pmatrix}1\\0\\-1\\-2\end{pmatrix}+t\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}+s\begin{pmatrix}2\\2\\0\\-2\end{pmatrix}\bigm| t,s\in\mathbb{R}\}$$

is a *degenerate* plane because it is actually a line.

$$L=\{\begin{pmatrix}1\\0\\-1\\-2\end{pmatrix}+r\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}\bigm| r\in\mathbb{R}\}$$

We shall see in the Linear Independence section of Chapter Two what relationships among vectors causes the linear surface they generate to be degenerate.

We finish this subsection by restating our conclusions from the first section in geometric terms. First, the solution set of a linear system with n unknowns is a linear surface in $\mathbb{R}^n$. Specifically, it is a k-dimensional linear surface, where k is the number of free variables in an echelon form version of the system. Second, the solution set of a homogeneous linear system is a linear surface passing through the origin. Finally, we can view the general solution set of any linear system as being the solution set of its associated homogeneous system offset from the origin by a vector, namely by any particular solution.

Exercises

✓ **1.1** Find the canonical name for each vector.

(a) the vector from $(2,1)$ to $(4,2)$ in $\mathbb{R}^2$
(b) the vector from $(3,3)$ to $(2,5)$ in $\mathbb{R}^2$
(c) the vector from $(1,0,6)$ to $(5,0,3)$ in $\mathbb{R}^3$
(d) the vector from $(6,8,8)$ to $(6,8,8)$ in $\mathbb{R}^3$

✓ **1.2** Decide if the two vectors are equal.

(a) the vector from $(5,3)$ to $(6,2)$ and the vector from $(1,-2)$ to $(1,1)$

(b) the vector from $(2,1,1)$ to $(3,0,4)$ and the vector from $(5,1,4)$ to $(6,0,7)$

✓ **1.3** Does $(1,0,2,1)$ lie on the line through $(-2,1,1,0)$ and $(5,10,-1,4)$?

✓ **1.4** **(a)** Describe the plane through $(1,1,5,-1)$, $(2,2,2,0)$, and $(3,1,0,4)$.
(b) Is the origin in that plane?

1.5 Describe the plane that contains this point and line.

$$\begin{pmatrix}2\\0\\3\end{pmatrix} \qquad \{\begin{pmatrix}-1\\0\\-4\end{pmatrix}+\begin{pmatrix}1\\1\\2\end{pmatrix}t \mid t\in\mathbb{R}\}$$

✓ **1.6** Intersect these planes.

$$\{\begin{pmatrix}1\\1\\1\end{pmatrix}t+\begin{pmatrix}0\\1\\3\end{pmatrix}s \mid t,s\in\mathbb{R}\} \qquad \{\begin{pmatrix}1\\1\\0\end{pmatrix}+\begin{pmatrix}0\\3\\0\end{pmatrix}k+\begin{pmatrix}2\\0\\4\end{pmatrix}m \mid k,m\in\mathbb{R}\}$$

✓ **1.7** Intersect each pair, if possible.

(a) $\{\begin{pmatrix}1\\1\\2\end{pmatrix}+t\begin{pmatrix}0\\1\\1\end{pmatrix} \mid t\in\mathbb{R}\}$, $\{\begin{pmatrix}1\\3\\-2\end{pmatrix}+s\begin{pmatrix}0\\1\\2\end{pmatrix} \mid s\in\mathbb{R}\}$

(b) $\{\begin{pmatrix}2\\0\\1\end{pmatrix}+t\begin{pmatrix}1\\1\\-1\end{pmatrix} \mid t\in\mathbb{R}\}$, $\{s\begin{pmatrix}0\\1\\2\end{pmatrix}+w\begin{pmatrix}0\\4\\1\end{pmatrix} \mid s,w\in\mathbb{R}\}$

1.8 When a plane does not pass through the origin, performing operations on vectors whose bodies lie in it is more complicated than when the plane passes through the origin. Consider the picture in this subsection of the plane

$$\{\begin{pmatrix}2\\0\\0\end{pmatrix}+\begin{pmatrix}-0.5\\1\\0\end{pmatrix}y+\begin{pmatrix}-0.5\\0\\1\end{pmatrix}z \mid y,z\in\mathbb{R}\}$$

and the three vectors it shows, with endpoints $(2,0,0)$, $(1.5,1,0)$, and $(1.5,0,1)$.

(a) Redraw the picture, including the vector in the plane that is twice as long as the one with endpoint $(1.5,1,0)$. The endpoint of your vector is not $(3,2,0)$; what is it?
(b) Redraw the picture, including the parallelogram in the plane that shows the sum of the vectors ending at $(1.5,0,1)$ and $(1.5,1,0)$. The endpoint of the sum, on the diagonal, is not $(3,1,1)$; what is it?

1.9 Show that the line segments $\overline{(a_1,a_2)(b_1,b_2)}$ and $\overline{(c_1,c_2)(d_1,d_2)}$ have the same lengths and slopes if $b_1-a_1=d_1-c_1$ and $b_2-a_2=d_2-c_2$. Is that only if?

1.10 How should $\mathbb{R}^0$ be defined?

? ✓ **1.11** A person traveling eastward at a rate of 3 miles per hour finds that the wind appears to blow directly from the north. On doubling his speed it appears to come from the north east. What was the wind's velocity? [Math. Mag., Jan. 1957]

1.12 Euclid describes a plane as "a surface which lies evenly with the straight lines on itself". Commentators (e.g., Heron) have interpreted this to mean "(A plane surface is) such that, if a straight line pass through two points on it, the line coincides wholly with it at every spot, all ways". (Translations from [Heath], pp. 171-172.) Do planes, as described in this section, have that property? Does this description adequately define planes?

II.2 Length and Angle Measures

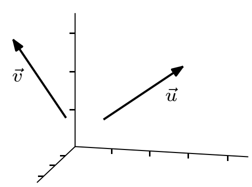
We've translated the first section's results about solution sets into geometric terms for insight into how those sets look. But we must watch out not to be mislead by our own terms; labeling subsets of $\mathbb{R}^k$ of the forms $\{\vec{p}+t\vec{v} \mid t\in\mathbb{R}\}$ and $\{\vec{p}+t\vec{v}+s\vec{w} \mid t,s\in\mathbb{R}\}$ as "lines" and "planes" doesn't make them act like the lines and planes of our prior experience. Rather, we must ensure that the names suit the sets. While we can't prove that the sets satisfy our intuition — we can't prove anything about intuition — in this subsection we'll observe that a result familiar from $\mathbb{R}^2$ and $\mathbb{R}^3$, when generalized to arbitrary $\mathbb{R}^k$, supports the idea that a line is straight and a plane is flat. Specifically, we'll see how to do Euclidean geometry in a "plane" by giving a definition of the angle between two $\mathbb{R}^n$ vectors in the plane that they generate.

2.1 Definition The *length* of a vector $\vec{v}\in\mathbb{R}^n$ is this.

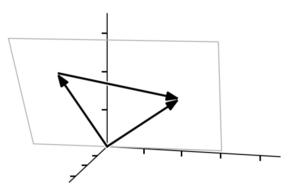
$$\|\vec{v}\,\| = \sqrt{v_1^2+\cdots+v_n^2}$$

2.2 Remark This is a natural generalization of the Pythagorean Theorem. A classic discussion is in [Polya].

We can use that definition to derive a formula for the angle between two vectors. For a model of what to do, consider two vectors in $\mathbb{R}^3$.



Put them in canonical position and, in the plane that they determine, consider the triangle formed by $\vec{u}$, $\vec{v}$, and $\vec{u}-\vec{v}$.



Apply the Law of Cosines, $\|\vec{u}-\vec{v}\,\|^2 = \|\vec{u}\,\|^2+\|\vec{v}\,\|^2 - 2\,\|\vec{u}\,\|\,\|\vec{v}\,\|\cos\theta$, where θ is the angle between the vectors. Expand both sides

$$(u_1-v_1)^2+(u_2-v_2)^2+(u_3-v_3)^2 = (u_1^2+u_2^2+u_3^2)+(v_1^2+v_2^2+v_3^2) - 2\,\|\vec{u}\,\|\,\|\vec{v}\,\|\cos\theta$$

and simplify.

$$\theta = \arccos\Bigl(\frac{u_1v_1+u_2v_2+u_3v_3}{\|\vec{u}\,\|\,\|\vec{v}\,\|}\Bigr)$$

In higher dimensions no picture suffices but we can make the same argument analytically. First, the form of the numerator is clear — it comes from the middle terms of the squares $(u_1 - v_1)^2$, $(u_2 - v_2)^2$, etc.

2.3 Definition The *dot product* (or *inner product*, or *scalar product*) of two n-component real vectors is the linear combination of their components.

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Note that the dot product of two vectors is a real number, not a vector, and that the dot product of a vector from $\mathbb{R}^n$ with a vector from $\mathbb{R}^m$ is defined only when n equals m. Note also this relationship between dot product and length: dotting a vector with itself gives its length squared $\vec{u} \cdot \vec{u} = u_1 u_1 + \cdots + u_n u_n = \|\vec{u}\,\|^2$.

2.4 Remark The wording in that definition allows one or both of the two to be a row vector instead of a column vector. Some books require that the first vector be a row vector and that the second vector be a column vector. We shall not be that strict.

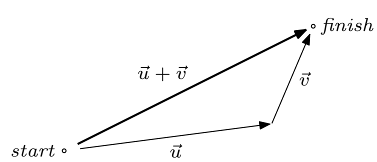
Still reasoning with letters, but guided by the pictures, we use the next theorem to argue that the triangle formed by $\vec{u}$, $\vec{v}$, and $\vec{u} - \vec{v}$ in $\mathbb{R}^n$ lies in the planar subset of $\mathbb{R}^n$ generated by $\vec{u}$ and $\vec{v}$.

2.5 Theorem (Triangle Inequality) For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$\|\vec{u} + \vec{v}\,\| \leq \|\vec{u}\,\| + \|\vec{v}\,\|$$

with equality if and only if one of the vectors is a nonnegative scalar multiple of the other one.

This inequality is the source of the familiar saying, "The shortest distance between two points is in a straight line."



Proof. (We'll use some algebraic properties of dot product that we have not yet checked, for instance that $\vec{u} \cdot (\vec{a} + \vec{b}) = \vec{u} \cdot \vec{a} + \vec{u} \cdot \vec{b}$ and that $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$. See Exercise 17.) The desired inequality holds if and only if its square holds.

$$\begin{aligned}
\|\vec{u} + \vec{v}\,\|^2 &\leq (\,\|\vec{u}\,\| + \|\vec{v}\,\|\,)^2 \\
(\,\vec{u} + \vec{v}\,) \cdot (\,\vec{u} + \vec{v}\,) &\leq \|\vec{u}\,\|^2 + 2\,\|\vec{u}\,\|\,\|\vec{v}\,\| + \|\vec{v}\,\|^2 \\
\vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} &\leq \vec{u} \cdot \vec{u} + 2\,\|\vec{u}\,\|\,\|\vec{v}\,\| + \vec{v} \cdot \vec{v} \\
2\,\vec{u} \cdot \vec{v} &\leq 2\,\|\vec{u}\,\|\,\|\vec{v}\,\|
\end{aligned}$$

That, in turn, holds if and only if the relationship obtained by multiplying both sides by the nonnegative numbers $\|\vec{u}\,\|$ and $\|\vec{v}\,\|$

$$2\,(\,\|\vec{v}\,\|\,\vec{u}\,)\cdot(\,\|\vec{u}\,\|\,\vec{v}\,)\leq 2\,\|\vec{u}\,\|^2\,\|\vec{v}\,\|^2$$

and rewriting

$$0\leq \|\vec{u}\,\|^2\,\|\vec{v}\,\|^2-2\,(\,\|\vec{v}\,\|\,\vec{u}\,)\cdot(\,\|\vec{u}\,\|\,\vec{v}\,)+\|\vec{u}\,\|^2\,\|\vec{v}\,\|^2$$

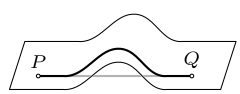
is true. But factoring

$$0\leq(\,\|\vec{u}\,\|\,\vec{v}-\|\vec{v}\,\|\,\vec{u}\,)\cdot(\,\|\vec{u}\,\|\,\vec{v}-\|\vec{v}\,\|\,\vec{u}\,)$$

shows that this certainly is true since it only says that the square of the length of the vector $\|\vec{u}\,\|\,\vec{v}-\|\vec{v}\,\|\,\vec{u}$ is not negative.

As for equality, it holds when, and only when, $\|\vec{u}\,\|\,\vec{v}-\|\vec{v}\,\|\,\vec{u}$ is $\vec{0}$. The check that $\|\vec{u}\,\|\,\vec{v}=\|\vec{v}\,\|\,\vec{u}$ if and only if one vector is a nonnegative real scalar multiple of the other is easy. QED

This result supports the intuition that even in higher-dimensional spaces, lines are straight and planes are flat. For any two points in a linear surface, the line segment connecting them is contained in that surface (this is easily checked from the definition). But if the surface has a bend then that would allow for a shortcut (shown here grayed, while the segment from P to Q that is contained in the surface is solid).



Because the Triangle Inequality says that in any $\mathbb{R}^n$, the shortest cut between two endpoints is simply the line segment connecting them, linear surfaces have no such bends.

Back to the definition of angle measure. The heart of the Triangle Inequality's proof is the '$\vec{u}\cdot\vec{v}\leq\|\vec{u}\,\|\,\|\vec{v}\,\|$' line. At first glance, a reader might wonder if some pairs of vectors satisfy the inequality in this way: while $\vec{u}\cdot\vec{v}$ is a large number, with absolute value bigger than the right-hand side, it is a negative large number. The next result says that no such pair of vectors exists.

2.6 Corollary (Cauchy-Schwartz Inequality) For any $\vec{u},\vec{v}\in\mathbb{R}^n$,

$$|\,\vec{u}\cdot\vec{v}\,|\leq\|\,\vec{u}\,\|\,\|\vec{v}\,\|$$

with equality if and only if one vector is a scalar multiple of the other.

Proof. The Triangle Inequality's proof shows that $\vec{u}\cdot\vec{v}\leq\|\vec{u}\,\|\,\|\vec{v}\,\|$ so if $\vec{u}\cdot\vec{v}$ is positive or zero then we are done. If $\vec{u}\cdot\vec{v}$ is negative then this holds.

$$|\,\vec{u}\cdot\vec{v}\,|=-(\,\vec{u}\cdot\vec{v}\,)=(-\vec{u}\,)\cdot\vec{v}\leq\|-\vec{u}\,\|\,\|\vec{v}\,\|=\|\vec{u}\,\|\,\|\vec{v}\,\|$$

The equality condition is Exercise 18. QED

The Cauchy-Schwartz inequality assures us that the next definition makes sense because the fraction has absolute value less than or equal to one.

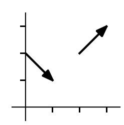
2.7 Definition The *angle* between two nonzero vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} \right)$$

(the angle between the zero vector and any other vector is defined to be a right angle).

Thus vectors from $\mathbb{R}^n$ are orthogonal if and only if their dot product is zero.

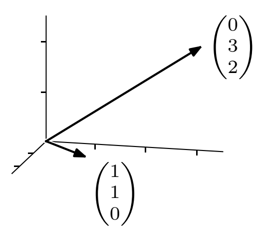
2.8 Example These vectors are orthogonal.



$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

The arrows are shown away from canonical position but nevertheless the vectors are orthogonal.

2.9 Example The $\mathbb{R}^3$ angle formula given at the start of this subsection is a special case of the definition. Between these two



the angle is

$$\arccos\left(\frac{(1)(0)+(1)(3)+(0)(2)}{\sqrt{1^2+1^2+0^2}\sqrt{0^2+3^2+2^2}}\right) = \arccos\left(\frac{3}{\sqrt{2}\sqrt{13}}\right)$$

approximately 0.94 radians. Notice that these vectors are not orthogonal. Although the yz-plane may appear to be perpendicular to the xy-plane, in fact the two planes are that way only in the weak sense that there are vectors in each orthogonal to all vectors in the other. Not every vector in each is orthogonal to all vectors in the other.

Exercises

✓ **2.10** Find the length of each vector.

(a) $\begin{pmatrix}3\\1\end{pmatrix}$ **(b)** $\begin{pmatrix}-1\\2\end{pmatrix}$ **(c)** $\begin{pmatrix}4\\1\\1\end{pmatrix}$ **(d)** $\begin{pmatrix}0\\0\\0\end{pmatrix}$ **(e)** $\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}$

✓ **2.11** Find the angle between each two, if it is defined.

(a) $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}$ **(b)** $\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\4\\1\end{pmatrix}$ **(c)** $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\4\\-1\end{pmatrix}$

✓ **2.12** During maneuvers preceding the Battle of Jutland, the British battle cruiser *Lion* moved as follows (in nautical miles): 1.2 miles north, 6.1 miles 38 degrees east of south, 4.0 miles at 89 degrees east of north, and 6.5 miles at 31 degrees east of north. Find the distance between starting and ending positions. [Ohanian]

2.13 Find k so that these two vectors are perpendicular.

$$\begin{pmatrix}k\\1\end{pmatrix} \qquad \begin{pmatrix}4\\3\end{pmatrix}$$

2.14 Describe the set of vectors in $\mathbb{R}^3$ orthogonal to this one.

$$\begin{pmatrix}1\\3\\-1\end{pmatrix}$$

✓ **2.15** **(a)** Find the angle between the diagonal of the unit square in $\mathbb{R}^2$ and one of the axes.

(b) Find the angle between the diagonal of the unit cube in $\mathbb{R}^3$ and one of the axes.

(c) Find the angle between the diagonal of the unit cube in $\mathbb{R}^n$ and one of the axes.

(d) What is the limit, as n goes to ∞, of the angle between the diagonal of the unit cube in $\mathbb{R}^n$ and one of the axes?

2.16 Is any vector perpendicular to itself?

✓ **2.17** Describe the algebraic properties of dot product.

(a) Is it right-distributive over addition: $(\vec{u}+\vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$?

(b) Is is left-distributive (over addition)?

(c) Does it commute?

(d) Associate?

(e) How does it interact with scalar multiplication?

As always, any assertion must be backed by either a proof or an example.

2.18 Verify the equality condition in Corollary 2.6, the Cauchy-Schwartz Inequality.

(a) Show that if $\vec{u}$ is a negative scalar multiple of $\vec{v}$ then $\vec{u} \cdot \vec{v}$ and $\vec{v} \cdot \vec{u}$ are less than or equal to zero.

(b) Show that $|\vec{u} \cdot \vec{v}| = \|\vec{u}\|\,\|\vec{v}\|$ if and only if one vector is a scalar multiple of the other.

2.19 Suppose that $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$ and $\vec{u} \neq \vec{0}$. Must $\vec{v} = \vec{w}$?

✓ **2.20** Does any vector have length zero except a zero vector? (If “yes”, produce an example. If “no”, prove it.)

✓ **2.21** Find the midpoint of the line segment connecting (x_1, y_1) with (x_2, y_2) in $\mathbb{R}^2$. Generalize to $\mathbb{R}^n$.

2.22 Show that if $\vec{v} \neq \vec{0}$ then $\vec{v}/\|\vec{v}\|$ has length one. What if $\vec{v} = \vec{0}$?

2.23 Show that if $r \geq 0$ then $r\vec{v}$ is r times as long as $\vec{v}$. What if $r < 0$?

✓ **2.24** A vector $\vec{v} \in \mathbb{R}^n$ of length one is a *unit* vector. Show that the dot product of two unit vectors has absolute value less than or equal to one. Can 'less than' happen? Can 'equal to'?

2.25 Prove that $\|\vec{u}+\vec{v}\,\|^2 + \|\vec{u}-\vec{v}\,\|^2 = 2\|\vec{u}\,\|^2 + 2\|\vec{v}\,\|^2$.

2.26 Show that if $\vec{x} \cdot \vec{y} = 0$ for every $\vec{y}$ then $\vec{x} = \vec{0}$.

2.27 Is $\|\vec{u}_1 + \cdots + \vec{u}_n\| \leq \|\vec{u}_1\| + \cdots + \|\vec{u}_n\|$? If it is true then it would generalize the Triangle Inequality.

2.28 What is the ratio between the sides in the Cauchy-Schwartz inequality?

2.29 Why is the zero vector defined to be perpendicular to every vector?

2.30 Describe the angle between two vectors in $\mathbb{R}^1$.

2.31 Give a simple necessary and sufficient condition to determine whether the angle between two vectors is acute, right, or obtuse.

✓ **2.32** Generalize to $\mathbb{R}^n$ the converse of the Pythagorean Theorem, that if $\vec{u}$ and $\vec{v}$ are perpendicular then $\|\vec{u}+\vec{v}\,\|^2 = \|\vec{u}\,\|^2 + \|\vec{v}\,\|^2$.

2.33 Show that $\|\vec{u}\,\| = \|\vec{v}\,\|$ if and only if $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are perpendicular. Give an example in $\mathbb{R}^2$.

2.34 Show that if a vector is perpendicular to each of two others then it is perpendicular to each vector in the plane they generate. (*Remark.* They could generate a degenerate plane — a line or a point — but the statement remains true.)

2.35 Prove that, where $\vec{u}, \vec{v} \in \mathbb{R}^n$ are nonzero vectors, the vector

$$\frac{\vec{u}}{\|\vec{u}\,\|} + \frac{\vec{v}}{\|\vec{v}\,\|}$$

bisects the angle between them. Illustrate in $\mathbb{R}^2$.

2.36 Verify that the definition of angle is dimensionally correct: (1) if $k > 0$ then the cosine of the angle between $k\vec{u}$ and $\vec{v}$ equals the cosine of the angle between $\vec{u}$ and $\vec{v}$, and (2) if $k < 0$ then the cosine of the angle between $k\vec{u}$ and $\vec{v}$ is the negative of the cosine of the angle between $\vec{u}$ and $\vec{v}$.

✓ **2.37** Show that the inner product operation is *linear*: for $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $k, m \in \mathbb{R}$, $\vec{u} \cdot (k\vec{v} + m\vec{w}) = k(\vec{u} \cdot \vec{v}) + m(\vec{u} \cdot \vec{w})$.

✓ **2.38** The *geometric mean* of two positive reals x, y is $\sqrt{xy}$. It is analogous to the *arithmetic mean* $(x+y)/2$. Use the Cauchy-Schwartz inequality to show that the geometric mean of any $x, y \in \mathbb{R}$ is less than or equal to the arithmetic mean.

? **2.39** A ship is sailing with speed and direction $\vec{v}_1$; the wind blows apparently (judging by the vane on the mast) in the direction of a vector $\vec{a}$; on changing the direction and speed of the ship from $\vec{v}_1$ to $\vec{v}_2$ the apparent wind is in the direction of a vector $\vec{b}$.

Find the vector velocity of the wind. [Am. Math. Mon., Feb. 1933]

2.40 Verify the Cauchy-Schwartz inequality by first proving Lagrange's identity:

$$\left(\sum_{1\leq j\leq n} a_j b_j\right)^2 = \left(\sum_{1\leq j\leq n} a_j^2\right)\left(\sum_{1\leq j\leq n} b_j^2\right) - \sum_{1\leq k<j\leq n} (a_k b_j - a_j b_k)^2$$

and then noting that the final term is positive. (Recall the meaning

$$\sum_{1\leq j\leq n} a_j b_j = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

and

$$\sum_{1\leq j\leq n} {a_j}^2 = {a_1}^2 + {a_2}^2 + \cdots + {a_n}^2$$

of the Σ notation.) This result is an improvement over Cauchy-Schwartz because it gives a formula for the difference between the two sides. Interpret that difference in $\mathbb{R}^2$.

III Reduced Echelon Form

After developing the mechanics of Gauss' method, we observed that it can be done in more than one way. One example is that we sometimes have to swap rows and there can be more than one row to choose from. Another example is that from this matrix

$$\begin{pmatrix}2&2\\4&3\end{pmatrix}$$

Gauss' method could derive any of these echelon form matrices.

$$\begin{pmatrix}2&2\\0&-1\end{pmatrix}\qquad\begin{pmatrix}1&1\\0&-1\end{pmatrix}\qquad\begin{pmatrix}2&0\\0&-1\end{pmatrix}$$

The first results from $-2\rho_1+\rho_2$. The second comes from following $(1/2)\rho_1$ with $-4\rho_1+\rho_2$. The third comes from $-2\rho_1+\rho_2$ followed by $2\rho_2+\rho_1$ (after the first pivot the matrix is already in echelon form so the second one is extra work but it is nonetheless a legal row operation).

The fact that the echelon form outcome of Gauss' method is not unique leaves us with some questions. Will any two echelon form versions of a system have the same number of free variables? Will they in fact have exactly the same variables free? In this section we will answer both questions "yes". We will do more than answer the questions. We will give a way to decide if one linear system can be derived from another by row operations. The answers to the two questions will follow from this larger result.

III.1 Gauss-Jordan Reduction

Gaussian elimination coupled with back-substitution solves linear systems, but it's not the only method possible. Here is an extension of Gauss' method that has some advantages.

1.1 Example To solve

$$\begin{aligned}x+y-2z&=-2\\ y+3z&=7\\ x-z&=-1\end{aligned}$$

we can start by going to echelon form as usual.

$$\xrightarrow{-\rho_1+\rho_3}\left(\begin{array}{ccc|c}1&1&-2&-2\\0&1&3&7\\0&-1&1&1\end{array}\right)\xrightarrow{\rho_2+\rho_3}\left(\begin{array}{ccc|c}1&1&-2&-2\\0&1&3&7\\0&0&4&8\end{array}\right)$$

We can keep going to a second stage by making the leading entries into ones

$$\xrightarrow{(1/4)\rho_3}\left(\begin{array}{ccc|c}1&1&-2&-2\\0&1&3&7\\0&0&1&2\end{array}\right)$$

and then to a third stage that uses the leading entries to eliminate all of the other entries in each column by pivoting upwards.

$$\xrightarrow[2\rho_3+\rho_1]{-3\rho_3+\rho_2}\left(\begin{array}{ccc|c}1&1&0&2\\0&1&0&1\\0&0&1&2\end{array}\right)\xrightarrow{-\rho_2+\rho_1}\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&1\\0&0&1&2\end{array}\right)$$

The answer is $x = 1$, $y = 1$, and $z = 2$.

Note that the pivot operations in the first stage proceed from column one to column three while the pivot operations in the third stage proceed from column three to column one.

1.2 Example We often combine the operations of the middle stage into a single step, even though they are operations on different rows.

$$\begin{aligned}\left(\begin{array}{cc|c}2&1&7\\4&-2&6\end{array}\right)&\xrightarrow{-2\rho_1+\rho_2}\left(\begin{array}{cc|c}2&1&7\\0&-4&-8\end{array}\right)\\&\xrightarrow[(-1/4)\rho_2]{(1/2)\rho_1}\left(\begin{array}{cc|c}1&1/2&7/2\\0&1&2\end{array}\right)\\&\xrightarrow{-(1/2)\rho_2+\rho_1}\left(\begin{array}{cc|c}1&0&5/2\\0&1&2\end{array}\right)\end{aligned}$$

The answer is $x = 5/2$ and $y = 2$.

This extension of Gauss' method is *Gauss-Jordan reduction*. It goes past echelon form to a more refined, more specialized, matrix form.

1.3 Definition A matrix is in *reduced echelon form* if, in addition to being in echelon form, each leading entry is a one and is the only nonzero entry in its column.

The disadvantage of using Gauss-Jordan reduction to solve a system is that the additional row operations mean additional arithmetic. The advantage is that the solution set can just be read off.

In any echelon form, plain or reduced, we can read off when a system has an empty solution set because there is a contradictory equation, we can read off when a system has a one-element solution set because there is no contradiction and every variable is the leading variable in some row, and we can read off when a system has an infinite solution set because there is no contradiction and at least one variable is free.

In reduced echelon form we can read off not just what kind of solution set the system has, but also its description. Whether or not the echelon form is reduced, we have no trouble describing the solution set when it is empty, of course. The two examples above show that when the system has a single solution then the solution can be read off from the right-hand column. In the case when the solution set is infinite, its parametrization can also be read off of the reduced echelon form. Consider, for example, this system that is shown brought to echelon form and then to reduced echelon form.

$$\begin{pmatrix} 2&6&1&2&\bigm|&5\\ 0&3&1&4&\bigm|&1\\ 0&3&1&2&\bigm|&5 \end{pmatrix} \xrightarrow{-\rho_2+\rho_3} \begin{pmatrix} 2&6&1&2&\bigm|&5\\ 0&3&1&4&\bigm|&1\\ 0&0&0&-2&\bigm|&4 \end{pmatrix}$$

$$\xrightarrow[\substack{(1/3)\rho_2\\ -(1/2)\rho_3}]{(1/2)\rho_1} \xrightarrow[-\rho_3+\rho_1]{(4/3)\rho_3+\rho_2} \xrightarrow{-3\rho_2+\rho_1} \begin{pmatrix} 1&0&-1/2&0&\bigm|&-9/2\\ 0&1&1/3&0&\bigm|&3\\ 0&0&0&1&\bigm|&-2 \end{pmatrix}$$

Starting with the middle matrix, the echelon form version, back substitution produces $-2x_4 = 4$ so that $x_4 = -2$, then another back substitution gives $3x_2 + x_3 + 4(-2) = 1$ implying that $x_2 = 3 - (1/3)x_3$, and then the final back substitution gives $2x_1 + 6(3 - (1/3)x_3) + x_3 + 2(-2) = 5$ implying that $x_1 = -(9/2) + (1/2)x_3$. Thus the solution set is this.

$$S = \{ \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{pmatrix} = \begin{pmatrix} -9/2\\ 3\\ 0\\ -2 \end{pmatrix} + \begin{pmatrix} 1/2\\ -1/3\\ 1\\ 0 \end{pmatrix} x_3 \bigm| x_3 \in \mathbb{R} \}$$

Now, considering the final matrix, the reduced echelon form version, note that adjusting the parametrization by moving the x_3 terms to the other side does indeed give the description of this infinite solution set.

Part of the reason that this works is straightforward. While a set can have many parametrizations that describe it, e.g., both of these also describe the above set S (take t to be $x_3/6$ and s to be $x_3 - 1$)

$$\{ \begin{pmatrix} -9/2\\ 3\\ 0\\ -2 \end{pmatrix} + \begin{pmatrix} 3\\ -2\\ 6\\ 0 \end{pmatrix} t \bigm| t \in \mathbb{R} \} \qquad \{ \begin{pmatrix} -4\\ 8/3\\ 1\\ -2 \end{pmatrix} + \begin{pmatrix} 1/2\\ -1/3\\ 1\\ 0 \end{pmatrix} s \bigm| s \in \mathbb{R} \}$$

nonetheless we have in this book stuck to a convention of parametrizing using the unmodified free variables (that is, $x_3 = x_3$ instead of $x_3 = 6t$). We can easily see that a reduced echelon form version of a system is equivalent to a parametrization in terms of unmodified free variables. For instance,

$$\begin{aligned} x_1 &= 4 - 2x_3\\ x_2 &= 3 - x_3 \end{aligned} \iff \begin{pmatrix} 1&0&2&\bigm|&4\\ 0&1&1&\bigm|&3\\ 0&0&0&\bigm|&0 \end{pmatrix}$$

(to move from left to right we also need to know how many equations are in the system). So, the convention of parametrizing with the free variables by solving each equation for its leading variable and then eliminating that leading variable from every other equation is exactly equivalent to the reduced echelon form conditions that each leading entry must be a one and must be the only nonzero entry in its column.

Not as straightforward is the other part of the reason that the reduced echelon form version allows us to read off the parametrization that we would have gotten had we stopped at echelon form and then done back substitution. The prior paragraph shows that reduced echelon form corresponds to some parametrization, but why the same parametrization? A solution set can be parametrized in many ways, and Gauss' method or the Gauss-Jordan method can be done in many ways, so a first guess might be that we could derive many different reduced echelon form versions of the same starting system and many different parametrizations. But we never do. Experience shows that starting with the same system and proceeding with row operations in many different ways always yields the same reduced echelon form and the same parametrization (using the unmodified free variables).

In the rest of this section we will show that the reduced echelon form version of a matrix is unique. It follows that the parametrization of a linear system in terms of its unmodified free variables is unique because two different ones would give two different reduced echelon forms.

We shall use this result, and the ones that lead up to it, in the rest of the book but perhaps a restatement in a way that makes it seem more immediately useful may be encouraging. Imagine that we solve a linear system, parametrize, and check in the back of the book for the answer. But the parametrization there appears different. Have we made a mistake, or could these be different-looking descriptions of the same set, as with the three descriptions above of S? The prior paragraph notes that we will show here that different-looking parametrizations (using the unmodified free variables) describe genuinely different sets.

Here is an informal argument that the reduced echelon form version of a matrix is unique. Consider again the example that started this section of a matrix that reduces to three different echelon form matrices. The first matrix of the three is the natural echelon form version. The second matrix is the same as the first except that a row has been halved. The third matrix, too, is just a cosmetic variant of the first. The definition of reduced echelon form outlaws this kind of fooling around. In reduced echelon form, halving a row is not possible because that would change the row's leading entry away from one, and neither is combining rows possible, because then a leading entry would no longer be alone in its column.

This informal justification is not a proof; we have argued that no two different reduced echelon form matrices are related by a single row operation step, but we have not ruled out the possibility that multiple steps might do. Before we go to that proof, we finish this subsection by rephrasing our work in a terminology that will be enlightening.

Many different matrices yield the same reduced echelon form matrix. The three echelon form matrices from the start of this section, and the matrix they were derived from, all give this reduced echelon form matrix.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We think of these matrices as related to each other. The next result speaks to this relationship.

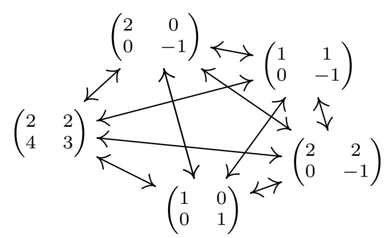
1.4 Lemma Elementary row operations are reversible.

Proof. For any matrix A, the effect of swapping rows is reversed by swapping them back, multiplying a row by a nonzero k is undone by multiplying by $1/k$, and adding a multiple of row i to row j (with $i \neq j$) is undone by subtracting the same multiple of row i from row j.

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} \xrightarrow{\rho_j \leftrightarrow \rho_i} A \qquad A \xrightarrow{k\rho_i} \xrightarrow{(1/k)\rho_i} A \qquad A \xrightarrow{k\rho_i + \rho_j} \xrightarrow{-k\rho_i + \rho_j} A$$

(The $i \neq j$ conditions is needed. See Exercise 13.) QED

This lemma suggests that 'reduces to' is misleading — where $A \longrightarrow B$, we shouldn't think of B as "after" A or "simpler than" A. Instead we should think of them as interreducible or interrelated. Below is a picture of the idea. The matrices from the start of this section and their reduced echelon form version are shown in a cluster. They are all interreducible; these relationships are shown also.



We say that matrices that reduce to each other are 'equivalent with respect to the relationship of row reducibility'. The next result verifies this statement using the definition of an equivalence.*

1.5 Lemma Between matrices, 'reduces to' is an equivalence relation.

Proof. We must check the conditions (i) reflexivity, that any matrix reduces to itself, (ii) symmetry, that if A reduces to B then B reduces to A, and (iii) transitivity, that if A reduces to B and B reduces to C then A reduces to C.

Reflexivity is easy; any matrix reduces to itself in zero row operations.

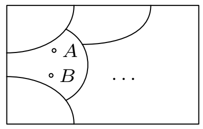
That the relationship is symmetric is Lemma 1.4 — if A reduces to B by some row operations then also B reduces to A by reversing those operations.

For transitivity, suppose that A reduces to B and that B reduces to C. Linking the reduction steps from $A \rightarrow \cdots \rightarrow B$ with those from $B \rightarrow \cdots \rightarrow C$ gives a reduction from A to C. QED

1.6 Definition Two matrices that are interreducible by the elementary row operations are *row equivalent*.

* More information on equivalence relations is in the appendix.

The diagram below shows the collection of all matrices as a box. Inside that box, each matrix lies in some class. Matrices are in the same class if and only if they are interreducible. The classes are disjoint — no matrix is in two distinct classes. The collection of matrices has been partitioned into *row equivalence classes.*[*]



One of the classes in this partition is the cluster of matrices shown above, expanded to include all of the nonsingular 2×2 matrices.

The next subsection proves that the reduced echelon form of a matrix is unique; that every matrix reduces to one and only one reduced echelon form matrix. Rephrased in terms of the row-equivalence relationship, we shall prove that every matrix is row equivalent to one and only one reduced echelon form matrix. In terms of the partition what we shall prove is: every equivalence class contains one and only one reduced echelon form matrix. So each reduced echelon form matrix serves as a representative of its class.

After that proof we shall, as mentioned in the introduction to this section, have a way to decide if one matrix can be derived from another by row reduction. We just apply the Gauss-Jordan procedure to both and see whether or not they come to the same reduced echelon form.

Exercises

✓ **1.7** Use Gauss-Jordan reduction to solve each system.

(a) $\begin{array}{rl} x+y &= 2 \\ x-y &= 0 \end{array}$ **(b)** $\begin{array}{rl} x \qquad -z &= 4 \\ 2x+2y \qquad &= 1 \end{array}$ **(c)** $\begin{array}{rl} 3x-2y &= 1 \\ 6x+y &= 1/2 \end{array}$

(d) $\begin{array}{rl} 2x-y \qquad &= -1 \\ x+3y-z &= 5 \\ y+2z &= 5 \end{array}$

✓ **1.8** Find the reduced echelon form of each matrix.

(a) $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & 4 \\ -1 & -3 & -3 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 0 & 3 & 1 & 2 \\ 1 & 4 & 2 & 1 & 5 \\ 3 & 4 & 8 & 1 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 & 3 & 2 \\ 0 & 0 & 5 & 6 \\ 1 & 5 & 1 & 5 \end{pmatrix}$

✓ **1.9** Find each solution set by using Gauss-Jordan reduction, then reading off the parametrization.

(a) $\begin{array}{rl} 2x+y-z &= 1 \\ 4x-y \qquad &= 3 \end{array}$ **(b)** $\begin{array}{rl} x \qquad - z \qquad &= 1 \\ y+2z-w &= 3 \\ x+2y+3z-w &= 7 \end{array}$ **(c)** $\begin{array}{rl} x-y+z \qquad &= 0 \\ y \qquad +w &= 0 \\ 3x-2y+3z+w &= 0 \\ -y \qquad -w &= 0 \end{array}$

(d) $\begin{array}{rl} a+2b+3c+d-e &= 1 \\ 3a-b+c+d+e &= 3 \end{array}$

[*] More information on partitions and class representatives is in the appendix.

1.10 Give two distinct echelon form versions of this matrix.

$$\begin{pmatrix} 2 & 1 & 1 & 3 \\ 6 & 4 & 1 & 2 \\ 1 & 5 & 1 & 5 \end{pmatrix}$$

✓ **1.11** List the reduced echelon forms possible for each size.
(a) 2×2 **(b)** 2×3 **(c)** 3×2 **(d)** 3×3

✓ **1.12** What results from applying Gauss-Jordan reduction to a nonsingular matrix?

1.13 The proof of Lemma 1.4 contains a reference to the $i \neq j$ condition on the row pivoting operation.
(a) The definition of row operations has an $i \neq j$ condition on the swap operation $\rho_i \leftrightarrow \rho_j$. Show that in $A \xrightarrow{\rho_i \leftrightarrow \rho_j} \xrightarrow{\rho_i \leftrightarrow \rho_j} A$ this condition is not needed.
(b) Write down a 2×2 matrix with nonzero entries, and show that the $-1 \cdot \rho_1 + \rho_1$ operation is not reversed by $1 \cdot \rho_1 + \rho_1$.
(c) Expand the proof of that lemma to make explicit exactly where the $i \neq j$ condition on pivoting is used.

III.2 Row Equivalence

We will close this section and this chapter by proving that every matrix is row equivalent to one and only one reduced echelon form matrix. The ideas that appear here will reappear, and be further developed, in the next chapter.

The underlying theme here is that one way to understand a mathematical situation is by being able to classify the cases that can happen. We have met this theme several times already. We have classified solution sets of linear systems into the no-elements, one-element, and infinitely-many elements cases. We have also classified linear systems with the same number of equations as unknowns into the nonsingular and singular cases. We adopted these classifications because they give us a way to understand the situations that we were investigating. Here, where we are investigating row equivalence, we know that the set of all matrices breaks into the row equivalence classes. When we finish the proof here, we will have a way to understand each of those classes — its matrices can be thought of as derived by row operations from the unique reduced echelon form matrix in that class.

To understand how row operations act to transform one matrix into another, we consider the effect that they have on the parts of a matrix. The crucial observation is that row operations combine the rows linearly.

2.1 Definition A *linear combination* of $x_1, \ldots, x_m$ is an expression of the form $c_1x_1 + c_2x_2 + \cdots + c_mx_m$ where the c's are scalars.

(We have already used the phrase 'linear combination' in this book. The meaning is unchanged, but the next result's statement makes a more formal definition in order.)

2.2 Lemma (Linear Combination Lemma) A linear combination of linear combinations is a linear combination.

Proof. Given the linear combinations $c_{1,1}x_1 + \cdots + c_{1,n}x_n$ through $c_{m,1}x_1 + \cdots + c_{m,n}x_n$, consider a combination of those

$$d_1(c_{1,1}x_1 + \cdots + c_{1,n}x_n) + \cdots + d_m(c_{m,1}x_1 + \cdots + c_{m,n}x_n)$$

where the d's are scalars along with the c's. Distributing those d's and regrouping gives

$$\begin{aligned} &= d_1c_{1,1}x_1 + \cdots + d_1c_{1,n}x_n + d_2c_{2,1}x_1 + \cdots + d_mc_{1,1}x_1 + \cdots + d_mc_{1,n}x_n \\ &= (d_1c_{1,1} + \cdots + d_mc_{m,1})x_1 + \cdots + (d_1c_{1,n} + \cdots + d_mc_{m,n})x_n \end{aligned}$$

which is indeed a linear combination of the x's. QED

In this subsection we will use the convention that, where a matrix is named with an upper case roman letter, the matching lower-case greek letter names the rows.

$$A = \begin{pmatrix} \cdots \alpha_1 \cdots \\ \cdots \alpha_2 \cdots \\ \vdots \\ \cdots \alpha_m \cdots \end{pmatrix} \qquad B = \begin{pmatrix} \cdots \beta_1 \cdots \\ \cdots \beta_2 \cdots \\ \vdots \\ \cdots \beta_m \cdots \end{pmatrix}$$

2.3 Corollary Where one matrix row reduces to another, each row of the second is a linear combination of the rows of the first.

The proof below uses induction on the number of row operations used to reduce one matrix to the other. Before we proceed, here is an outline of the argument (readers unfamiliar with induction may want to compare this argument with the one used in the 'General = Particular + Homogeneous' proof).* First, for the base step of the argument, we will verify that the proposition is true when reduction can be done in zero row operations. Second, for the inductive step, we will argue that if being able to reduce the first matrix to the second in some number $t \geq 0$ of operations implies that each row of the second is a linear combination of the rows of the first, then being able to reduce the first to the second in $t+1$ operations implies the same thing. Together, this base step and induction step prove this result because by the base step the proposition is true in the zero operations case, and by the inductive step the fact that it is true in the zero operations case implies that it is true in the one operation case, and the inductive step applied again gives that it is therefore true in the two operations case, etc.

Proof. We proceed by induction on the minimum number of row operations that take a first matrix A to a second one B.

* More information on mathematical induction is in the appendix.

In the base step, that zero reduction operations suffice, the two matrices are equal and each row of B is obviously a combination of A's rows: $\vec{\beta}_i = 0 \cdot \vec{\alpha}_1 + \cdots + 1 \cdot \vec{\alpha}_i + \cdots + 0 \cdot \vec{\alpha}_m$.

For the inductive step, assume the inductive hypothesis: with $t \geq 0$, if a matrix can be derived from A in t or fewer operations then its rows are linear combinations of the A's rows. Consider a B that takes $t+1$ operations. Because there are more than zero operations, there must be a next-to-last matrix G so that $A \longrightarrow \cdots \longrightarrow G \longrightarrow B$. This G is only t operations away from A and so the inductive hypothesis applies to it, that is, each row of G is a linear combination of the rows of A.

If the last operation, the one from G to B, is a row swap then the rows of B are just the rows of G reordered and thus each row of B is also a linear combination of the rows of A. The other two possibilities for this last operation, that it multiplies a row by a scalar and that it adds a multiple of one row to another, both result in the rows of B being linear combinations of the rows of G. But therefore, by the Linear Combination Lemma, each row of B is a linear combination of the rows of A.

With that, we have both the base step and the inductive step, and so the proposition follows. QED

2.4 Example In the reduction

$$\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \xrightarrow{(1/2)\rho_2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \xrightarrow{-\rho_2 + \rho_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

call the matrices A, D, G, and B. The methods of the proof show that there are three sets of linear relationships.

$$\begin{array}{lll} \delta_1 = 0 \cdot \alpha_1 + 1 \cdot \alpha_2 & \gamma_1 = 0 \cdot \alpha_1 + 1 \cdot \alpha_2 & \beta_1 = (-1/2)\alpha_1 + 1 \cdot \alpha_2 \\ \delta_2 = 1 \cdot \alpha_1 + 0 \cdot \alpha_2 & \gamma_2 = (1/2)\alpha_1 + 0 \cdot \alpha_2 & \beta_2 = (1/2)\alpha_1 + 0 \cdot \alpha_2 \end{array}$$

The prior result gives us the insight that Gauss' method works by taking linear combinations of the rows. But to what end; why do we go to echelon form as a particularly simple, or basic, version of a linear system? The answer, of course, is that echelon form is suitable for back substitution, because we have isolated the variables. For instance, in this matrix

$$R = \begin{pmatrix} 2 & 3 & 7 & 8 & 0 & 0 \\ 0 & 0 & 1 & 5 & 1 & 1 \\ 0 & 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

x_1 has been removed from x_5's equation. That is, Gauss' method has made x_5's row independent of x_1's row.

Independence of a collection of row vectors, or of any kind of vectors, will be precisely defined and explored in the next chapter. But a first take on it is that we can show that, say, the third row above is not comprised of the other rows, that $\rho_3 \neq c_1\rho_1 + c_2\rho_2 + c_4\rho_4$. For, suppose that there are scalars c_1, c_2, and c_4 such that this relationship holds.

$$\begin{aligned}\begin{pmatrix}0&0&0&3&3&0\end{pmatrix} = c_1 &\begin{pmatrix}2&3&7&8&0&0\end{pmatrix}\\ &+ c_2\begin{pmatrix}0&0&1&5&1&1\end{pmatrix}\\ &+ c_4\begin{pmatrix}0&0&0&0&2&1\end{pmatrix}\end{aligned}$$

The first row's leading entry is in the first column and narrowing our consideration of the above relationship to consideration only of the entries from the first column $0 = 2c_1+0c_2+0c_4$ gives that $c_1 = 0$. The second row's leading entry is in the third column and the equation of entries in that column $0 = 7c_1 + 1c_2 + 0c_4$, along with the knowledge that $c_1 = 0$, gives that $c_2 = 0$. Now, to finish, the third row's leading entry is in the fourth column and the equation of entries in that column $3 = 8c_1 + 5c_2 + 0c_4$, along with $c_1 = 0$ and $c_2 = 0$, gives an impossibility.

The following result shows that this effect always holds. It shows that what Gauss' linear elimination method eliminates is linear relationships among the rows.

2.5 Lemma In an echelon form matrix, no nonzero row is a linear combination of the other rows.

PROOF. Let R be in echelon form. Suppose, to obtain a contradiction, that some nonzero row is a linear combination of the others.

$$\rho_i = c_1\rho_1 + \ldots + c_{i-1}\rho_{i-1} + c_{i+1}\rho_{i+1} + \ldots + c_m\rho_m$$

We will first use induction to show that the coefficients $c_1, \ldots, c_{i-1}$ associated with rows above ρ_i are all zero. The contradiction will come from consideration of ρ_i and the rows below it.

The base step of the induction argument is to show that the first coefficient c_1 is zero. Let the first row's leading entry be in column number ℓ_1 be the column number of the leading entry of the first row and consider the equation of entries in that column.

$$\rho_{i,\ell_1} = c_1\rho_{1,\ell_1} + \ldots + c_{i-1}\rho_{i-1,\ell_1} + c_{i+1}\rho_{i+1,\ell_1} + \ldots + c_m\rho_{m,\ell_1}$$

The matrix is in echelon form so the entries $\rho_{2,\ell_1}, \ldots, \rho_{m,\ell_1}$, including ρ_{i,ℓ_1}, are all zero.

$$0 = c_1\rho_{1,\ell_1} + \cdots + c_{i-1}\cdot 0 + c_{i+1}\cdot 0 + \cdots + c_m \cdot 0$$

Because the entry ρ_{1,ℓ_1} is nonzero as it leads its row, the coefficient c_1 must be zero.

The inductive step is to show that for each row index k between 1 and $i-2$, if the coefficient c_1 and the coefficients $c_2, \ldots, c_k$ are all zero then c_{k+1} is also zero. That argument, and the contradiction that finishes this proof, is saved for Exercise 21. QED

We can now prove that each matrix is row equivalent to one and only one reduced echelon form matrix. We will find it convenient to break the first half of the argument off as a preliminary lemma. For one thing, it holds for any echelon form whatever, not just reduced echelon form.

2.6 Lemma If two echelon form matrices are row equivalent then the leading entries in their first rows lie in the same column. The same is true of all the nonzero rows — the leading entries in their second rows lie in the same column, etc.

For the proof we rephrase the result in more technical terms. Define the *form* of an $m \times n$ matrix to be the sequence $\langle \ell_1, \ell_2, \ldots, \ell_m \rangle$ where ℓ_i is the column number of the leading entry in row i and $\ell_i = \infty$ if there is no leading entry in that column. The lemma says that if two echelon form matrices are row equivalent then their forms are equal sequences.

Proof. Let B and D be echelon form matrices that are row equivalent. Because they are row equivalent they must be the same size, say $n \times m$. Let the column number of the leading entry in row i of B be ℓ_i and let the column number of the leading entry in row j of D be k_j. We will show that $\ell_1 = k_1$, that $\ell_2 = k_2$, etc., by induction.

This induction argument relies on the fact that the matrices are row equivalent, because the Linear Combination Lemma and its corollary therefore give that each row of B is a linear combination of the rows of D and vice versa:

$$\beta_i = s_{i,1}\delta_1 + s_{i,2}\delta_2 + \cdots + s_{i,m}\delta_m \quad \text{and} \quad \delta_j = t_{j,1}\beta_1 + t_{j,2}\beta_2 + \cdots + t_{j,m}\beta_m$$

where the s's and t's are scalars.

The base step of the induction is to verify the lemma for the first rows of the matrices, that is, to verify that $\ell_1 = k_1$. If either row is a zero row then the entire matrix is a zero matrix since it is in echelon form, and hterefore both matrices are zero matrices (by Corollary 2.3), and so both ℓ_1 and k_1 are ∞. For the case where neither β_1 nor δ_1 is a zero row, consider the $i = 1$ instance of the linear relationship above.

$$\begin{aligned}
\beta_1 &= s_{1,1}\delta_1 + s_{1,2}\delta_2 + \cdots + s_{1,m}\delta_m \\
\begin{pmatrix} 0 & \cdots & b_{1,\ell_1} & \cdots \end{pmatrix} &= s_{1,1}\begin{pmatrix} 0 & \cdots & d_{1,k_1} & \cdots \end{pmatrix} \\
&\quad + s_{1,2}\begin{pmatrix} 0 & \cdots & 0 & \cdots \end{pmatrix} \\
&\quad \vdots \\
&\quad + s_{1,m}\begin{pmatrix} 0 & \cdots & 0 & \cdots \end{pmatrix}
\end{aligned}$$

First, note that $\ell_1 < k_1$ is impossible: in the columns of D to the left of column k_1 the entries are are all zeroes (as d_{1,k_1} leads the first row) and so if $\ell_1 < k_1$ then the equation of entries from column ℓ_1 would be $b_{1,\ell_1} = s_{1,1} \cdot 0 + \cdots + s_{1,m} \cdot 0$, but b_{1,ℓ_1} isn't zero since it leads its row and so this is an impossibility. Next, a symmetric argument shows that $k_1 < \ell_1$ also is impossible. Thus the $\ell_1 = k_1$ base case holds.

The inductive step is to show that if $\ell_1 = k_1$, and $\ell_2 = k_2$, ..., and $\ell_r = k_r$, then also $\ell_{r+1} = k_{r+1}$ (for r in the interval $1 .. m-1$). This argument is saved for Exercise 22. QED

That lemma answers two of the questions that we have posed (i) any two echelon form versions of a matrix have the same free variables, and consequently (ii) any two echelon form versions have the same number of free variables. There is no linear system and no combination of row operations such that, say, we could solve the system one way and get y and z free but solve it another way and get y and w free, or solve it one way and get two free variables while solving it another way yields three.

We finish now by specializing to the case of reduced echelon form matrices.

2.7 Theorem Each matrix is row equivalent to a unique reduced echelon form matrix.

Proof. Clearly any matrix is row equivalent to at least one reduced echelon form matrix, via Gauss-Jordan reduction. For the other half, that any matrix is equivalent to at most one reduced echelon form matrix, we will show that if a matrix Gauss-Jordan reduces to each of two others then those two are equal.

Suppose that a matrix is row equivalent the two reduced echelon form matrices B and D, which are therefore row equivalent to each other. The Linear Combination Lemma and its corollary allow us to write the rows of one, say B, as a linear combination of the rows of the other $\beta_i = c_{i,1}\delta_1 + \cdots + c_{i,m}\delta_m$. The preliminary result, Lemma 2.6, says that in the two matrices, the same collection of rows are nonzero. Thus, if β_1 through β_r are the nonzero rows of B then the nonzero rows of D are δ_1 through δ_r. Zero rows don't contribute to the sum so we can rewrite the relationship to include just the nonzero rows.

$$\beta_i = c_{i,1}\delta_1 + \cdots + c_{i,r}\delta_r \qquad (*)$$

The preliminary result also says that for each row j between 1 and r, the leading entries of the j-th row of B and D appear in the same column, denoted ℓ_j. Rewriting the above relationship to focus on the entries in the ℓ_j-th column

$$\begin{aligned}
\begin{pmatrix} \cdots & b_{i,\ell_j} & \cdots \end{pmatrix} = c_{i,1} &\begin{pmatrix} \cdots & d_{1,\ell_j} & \cdots \end{pmatrix} \\
+ c_{i,2} &\begin{pmatrix} \cdots & d_{2,\ell_j} & \cdots \end{pmatrix} \\
&\vdots \\
+ c_{i,r} &\begin{pmatrix} \cdots & d_{r,\ell_j} & \cdots \end{pmatrix}
\end{aligned}$$

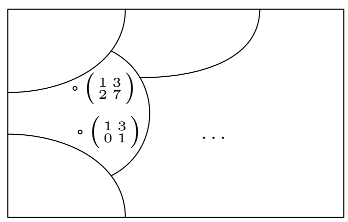
gives this set of equations for $i = 1$ up to $i = r$.

$$\begin{aligned}
b_{1,\ell_j} &= c_{1,1}d_{1,\ell_j} + \cdots + c_{1,j}d_{j,\ell_j} + \cdots + c_{1,r}d_{r,\ell_j} \\
&\vdots \\
b_{j,\ell_j} &= c_{j,1}d_{1,\ell_j} + \cdots + c_{j,j}d_{j,\ell_j} + \cdots + c_{j,r}d_{r,\ell_j} \\
&\vdots \\
b_{r,\ell_j} &= c_{r,1}d_{1,\ell_j} + \cdots + c_{r,j}d_{j,\ell_j} + \cdots + c_{r,r}d_{r,\ell_j}
\end{aligned}$$

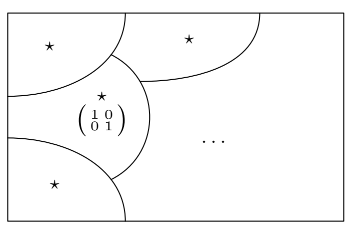
Since D is in reduced echelon form, all of the d's in column ℓ_j are zero except for d_{j,ℓ_j}, which is 1. Thus each equation above simplifies to $b_{i,\ell_j} = c_{i,j}d_{j,\ell_j} = c_{i,j}\cdot 1$. But B is also in reduced echelon form and so all of the b's in column ℓ_j are zero except for b_{j,ℓ_j}, which is 1. Therefore, each $c_{i,j}$ is zero, except that $c_{1,1} = 1$, and $c_{2,2} = 1$, ..., and $c_{r,r} = 1$.

We have shown that the only nonzero coefficient in the linear combination labelled $(*)$ is $c_{j,j}$, which is 1. Therefore $\beta_j = \delta_j$. Because this holds for all nonzero rows, $B = D$. QED

We end with a recap. In Gauss' method we start with a matrix and then derive a sequence of other matrices. We defined two matrices to be related if one can be derived from the other. That relation is an equivalence relation, called row equivalence, and so partitions the set of all matrices into row equivalence classes.



(There are infinitely many matrices in the pictured class, but we've only got room to show two.) We have proved there is one and only one reduced echelon form matrix in each row equivalence class. So the reduced echelon form is a *canonical form** for row equivalence: the reduced echelon form matrices are representatives of the classes.



We can answer questions about the classes by translating them into questions about the representatives.

2.8 Example We can decide if matrices are interreducible by seeing if Gauss-Jordan reduction produces the same reduced echelon form result. Thus, these are not row equivalent

$$\begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix} \qquad \begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix}$$

* More information on canonical representatives is in the appendix.

because their reduced echelon forms are not equal.

$$\begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2.9 Example Any nonsingular 3×3 matrix Gauss-Jordan reduces to this.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.10 Example We can describe the classes by listing all possible reduced echelon form matrices. Any 2×2 matrix lies in one of these: the class of matrices row equivalent to this,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

the infinitely many classes of matrices row equivalent to one of this type

$$\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$$

where $a \in \mathbb{R}$ (including $a = 0$), the class of matrices row equivalent to this,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and the class of matrices row equivalent to this

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(this the class of nonsingular 2×2 matrices).

Exercises

✓ **2.11** Decide if the matrices are row equivalent.

(a) $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 4 & 3 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \end{pmatrix}$ **(d)** $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 3 & -1 \\ 2 & 2 & 5 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$

2.12 Describe the matrices in each of the classes represented in Example 2.10.

2.13 Describe all matrices in the row equivalence class of these.

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$

2.14 How many row equivalence classes are there?

2.15 Can row equivalence classes contain different-sized matrices?

2.16 How big are the row equivalence classes?
(a) Show that the class of any zero matrix is finite.
(b) Do any other classes contain only finitely many members?

✓ **2.17** Give two reduced echelon form matrices that have their leading entries in the same columns, but that are not row equivalent.

✓ **2.18** Show that any two $n\times n$ nonsingular matrices are row equivalent. Are any two singular matrices row equivalent?

✓ **2.19** Describe all of the row equivalence classes containing these.
(a) 2×2 matrices **(b)** 2×3 matrices **(c)** 3×2 matrices
(d) 3×3 matrices

2.20 **(a)** Show that a vector $\vec{\beta}_0$ is a linear combination of members of the set $\{\vec{\beta}_1,\ldots,\vec{\beta}_n\}$ if and only there is a linear relationship $\vec{0}=c_0\vec{\beta}_0+\cdots+c_n\vec{\beta}_n$ where c_0 is not zero. (Watch out for the $\vec{\beta}_0=\vec{0}$ case.)
(b) Derive Lemma 2.5.

✓ **2.21** Finish the proof of Lemma 2.5.
(a) First illustrate the inductive step by showing that $\ell_2=k_2$.
(b) Do the full inductive step: assume that c_k is zero for $1\leq k<i-1$, and deduce that c_{k+1} is also zero.
(c) Find the contradiction.

2.22 Finish the induction argument in Lemma 2.6.
(a) State the inductive hypothesis, Also state what must be shown to follow from that hypothesis.
(b) Check that the inductive hypothesis implies that in the relationship $\beta_{r+1}=s_{r+1,1}\delta_1+s_{r+2,2}\delta_2+\cdots+s_{r+1,m}\delta_m$ the coefficients $s_{r+1,1},\ldots,s_{r+1,r}$ are each zero.
(c) Finish the inductive step by arguing, as in the base case, that $\ell_{r+1}<k_{r+1}$ and $k_{r+1}<\ell_{r+1}$ are impossible.

2.23 Why, in the proof of Theorem 2.7, do we bother to restrict to the nonzero rows? Why not just stick to the relationship that we began with, $\beta_i=c_{i,1}\delta_1+\cdots+c_{i,m}\delta_m$, with m instead of r, and argue using it that the only nonzero coefficient is $c_{i,i}$, which is 1?

✓ **2.24** Three truck drivers went into a roadside cafe. One truck driver purchased four sandwiches, a cup of coffee, and ten doughnuts for $8.45. Another driver purchased three sandwiches, a cup of coffee, and seven doughnuts for $6.30. What did the third truck driver pay for a sandwich, a cup of coffee, and a doughnut? [Trono]

2.25 The fact that Gaussian reduction disallows multiplication of a row by zero is needed for the proof of uniqueness of reduced echelon form, or else every matrix would be row equivalent to a matrix of all zeros. Where is it used?

✓ **2.26** The Linear Combination Lemma says which equations can be gotten from Gaussian reduction from a given linear system.

(1) Produce an equation not implied by this system.

$$\begin{aligned}3x+4y&=8\\2x+y&=3\end{aligned}$$

(2) Can any equation be derived from an inconsistent system?

2.27 Extend the definition of row equivalence to linear systems. Under your definition, do equivalent systems have the same solution set? [Hoffman & Kunze]

✓ **2.28** In this matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 3 \\ 1 & 4 & 5 \end{pmatrix}$$

the first and second columns add to the third.

(a) Show that remains true under any row operation.
(b) Make a conjecture.
(c) Prove that it holds.

Topic: Computer Algebra Systems

The linear systems in this chapter are small enough that their solution by hand is easy. But large systems are easiest, and safest, to do on a computer. There are special purpose programs such as LINPACK for this job. Another popular tool is a general purpose computer algebra system, including both commercial packages such as Maple, Mathematica, or MATLAB, or free packages such as SciLab,, MuPAD, or Octave.

For example, in the Topic on Networks, we need to solve this.

$$\begin{array}{rcrcrcrcrcrcrcl}
i_0 & - & i_1 & - & i_2 & & & & & & & & & = & 0 \\
 & & i_1 & & & - & i_3 & & & - & i_5 & & & = & 0 \\
 & & & & i_2 & & & - & i_4 & + & i_5 & & & = & 0 \\
 & & & & & & i_3 & + & i_4 & & & - & i_6 & = & 0 \\
 & & 5i_1 & & & + & 10i_3 & & & & & & & = & 10 \\
 & & & & 2i_2 & & & + & 4i_4 & & & & & = & 10 \\
 & & 5i_1 & - & 2i_2 & & & & & + & 50i_5 & & & = & 0
\end{array}$$

It can be done by hand, but it would take a while and be error-prone. Using a computer is better.

We illustrate by solving that system under Maple (for another system, a user's manual would obviously detail the exact syntax needed). The array of coefficients can be entered in this way

```
> A:=array( [[1,-1,-1,0,0,0,0],
             [0,1,0,-1,0,-1,0],
             [0,0,1,0,-1,1,0],
             [0,0,0,1,1,0,-1],
             [0,5,0,10,0,0,0],
             [0,0,2,0,4,0,0],
             [0,5,-1,0,0,10,0]] );
```

(putting the rows on separate lines is not necessary, but is done for clarity). The vector of constants is entered similarly.

```
> u:=array( [0,0,0,0,10,10,0] );
```

Then the system is solved, like magic.

```
> linsolve(A,u);
      7  2  5  2  5     7
    [ -, -, -, -, -, 0, - ]
      3  3  3  3  3     3
```

Systems with infinitely many solutions are solved in the same way — the computer simply returns a parametrization.

Exercises

Answers for this Topic use Maple as the computer algebra system. In particular, all of these were tested on Maple V running under MS-DOS NT version 4.0. *(On all of them, the preliminary command to load the linear algebra package along with Maple's responses to the Enter key, have been omitted.) Other systems have similar commands.*

1 Use the computer to solve the two problems that opened this chapter.

(a) This is the Statics problem.

$$\begin{aligned} 40h + 15c &= 100 \\ 25c &= 50 + 50h \end{aligned}$$

(b) This is the Chemistry problem.

$$\begin{aligned} 7h &= 7j \\ 8h + 1i &= 5j + 2k \\ 1i &= 3j \\ 3i &= 6j + 1k \end{aligned}$$

2 Use the computer to solve these systems from the first subsection, or conclude 'many solutions' or 'no solutions'.

(a) $\begin{aligned} 2x + 2y &= 5 \\ x - 4y &= 0 \end{aligned}$ **(b)** $\begin{aligned} -x + y &= 1 \\ x + y &= 2 \end{aligned}$ **(c)** $\begin{aligned} x - 3y + z &= 1 \\ x + y + 2z &= 14 \end{aligned}$

(d) $\begin{aligned} -x - y &= 1 \\ -3x - 3y &= 2 \end{aligned}$ **(e)** $\begin{aligned} 4y + z &= 20 \\ 2x - 2y + z &= 0 \\ x + z &= 5 \\ x + y - z &= 10 \end{aligned}$ **(f)** $\begin{aligned} 2x + z + w &= 5 \\ y - w &= -1 \\ 3x - z - w &= 0 \\ 4x + y + 2z + w &= 9 \end{aligned}$

3 Use the computer to solve these systems from the second subsection.

(a) $\begin{aligned} 3x + 6y &= 18 \\ x + 2y &= 6 \end{aligned}$ **(b)** $\begin{aligned} x + y &= 1 \\ x - y &= -1 \end{aligned}$ **(c)** $\begin{aligned} x_1 + x_3 &= 4 \\ x_1 - x_2 + 2x_3 &= 5 \\ 4x_1 - x_2 + 5x_3 &= 17 \end{aligned}$

(d) $\begin{aligned} 2a + b - c &= 2 \\ 2a + c &= 3 \\ a - b &= 0 \end{aligned}$ **(e)** $\begin{aligned} x + 2y - z &= 3 \\ 2x + y + w &= 4 \\ x - y + z + w &= 1 \end{aligned}$ **(f)** $\begin{aligned} x + z + w &= 4 \\ 2x + y - w &= 2 \\ 3x + y + z &= 7 \end{aligned}$

4 What does the computer give for the solution of the general 2×2 system?

$$\begin{aligned} ax + cy &= p \\ bx + dy &= q \end{aligned}$$

Topic: Input-Output Analysis

An economy is an immensely complicated network of interdependences. Changes in one part can ripple out to affect other parts. Economists have struggled to be able to describe, and to make predictions about, such a complicated object. Mathematical models using systems of linear equations have emerged as a key tool. One is Input-Output Analysis, pioneered by W. Leontief, who won the 1973 Nobel Prize in Economics.

Consider an economy with many parts, two of which are the steel industry and the auto industry. As they work to meet the demand for their product from other parts of the economy, that is, from users external to the steel and auto sectors, these two interact tightly. For instance, should the external demand for autos go up, that would lead to an increase in the auto industry's usage of steel. Or, should the external demand for steel fall, then it would lead to a fall in steel's purchase of autos. The type of Input-Output model we will consider takes in the external demands and then predicts how the two interact to meet those demands.

We start with a listing of production and consumption statistics. (These numbers, giving dollar values in millions, are excerpted from [Leontief 1965], describing the 1958 U.S. economy. Today's statistics would be quite different, both because of inflation and because of technical changes in the industries.)

	used by steel	*used by auto*	*used by others*	*total*
value of steel	5 395	2 664		25 448
value of auto	48	9 030		30 346

For instance, the dollar value of steel used by the auto industry in this year is 2,664 million. Note that industries may consume some of their own output.

We can fill in the blanks for the external demand. This year's value of the steel used by others this year is 17,389 and this year's value of the auto used by others is 21,268. With that, we have a complete description of the external demands and of how auto and steel interact, this year, to meet them.

Now, imagine that the external demand for steel has recently been going up by 200 per year and so we estimate that next year it will be 17,589. Imagine also that for similar reasons we estimate that next year's external demand for autos will be down 25 to 21,243. We wish to predict next year's total outputs.

That prediction isn't as simple as adding 200 to this year's steel total and subtracting 25 from this year's auto total. For one thing, a rise in steel will cause that industry to have an increased demand for autos, which will mitigate, to some extent, the loss in external demand for autos. On the other hand, the drop in external demand for autos will cause the auto industry to use less steel, and so lessen somewhat the upswing in steel's business. In short, these two industries form a system, and we need to predict the totals at which the system as a whole will settle.

For that prediction, let s be next years total production of steel and let a be next year's total output of autos. We form these equations.

$$\begin{aligned}\text{next year's production of steel} &= \text{next year's use of steel by steel}\\ &\quad + \text{next year's use of steel by auto}\\ &\quad + \text{next year's use of steel by others}\\ \text{next year's production of autos} &= \text{next year's use of autos by steel}\\ &\quad + \text{next year's use of autos by auto}\\ &\quad + \text{next year's use of autos by others}\end{aligned}$$

On the left side of those equations go the unknowns s and a. At the ends of the right sides go our external demand estimates for next year $17,589$ and $21,243$. For the remaining four terms, we look to the table of this year's information about how the industries interact.

For instance, for next year's use of steel by steel, we note that this year the steel industry used 5395 units of steel input to produce $25,448$ units of steel output. So next year, when the steel industry will produce s units out, we expect that doing so will take $s \cdot (5395)/(25\,448)$ units of steel input — this is simply the assumption that input is proportional to output. (We are assuming that the ratio of input to output remains constant over time; in practice, models may try to take account of trends of change in the ratios.)

Next year's use of steel by the auto industry is similar. This year, the auto industry uses 2664 units of steel input to produce 30346 units of auto output. So next year, when the auto industry's total output is a, we expect it to consume $a \cdot (2664)/(30346)$ units of steel.

Filling in the other equation in the same way, we get this system of linear equation.

$$\begin{aligned}\frac{5\,395}{25\,448} \cdot s + \frac{2\,664}{30\,346} \cdot a + 17\,589 &= s\\ \frac{48}{25\,448} \cdot s + \frac{9\,030}{30\,346} \cdot a + 21\,243 &= a\end{aligned}$$

Rounding to four decimal places and putting it into the form for Gauss' method gives this.

$$\begin{aligned}0.7880s - 0.0879a &= 17\,589\\ -0.0019s + 0.7024a &= 21\,268\end{aligned}$$

The solution is $s = 25\,708$ and $a = 30\,350$.

Looking back, recall that above we described why the prediction of next year's totals isn't as simple as adding 200 to last year's steel total and subtracting 25 from last year's auto total. In fact, comparing these totals for next year to the ones given at the start for the current year shows that, despite the drop in external demand, the total production of the auto industry is predicted to rise. The increase in internal demand for autos caused by steel's sharp rise in business more than makes up for the loss in external demand for autos.

One of the advantages of having a mathematical model is that we can ask "What if ...?" questions. For instance, we can ask "What if the estimates for next year's external demands are somewhat off?" To try to understand how much the model's predictions change in reaction to changes in our estimates, we can try revising our estimate of next year's external steel demand from 17, 589 down to 17, 489, while keeping the assumption of next year's external demand for autos fixed at 21, 243. The resulting system

$$\begin{aligned} 0.7880s - 0.0879a &= 17\,489 \\ -0.0019s + 0.7024a &= 21\,243 \end{aligned}$$

when solved gives $s = 25\,577$ and $a = 30\,314$. This kind of exploration of the model is *sensitivity analysis*. We are seeing how sensitive the predictions of our model are to the accuracy of the assumptions.

Obviously, we can consider larger models that detail the interactions among more sectors of an economy. These models are typically solved on a computer, using the techniques of matrix algebra that we will develop in Chapter Three. Some examples are given in the exercises. Obviously also, a single model does not suit every case; expert judgment is needed to see if the assumptions underlying the model can are reasonable ones to apply to a particular case. With those caveats, however, this model has proven in practice to be a useful and accurate tool for economic analysis. For further reading, try [Leontief 1951] and [Leontief 1965].

Exercises

Hint: these systems are easiest to solve on a computer.

1 With the steel-auto system given above, estimate next year's total productions in these cases.

(a) Next year's external demands are: up 200 from this year for steel, and unchanged for autos.

(b) Next year's external demands are: up 100 for steel, and up 200 for autos.

(c) Next year's external demands are: up 200 for steel, and up 200 for autos.

2 Imagine that a new process for making autos is pioneered. The ratio for use of steel by the auto industry falls to .0500 (that is, the new process is more efficient in its use of steel).

(a) How will the predictions for next year's total productions change compared to the first example discussed above (i.e., taking next year's external demands to be 17, 589 for steel and 21, 243 for autos)?

(b) Predict next year's totals if, in addition, the external demand for autos rises to be 21, 500 because the new cars are cheaper.

3 This table gives the numbers for the auto-steel system from a different year, 1947 (see [Leontief 1951]). The units here are billions of 1947 dollars.

	used by steel	*used by auto*	*used by others*	*total*
value of steel	6.90	1.28		18.69
value of autos	0	4.40		14.27

(a) Fill in the missing external demands, and compute the ratios.

(b) Solve for total output if next year's external demands are: steel's demand up 10% and auto's demand up 15%.

(c) How do the ratios compare to those given above in the discussion for the 1958 economy?

(d) Solve these equations with the 1958 external demands (note the difference in units; a 1947 dollar buys about what \$1.30 in 1958 dollars buys). How far off are the predictions for total output?

4 Predict next year's total productions of each of the three sectors of the hypothetical economy shown below

	used by farm	*used by rail*	*used by shipping*	*used by others*	*total*
value of farm	25	50	100		800
value of rail	25	50	50		300
value of shipping	15	10	0		500

if next year's external demands are as stated.

(a) 625 for farm, 200 for rail, 475 for shipping

(b) 650 for farm, 150 for rail, 450 for shipping

5 This table gives the interrelationships among three segments of an economy (see [Clark & Coupe]).

	used by food	*used by wholesale*	*used by retail*	*used by others*	*total*
value of food	0	2 318	4 679		11 869
value of wholesale	393	1 089	22 459		122 242
value of retail	3	53	75		116 041

We will do an Input-Output analysis on this system.

(a) Fill in the numbers for this year's external demands.

(b) Set up the linear system, leaving next year's external demands blank.

(c) Solve the system where next year's external demands are calculated by taking this year's external demands and inflating them 10%. Do all three sectors increase their total business by 10%? Do they all even increase at the same rate?

(d) Solve the system where next year's external demands are calculated by taking this year's external demands and reducing them 7%. (The study from which these numbers are taken concluded that because of the closing of a local military facility, overall personal income in the area would fall 7%, so this might be a first guess at what would actually happen.)

Topic: Accuracy of Computations

Gauss' method lends itself nicely to computerization. The code below illustrates. It operates on an $n{\times}n$ matrix `a`, pivoting with the first row, then with the second row, etc.

```
for(pivot_row=1;pivot_row<=n-1;pivot_row++){
  for(row_below=pivot_row+1;row_below<=n;row_below++){
    multiplier=a[row_below,pivot_row]/a[pivot_row,pivot_row];
    for(col=pivot_row;col<=n;col++){
      a[row_below,col]-=multiplier*a[pivot_row,col];
    }
  }
}
```

(This code is in the C language. Here is a brief translation. The loop construct `for(pivot_row=1;pivot_row<=n-1;pivot_row++){···}` sets `pivot_row` to 1 and then iterates while `pivot_row` is less than or equal to $n-1$, each time through incrementing `pivot_row` by one with the '`++`' operation. The other non-obvious construct is that the '`-=`' in the innermost loop amounts to the `a[row_below,col]` $=-$`multiplier` $*$ `a[pivot_row,col]` $+$ `a[row_below,col]` operation.)

While this code provides a quick take on how Gauss' method can be mechanized, it is not ready to use. It is naive in many ways. The most glaring way is that it assumes that a nonzero number is always found in the `pivot_row, pivot_row` position for use as the pivot entry. To make it practical, one way in which this code needs to be reworked is to cover the case where finding a zero in that location leads to a row swap, or to the conclusion that the matrix is singular.

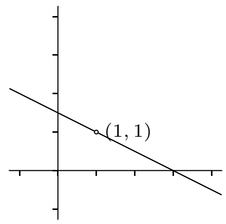
Adding some `if` $\cdots$ statements to cover those cases is not hard, but we will instead consider some more subtle ways in which the code is naive. There are pitfalls arising from the computer's reliance on finite-precision floating point arithmetic.

For example, we have seen above that we must handle as a separate case a system that is singular. But systems that are nearly singular also require care. Consider this one.

$$\begin{aligned} x + 2y &= 3 \\ 1.000\,000\,01x + 2y &= 3.000\,000\,01 \end{aligned}$$

By eye we get the solution $x=1$ and $y=1$. But a computer has more trouble. A computer that represents real numbers to eight significant places (as is common, usually called *single precision*) will represent the second equation internally as $1.000\,000\,0x + 2y = 3.000\,000\,0$, losing the digits in the ninth place. Instead of reporting the correct solution, this computer will report something that is not even close — this computer thinks that the system is singular because the two equations are represented internally as equal.

For some intuition about how the computer could think something that is so far off, we can graph the system.



At the scale of this graph, the two lines cannot be resolved apart. This system is nearly singular in the sense that the two lines are nearly the same line. Near-singularity gives this system the property that a small change in the system can cause a large change in its solution; for instance, changing the $3.000\,000\,01$ to $3.000\,000\,03$ changes the intersection point from $(1, 1)$ to $(3, 0)$. This system changes radically depending on a ninth digit, which explains why the eight-place computer has trouble. A problem that is very sensitive to inaccuracy or uncertainties in the input values is *ill-conditioned*.

The above example gives one way in which a system can be difficult to solve on a computer. It has the advantage that the picture of nearly-equal lines gives a memorable insight into one way that numerical difficulties can arise. Unfortunately this insight isn't very useful when we wish to solve some large system. We cannot, typically, hope to understand the geometry of an arbitrary large system. In addition, there are ways that a computer's results may be unreliable other than that the angle between some of the linear surfaces is quite small.

For an example, consider the system below, from [Hamming].

$$\begin{aligned} 0.001x + y &= 1 \\ x - y &= 0 \end{aligned} \qquad (*)$$

The second equation gives $x = y$, so $x = y = 1/1.001$ and thus both variables have values that are just less than 1. A computer using two digits represents the system internally in this way (we will do this example in two-digit floating point arithmetic, but a similar one with eight digits is easy to invent).

$$\begin{aligned} (1.0 \times 10^{-2})x + (1.0 \times 10^{0})y &= 1.0 \times 10^{0} \\ (1.0 \times 10^{0})x - (1.0 \times 10^{0})y &= 0.0 \times 10^{0} \end{aligned}$$

The computer's row reduction step $-1000\rho_1 + \rho_2$ produces a second equation $-1001y = -999$, which the computer rounds to two places as $(-1.0 \times 10^3)y = -1.0 \times 10^3$. Then the computer decides from the second equation that $y = 1$ and from the first equation that $x = 0$. This y value is fairly good, but the x is quite bad. Thus, another cause of unreliable output is a mixture of floating point arithmetic and a reliance on pivots that are small.

An experienced programmer may respond that we should go to *double precision* where sixteen significant digits are retained. This will indeed solve many problems. However, there are some difficulties with it as a general approach. For one thing, double precision takes longer than single precision (on a '486 chip, multiplication takes eleven ticks in single precision but fourteen in double precision [Programmer's Ref.]) and has twice the memory requirements. So attempting to do all calculations in double precision is just not practical. And besides, the above systems can obviously be tweaked to give the same trouble in the seventeenth digit, so double precision won't fix all problems. What we need is a strategy to minimize the numerical trouble arising from solving systems on a computer, and some guidance as to how far the reported solutions can be trusted.

Mathematicians have made a careful study of how to get the most reliable results. A basic improvement on the naive code above is to not simply take the entry in the *pivot_row*, *pivot_row* position for the pivot, but rather to look at all of the entries in the *pivot_row* column below the *pivot_row* row, and take the one that is most likely to give reliable results (e.g., take one that is not too small). This strategy is *partial pivoting*. For example, to solve the troublesome system ($*$) above, we start by looking at both equations for a best first pivot, and taking the 1 in the second equation as more likely to give good results. Then, the pivot step of $-.001\rho_2 + \rho_1$ gives a first equation of $1.001y = 1$, which the computer will represent as $(1.0\times 10^0)y = 1.0\times 10^0$, leading to the conclusion that $y = 1$ and, after back-substitution, $x = 1$, both of which are close to right. The code from above can be adapted to this purpose.

```
for(pivot_row=1;pivot_row<=n-1;pivot_row++){
/* find the largest pivot in this column (in row max) */
  max=pivot_row;
  for(row_below=pivot_row+1;pivot_row<=n;row_below++){
    if (abs(a[row_below,pivot_row]) > abs(a[max,row_below]))
      max=row_below;
  }
/* swap rows to move that pivot entry up */
  for(col=pivot_row;col<=n;col++){
    temp=a[pivot_row,col];
    a[pivot_row,col]=a[max,col];
    a[max,col]=temp;
  }
/* proceed as before */
  for(row_below=pivot_row+1;row_below<=n;row_below++){
    multiplier=a[row_below,pivot_row]/a[pivot_row,pivot_row];
     for(col=pivot_row;col<=n;col++){
       a[row_below,col]-=multiplier*a[pivot_row,col];
    }
  }
}
```

A full analysis of the best way to implement Gauss' method is outside the scope of the book (see [Wilkinson 1965]), but the method recommended by most experts is a variation on the code above that first finds the best pivot among the candidates, and then scales it to a number that is less likely to give trouble. This is *scaled partial pivoting*.

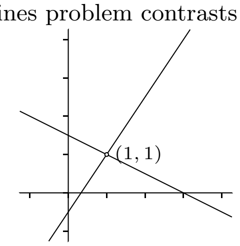
In addition to returning a result that is likely to be reliable, most well-done code will return a number, called the *conditioning number* that describes the factor by which uncertainties in the input numbers could be magnified to become inaccuracies in the results returned (see [Rice]).

The lesson of this discussion is that just because Gauss' method always works in theory, and just because computer code correctly implements that method, and just because the answer appears on green-bar paper, doesn't mean that the answer is reliable. In practice, always use a package where experts have worked hard to counter what can go wrong.

Exercises

1 Using two decimal places, add 253 and 2/3.

2 This intersect-the-lines problem contrasts with the example discussed above.



$$\begin{aligned} x + 2y &= 3 \\ 3x - 2y &= 1 \end{aligned}$$

Illustrate that in this system some small change in the numbers will produce only a small change in the solution by changing the constant in the bottom equation to 1.008 and solving. Compare it to the solution of the unchanged system.

3 Solve this system by hand ([Rice]).

$$\begin{aligned} 0.000\,3x + 1.556y &= 1.569 \\ 0.345\,4x - 2.346y &= 1.018 \end{aligned}$$

(a) Solve it accurately, by hand. **(b)** Solve it by rounding at each step to four significant digits.

4 Rounding inside the computer often has an effect on the result. Assume that your machine has eight significant digits.

(a) Show that the machine will compute $(2/3) + ((2/3) - (1/3))$ as unequal to $((2/3) + (2/3)) - (1/3)$. Thus, computer arithmetic is not associative.

(b) Compare the computer's version of $(1/3)x + y = 0$ and $(2/3)x + 2y = 0$. Is twice the first equation the same as the second?

5 Ill-conditioning is not only dependent on the matrix of coefficients. This example [Hamming] shows that it can arise from an interaction between the left and right sides of the system. Let ε be a small real.

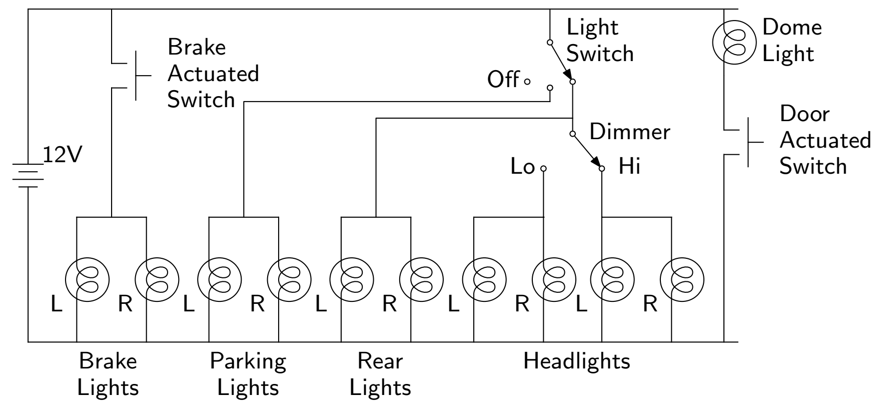
$$\begin{aligned} 3x + 2y + z &= 6 \\ 2x + 2\varepsilon y + 2\varepsilon z &= 2 + 4\varepsilon \\ x + 2\varepsilon y - \varepsilon z &= 1 + \varepsilon \end{aligned}$$

(a) Solve the system by hand. Notice that the ε's divide out only because there is an exact cancelation of the integer parts on the right side as well as on the left.

(b) Solve the system by hand, rounding to two decimal places, and with $\varepsilon = 0.001$.

Topic: Analyzing Networks

The diagram below shows some of a car's electrical network. The battery is on the left, drawn as stacked line segments. The wires are drawn as lines, shown straight and with sharp right angles for neatness. Each light is a circle enclosing a loop.



The designer of such a network needs to answer questions like: How much electricity flows when both the hi-beam headlights and the brake lights are on? Below, we will use linear systems to analyze simpler versions of electrical networks.

For the analysis we need two facts about electricity and two facts about electrical networks.

The first fact about electricity is that a battery is like a pump: it provides a force impelling the electricity to flow through the circuits connecting the battery's ends, if there are any such circuits. We say that the battery provides a *potential* to flow. Of course, this network accomplishes its function when, as the electricity flows through a circuit, it goes through a light. For instance, when the driver steps on the brake then the switch makes contact and a circuit is formed on the left side of the diagram, and the electrical current flowing through that circuit will make the brake lights go on, warning drivers behind.

The second electrical fact is that in some kinds of network components the amount of flow is proportional to the force provided by the battery. That is, for each such component there is a number, it's *resistance*, such that the potential is equal to the flow times the resistance. The units of measurement are: potential is described in *volts*, the rate of flow is in *amperes*, and resistance to the flow is in *ohms*. These units are defined so that volts $=$ amperes $\cdot$ ohms.

Components with this property, that the voltage-amperage response curve is a line through the origin, are called *resistors*. (Light bulbs such as the ones shown above are not this kind of component, because their ohmage changes as they heat up.) For example, if a resistor measures 2 ohms then wiring it to a 12 volt battery results in a flow of 6 amperes. Conversely, if we have flow of electrical current of 2 amperes through it then there must be a 4 volt potential difference between it's ends. This is the *voltage drop* across the resistor. One way to think of a electrical circuits like the one above is that the battery provides a voltage rise while the other components are voltage drops.

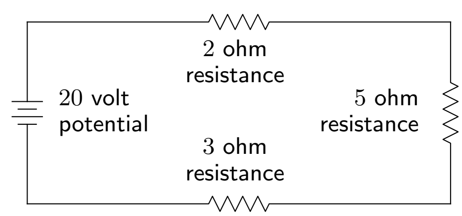
The two facts that we need about networks are Kirchhoff's Laws.

Current Law. For any point in a network, the flow in equals the flow out.

Voltage Law. Around any circuit the total drop equals the total rise.

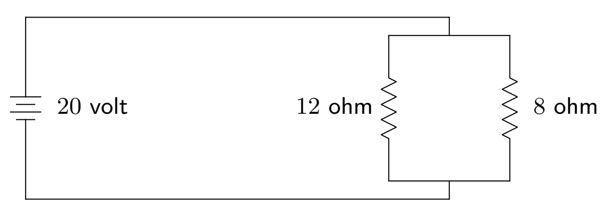
In the above network there is only one voltage rise, at the battery, but some networks have more than one.

For a start we can consider the network below. It has a battery that provides the potential to flow and three resistors (resistors are drawn as zig-zags). When components are wired one after another, as here, they are said to be in *series*.

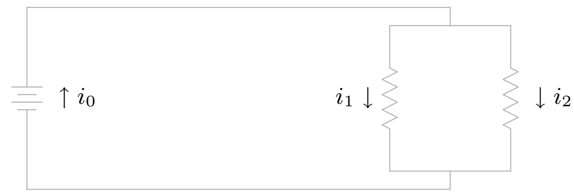


By Kirchhoff's Voltage Law, because the voltage rise is 20 volts, the total voltage drop must also be 20 volts. Since the resistance from start to finish is 10 ohms (the resistance of the wires is negligible), we get that the current is $(20/10) = 2$ amperes. Now, by Kirchhoff's Current Law, there are 2 amperes through each resistor. (And therefore the voltage drops are: 4 volts across the 2 oh m resistor, 10 volts across the 5 ohm resistor, and 6 volts across the 3 ohm resistor.)

The prior network is so simple that we didn't use a linear system, but the next network is more complicated. In this one, the resistors are in *parallel.* This network is more like the car lighting diagram shown earlier.



We begin by labeling the branches, shown below. Let the current through the left branch of the parallel portion be i_1 and that through the right branch be i_2, and also let the current through the battery be i_0. (We are following Kirchoff's Current Law; for instance, all points in the right branch have the same current, which we call i_2. Note that we don't need to know the actual direction of flow — if current flows in the direction opposite to our arrow then we will simply get a negative number in the solution.)

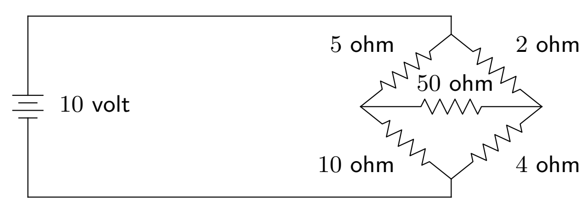


The Current Law, applied to the point in the upper right where the flow i_0 meets i_1 and i_2, gives that $i_0 = i_1 + i_2$. Applied to the lower right it gives $i_1 + i_2 = i_0$. In the circuit that loops out of the top of the battery, down the left branch of the parallel portion, and back into the bottom of the battery, the voltage rise is 20 while the voltage drop is $i_1 \cdot 12$, so the Voltage Law gives that $12i_1 = 20$. Similarly, the circuit from the battery to the right branch and back to the battery gives that $8i_2 = 20$. And, in the circuit that simply loops around in the left and right branches of the parallel portion (arbitrarily taken clockwise), there is a voltage rise of 0 and a voltage drop of $8i_2 - 12i_1$ so the Voltage Law gives that $8i_2 - 12i_1 = 0$.

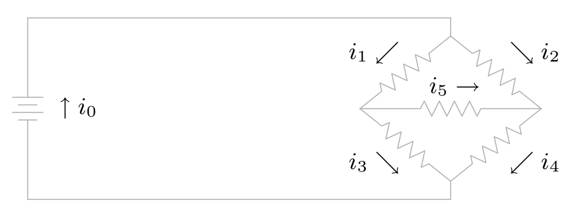
$$\begin{array}{rrrl} i_0 - & i_1 - & i_2 = & 0 \\ -i_0 + & i_1 + & i_2 = & 0 \\ & 12i_1 & = & 20 \\ & & 8i_2 = & 20 \\ & -12i_1 + & 8i_2 = & 0 \end{array}$$

The solution is $i_0 = 25/6$, $i_1 = 5/3$, and $i_2 = 5/2$, all in amperes. (Incidentally, this illustrates that redundant equations do indeed arise in practice.)

Kirchhoff's laws can be used to establish the electrical properties of networks of great complexity. The next diagram shows five resistors, wired in a *series-parallel* way.



This network is a *Wheatstone bridge* (see Exercise 4). To analyze it, we can place the arrows in this way.



Kirchoff's Current Law, applied to the top node, the left node, the right node, and the bottom node gives these.

$$
\begin{aligned}
i_0 &= i_1 + i_2 \\
i_1 &= i_3 + i_5 \\
i_2 + i_5 &= i_4 \\
i_3 + i_4 &= i_0
\end{aligned}
$$

Kirchhoff's Voltage Law, applied to the inside loop (the i_0 to i_1 to i_3 to i_0 loop), the outside loop, and the upper loop not involving the battery, gives these.

$$
\begin{aligned}
5i_1 + 10i_3 &= 10 \\
2i_2 + 4i_4 &= 10 \\
5i_1 + 50i_5 - 2i_2 &= 0
\end{aligned}
$$

Those suffice to determine the solution $i_0 = 7/3$, $i_1 = 2/3$, $i_2 = 5/3$, $i_3 = 2/3$, $i_4 = 5/3$, and $i_5 = 0$.

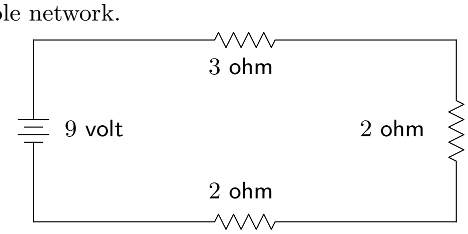
Networks of other kinds, not just electrical ones, can also be analyzed in this way. For instance, networks of streets are given in the exercises.

Exercises

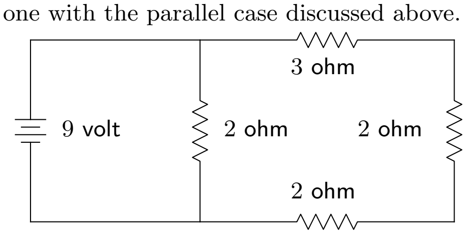
Many of the systems for these problems are mostly easily solved on a computer.

1 Calculate the amperages in each part of each network.

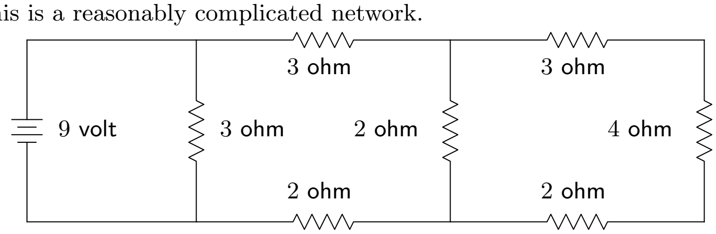
(a) This is a simple network.



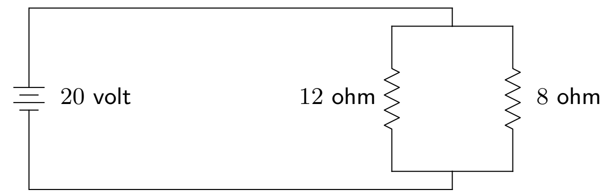
(b) Compare this one with the parallel case discussed above.



(c) This is a reasonably complicated network.



2 In the first network that we analyzed, with the three resistors in series, we just added to get that they acted together like a single resistor of 10 ohms. We can do a similar thing for parallel circuits. In the second circuit analyzed,



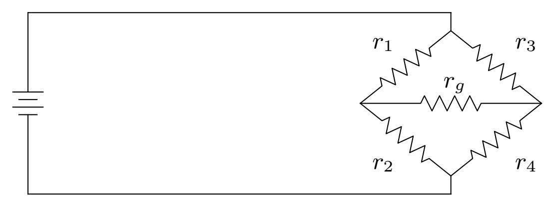
the electric current through the battery is $25/6$ amperes. Thus, the parallel portion is *equivalent* to a single resistor of $20/(25/6) = 4.8$ ohms.

(a) What is the equivalent resistance if we change the 12 ohm resistor to 5 ohms?
(b) What is the equivalent resistance if the two are each 8 ohms?
(c) Find the formula for the equivalent resistance if the two resistors in parallel are r_1 ohms and r_2 ohms.

3 For the car dashboard example that opens this Topic, solve for these amperages (assume that all resistances are 2 ohms).

(a) If the driver is stepping on the brakes, so the brake lights are on, and no other circuit is closed.
(b) If the hi-beam headlights and the brake lights are on.

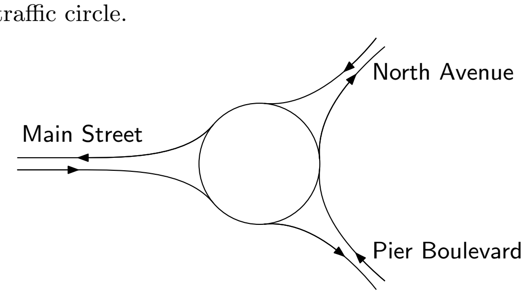
4 Show that, in this Wheatstone Bridge,



r_2/r_1 equals r_4/r_3 if and only if the current flowing through r_g is zero. (The way that this device is used in practice is that an unknown resistance at r_4 is compared to the other three r_1, r_2, and r_3. At r_g is placed a meter that shows the current. The three resistances r_1, r_2, and r_3 are varied — typically they each have a calibrated knob — until the current in the middle reads 0, and then the above equation gives the value of r_4.)

There are networks other than electrical ones, and we can ask how well Kirchoff's laws apply to them. The remaining questions consider an extension to networks of streets.

5 Consider this traffic circle.



This is the traffic volume, in units of cars per five minutes.

	North	*Pier*	*Main*
into	100	150	25
out of	75	150	50

We can set up equations to model how the traffic flows.

(a) Adapt Kirchoff's Current Law to this circumstance. Is it a reasonable modelling assumption?

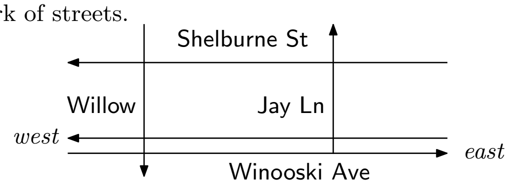
(b) Label the three between-road arcs in the circle with a variable. Using the (adapted) Current Law, for each of the three in-out intersections state an equation describing the traffic flow at that node.

(c) Solve that system.

(d) Interpret your solution.

(e) Restate the Voltage Law for this circumstance. How reasonable is it?

6 This is a network of streets.



The hourly flow of cars into this network's entrances, and out of its exits can be observed.

	east Winooski	*west Winooski*	*Willow*	*Jay*	*Shelburne*
into	80	50	65	–	40
out of	30	5	70	55	75

(Note that to reach Jay a car must enter the network via some other road first, which is why there is no 'into Jay' entry in the table. Note also that over a long period of time, the total in must approximately equal the total out, which is why both rows add to 235 cars.) Once inside the network, the traffic may flow in different ways, perhaps filling Willow and leaving Jay mostly empty, or perhaps flowing in some other way. Kirchhoff's Laws give the limits on that freedom.

(a) Determine the restrictions on the flow inside this network of streets by setting up a variable for each block, establishing the equations, and solving them. Notice that some streets are one-way only. (*Hint:* this will not yield a unique solution, since traffic can flow through this network in various ways; you should get at least one free variable.)

(b) Suppose that some construction is proposed for Winooski Avenue East between Willow and Jay, so traffic on that block will be reduced. What is the least amount of traffic flow that can be allowed on that block without disrupting the hourly flow into and out of the network?