Part II - Random Processes

Goals for this unit:

- Give overview of concepts from discrete probability
- Give analogous concepts from continuous probability
- See how the Monte Carlo method can be viewed as sampling technique
- See how Matlab can help us to simulate random processes
- See applications of the Monte Carlo method such as approximating an integral or finding extrema of a function.
- See how Random Walks can be used to simulate experiments that can't typically be done in any other way.

- Random processes are important because most real world problems exhibit random variations; the term stochastic processes is also used.
- The Monte Carlo Method is based on the principles of probability and statistics so we will first look at some basic ideas of probability.
- Similar to the situation where we looked at continuous and discrete problems (recall this was the main division in Algorithms I and II) we will consider concepts from both $discrete$ and $continuous$ probability.
- When we talk about discrete probability we consider experiments with a finite number of possible outcomes. For example, if we flip a coin or roll a die we have a fixed number of outcomes.
- When we talk about continuous probability we consider experiments where the random variable takes on all values in a range. For example, if we spin a spinner and see what point on the circle it lands, the random variable is the point and it takes on all values on the circle.

Historical Notes

Archaeologists have found evidence of games of chance on prehistoric digs, showing that gaming and gambling have been a major pastime for different peoples since the dawn of civilization. Given the Greek, Egyptian, Chinese, and Indian dynasties' other great mathematical discoveries (many of which predated the more often quoted European works) and the propensity of people to gamble, one would expect the mathematics of chance to have been one of the earliest developed. Surprisingly, it wasn't until the 17th century that a rigorous mathematics of probability was developed by French mathematicians Pierre de Fermat and Blaise Pascal. [1]

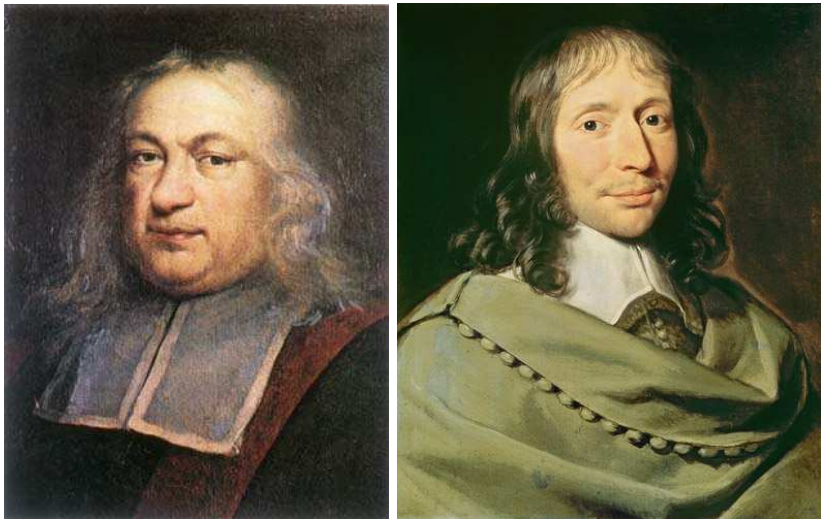
[1] Information taken from MathForum, http://mathforum.org/isaac/problems/prob1.html

The problem that inspired the development of mathematical probability in Renaissance Europe was the problem of points. It can be stated this way:

Two equally skilled players are interrupted while playing a game of chance for a certain amount of money. Given the score of the game at that point, how should the stakes be divided?

In this case "equally skilled" indicates that each player started the game with an equal chance of winning, for whatever reason. For the sake of illustration, imagine the following scenario.

Pascal and Fermat are sitting in a cafe in Paris and decide, after many arduous hours discussing more difficult scenarios, to play the simplest of all games, flipping a coin. If the coin comes up heads, Fermat gets a point. If it comes up tails, Pascal gets a point. The first to get 10 points wins. Knowing that they'll just end up taking each other out to dinner anyway, they each ante up a generous 50 Francs, making the total pot worth 100. They are, of course, playing 'winner takes all'. But then a strange thing happens. Fermat is winning, 8 points to 7, when he receives an urgent message that a friend is sick, and he must rush to his



home town of Toulouse. The carriage man who has delivered the message offers to take him, but only if they leave immediately. Of course Pascal understands, but later, in correspondence, the problem arises: how should the 100 Francs be divided?

How would you divide the 100 Francs? After we discuss probability you should be able to divide the money and argue that it is a fair division.

What do we mean by probability and events?

- Probability measures the chance of a given event occurring. It is a number between 0 and 1.
- The complement of an event is everything that is not in the event; for example the complement of 3 or less hurricanes hitting Florida in 2011 is 4 or more hurricanes hitting Florida.
- If the probability of an event is 1 then it is certain to occur.
- A probability of 0 represents an event that cannot occur, i.e., is impossible.
- Probabilities in between 0 and 1 are events that may occur and the bigger the number the more likely that the event will occur.
- "Rain tomorrow", "crop yield", "number of hurricanes to hit Florida's coast in 2011", " the number of times the sum of two dice thrown is even" are all examples of events for which we may be interested in determining their probabilities.

Notation & Terminology:

- Let X denote the random variable, for example the sum of two dice when they are thrown.
- The set of all possible outcomes is denoted Ω. For example, if we roll one die the possible outcomes are $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- We will denote an event as $\mathcal{E}$. So for example, if we roll a single die and want to determine the likelihood of the result being an even number, then $\mathcal{E} = \{2, 4, 6\}$; note that $\mathcal{E} \subset \Omega$.
- We are interested in determining the probability of an event and will denote it $p(\mathcal{E})$.

  The sum of all probabilities for all possible outcomes is 1.

  The probability of an event will be the sum of the probabilities of each outcome in the event.

  The probability of the complement of an event $\mathcal{E}$ will be $1 - p(\mathcal{E})$.

What is meant by the probability of an event?

Suppose the weather forecast is that the probability of rain tomorrow is $1/4$=0.25. We say the likelihood of rain is 25%. What does this mean?

The complement of this event is that there is NO rain tomorrow so the probability of no rain is

$$1 - \frac{1}{4} = 0.75$$

So there is a 75% chance of no rain tomorrow and thus it is 3 times more likely that there is no rain tomorrow than there is rain.

Unions and Intersections

Consider the following two outcomes:

- A= { 3 or less hurricanes to hit Florida in 2011}
- B= { 4 or 5 hurricanes to hit Florida in 2011}

Then $A \cup B$ is { 5 or less hurricanes to hit Florida in 2011}.

The probability of $A \cup B$ is $p(A \cup B) = p(A) + p(B)$ because the events are $mutually\ exclusive$.

Then $A \cap B$ is zero because both events can't be satisfied.

Now consider two events which are not mutually exclusive.

- $C = \{$ rain tomorrow$\}$
- $D = \{$ rain the day after tomorrow$\}$

Then $C \cup D = \{$ rain in the next two days$\}$ (i.e., rain tomorrow or rain the next day).

Now the intersection of C and D is everything that is in event C and in event D so

$$C \cap D = \{ \text{ rain tomorrow } and \text{ rain the next day} \}$$

.

What is the probability of $C \cup D$? We can't simply sum the probabilities. For example if there was a probability of 1/4 each day (i.e., 25% chance) then we can't say there is a 50% chance that it will rain in the next two days. In the case where the events are not mutually exclusive we have

$$p(C \cup D) = p(C) + p(D) - p(C \cap D)$$

Conditional Probability

- If we know that one event has occurred it may change our view of the probability of another event. For example, let

  $A = \{\text{rain today}\}, \quad B = \{\text{rain tomorrow}\}, \quad C = \{\text{rain in 90 days from now}\}$

  Assume that knowledge that A has occurred will change your view of the probability that B will occur, but not of the probability that C will occur.

- We write $p(B|A)$ to denote the conditional probability of B, given A.
- With our assumptions, $p(C|A) = p(C)$ but $p(B|A) \neq p(B)$.
- We say that A and C are independent, but A and B are not.
- For independent events A and C, the probability of $p(A \cap C) = p(A)p(C)$.

Example If we roll a single die (with sides 1, 2, 3, 4, 5, 6) what is the probability that you will roll the number 4? If we roll two dice, what is the probability that they will both be 4?

Clearly all numbers are equally probable and since the probabilities have to sum to 1 and there are 6 possible outcomes, $p(X = 4) = 1/6$. To determine the probability that a 4 occurs on both dice, we note that they are independent events so the probability is $\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$.

Example Let's return to the Fermat/Pascal problem of flipping a coin. We know that heads and tails are each equally likely to occur. If heads occurs then Fermat gets a point, otherwise Pascal gets one. When the game is interrupted the score is Fermat 8 and Pascal 7. There is 100 francs at stake so how do you fairly divide the winnings?

We first ask ourselves what is the maximum number of flips of the coin that guarantee someone will win? Clearly Fermat needs 2 points and Pascal 3. The worse case scenario would be for Fermat to get 1 and Pascal to get 3 so in 4 more flips we are guaranteed a winner.

If we determine the possible outcomes in these 4 flips then we can assign a winner to each. Then Fermat's probability of winning will be the number of outcomes he would win over the total number of outcomes; similarly for Pascal. Then we will know how to divide the money.

Here are the 2^4 possible outcomes for 4 flips and we have indicated the winner in each case remembering that Fermat needs 2 heads to win and Pascal 3 tails.

HHHH (F)	HHHT (F)	HHTH (F)	HHTT (F)
HTHH (F)	HTTH (F)	HTHT (F)	HTTT (P)
TTTT (P)	TTHT (P)	TTTH (P)	TTHH (F)
THTT (P)	THHH (F)	THTH (F)	THHT (F)

So Fermat will win 11 out of 16 times so his probability is 11/16. Pascal will win 5 out of 16 times so his probability is 5/16. Fermat should get 11/16 of the money and Pascal the rest.

Simulating random processes using Matlab's `rand` command

- Random events are easily simulated in Matlab with the function `rand`, which you have already encountered. A computer cannot generate a truly random number, rather a pseudorandom number, but for most purposes it suffices because it is practically unpredictable.

  We have seen that if you use the command `x=rand()` you get a number between 0 and 1. It can also generate row or column vectors. For example, `rand(1,5)` returns a row vector of five random numbers (1 row, 5 columns) such as

  $$0.2416 \quad 0.9644 \quad 0.4519 \quad 0.3278 \quad 0.8913$$

```
a = rand ( )          <-- a scalar value;
b = rand ( 5, 1 )   <-- a column vector;
c = rand ( 1, 5 )   <-- a row vector;
d = rand ( 3, 4 )   <-- a matrix;
e = rand ( 5 )       <-- a 5x5 matrix;
```

- If we want to generate n integers between say 0 and k then we first generate the numbers between 0 and 1 using `x=rand(1,n)` and then use either of

the following commands:

```
x=floor ( (k+1)*x ) )
x=ceil ( k*x )
```

For example, if we generate 5 random numbers

$$.8147, \quad .9058 \quad .1770 \quad .9134 \quad .6324$$

and we want integers ≤ 10 then `x=floor (11*x ) )` gives $\{8, 9, 1, 10, 6\}$ and `x=ceil ( 10*x )` gives $\{9, 10, 2, 10, 7\}$.

- The only thing that may be disconcerting is that if we run our program which is based on generating random numbers two times in a row, we will get different numbers! It is sometimes useful (such as in debugging) to be able to reproduce your results exactly. If you continually generate random numbers during the same Matlab session, you will get a different sequence each time, as you would expect.
- However, each time you start a session, the random number sequence begins at the same place (0.9501) and continues in the same way. This is not true to life, as every gambler knows.

- Clearly we don't want to terminate our session every time we want to reproduce results so we need a way to "fix" this. To do this, we provide a $seed$ for our random number generator. Just like Matlab has a seed at the beginning of a session we will give it a seed to start with at the beginning of our code.

  Seeding `rand`

- The random number generator rand can be seeded with the statement

$$\texttt{rand}(\texttt{state}, \texttt{n})$$

  where n is any integer. By default, n is set to 0 when a Matlab session starts.

- Note that this statement does not generate any random numbers; it only initializes the generator.

- Sometimes people will use the system clock to make sure the seed is being reset

$$\texttt{rand}(\texttt{state}, \texttt{sum}(100 * \texttt{clock}))$$

- Try setting the seed for `rand`, generating a random number, then resetting the seed and generate another random number; see that you get the same

random number two times in a row; e.g.,

$$\texttt{rand}('\texttt{state}', 12345) \qquad \texttt{rand}(1)$$

Exercise Use two calls to Matlab's `rand` command to generate two random row vectors of length 4. Now repeat the exercise where you reset the seed before each call to `rand`. You should get the same vector each time in this latter case.

Probability Density Function

- If we measure a random variable many times then we can build up a distribution of the values it can take.
- As we take more and more measurements and plot them, we will see an underlying structure or distribution of the values.
- This underlying structure is called the $probability\ density\ function$ (PDF).
- For a discrete probability distribution the PDF associates a probability with each value of a discrete random variable.

  For example, let the random variable X be the $number\ of\ rainy\ days\ in\ a\ 10\ day\ period$. The number of outcomes is 11 because we can have 0 days, 1 day, . . ., 10 days. The discrete PDF would be a plot of each probability.
- For a continuous probability distribution the random variables can take all values in a given range so we can not assign probabilities to individual values. Instead we have a continuous curve, called our continuous PDF, which allows

us to calculate the probability of a value within any interval.

- How will we determine the probability for a range of values from the continuous PDF? We will see that the probability is calculated as the area under the curve between the values of interest.
- We first look at discrete PDFs.

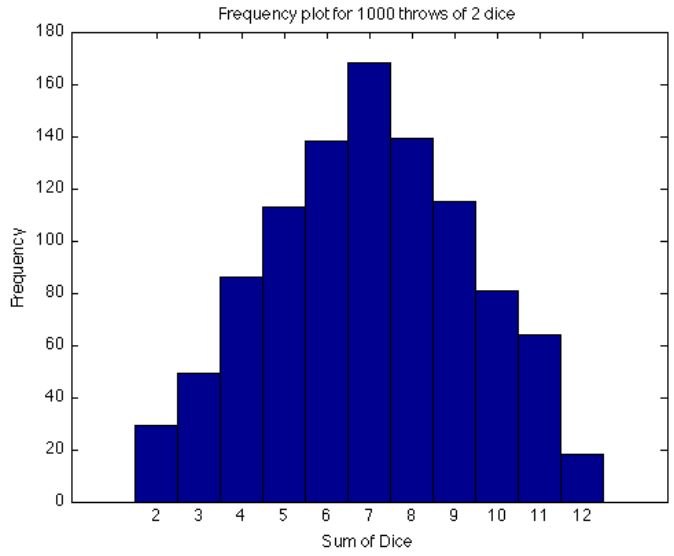
Estimating the Probabilities & the Discrete PDF

- If we have an experiment (like rolling dice or flipping a coin) we may be able to compute the $exact$ probabilities for each outcome (as we did in the Fermat-Pascal problem) and then plot the PDF.
- But what if we don't have the exact probabilities? Can we estimate them and then use the estimates to plot the PDF?

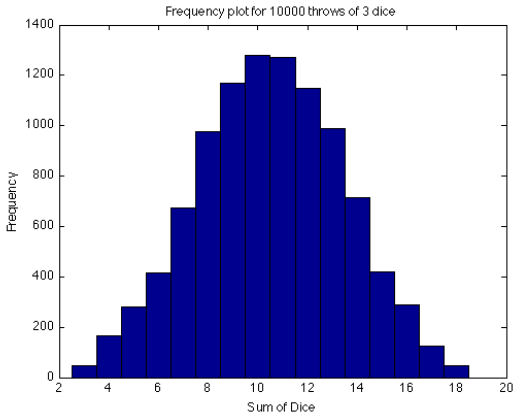
  The answer is yes. We essentially take many measurements of the random variable and accumulate the frequency that each outcome occurred. The relative frequency of each outcome gives us an estimate of its probability.

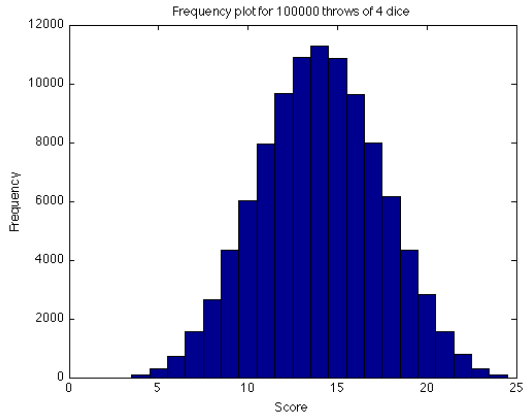
  For example, if a particular outcomes occurs 50% of the time, then its probability is 0.5.

As a concrete example, suppose we roll two dice and sum the numbers that appear. The individual outcomes on each die are still equally likely but now we are going to look at the sum of the two dice which is not equally likely. The possible outcomes are 2 through 12. Suppose we roll the dice 1000 times and count the frequency that each sum appears. When we look at the frequency plot for the sums we notice that it shows a nonuniformity. This is because there is only one way to score 2, but many ways to score 7.



When we repeat the calculations using 3 and 4 dice and draw a frequency plot for the sums, we begin to notice a trend in the shape of the plot and suspect that there is an underlying function $f(x)$ which is determining its shape.





- In fact, this plot suggests the normal curve, or "bell shaped distribution", even though we expect to see that curve only for continuous variables.

How can we simulate rolling the dice and generating these plots using Matlab?

Here is a version where we don't generate all the random numbers at once.

```
n_dice = 2;
freq(1:12)=0;

for k = 1:n

   x=rand(1,n_dice);   % generate 2 random numbers between 0 and 1
   x=ceil( 6*x );        % turn them into an integer 1 to 6
   value = sum(x);    % sum the two dice
   freq(value  ) =  freq(value  )  + 1;  % increment the frequency

end
```

We can also generate all the random numbers for all the throws for the two dice using

$$\texttt{throws} = \texttt{rand}(\texttt{n}, \texttt{n\_dice});$$

$$\texttt{throws} = \texttt{ceil}(6 * \texttt{throws});$$

How can we use the frequency plot to estimate the probabilities?

- A frequency plot is simply a record of what happened.
- However, if we keep records long enough, we may see that the frequency plot can be used to make statements about what is likely to happen in the future. For two dice, we can say that a score of 7 is very likely and can even guess how much more likely it is than say 3.
- This suggests that there is an underlying probability that influences which outcomes occur. In cases where we are simply collecting data, we can turn our frequencies into estimates of probability by normalizing by the total number of cases we observed:

$$\text{estimated probability of result \#i} = \frac{\text{frequency of result \#i}}{\text{total number of results}}$$

Now let's compare the $approximate$ probabilities found using this formula for the case where we used n throws of 2 dice and then compare this with their $exact$ probabilities.

To calculate the actual probabilities of each sum we determine the possible outcomes of rolling the dice. Because there are more outcomes in this case, it's better to make an outcome table where we list all possible outcomes of the roll of 2 (fair) dice.

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- We see that there are a total of 36 combinations of the dice.
- If we want to compute the probability of an outcome, how would we do it?

As an example, consider the probability of the outcome (3,5). Remember that these events are independent and because the probability of rolling a 3 with the first die is 1 out of 6 and the probability of rolling a 5 with the second die is also 1 out of 6 so the probability of the outcome (3,5) is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

- However, we want to determine the probability that a particular sum will occur. The number of outcomes resulting in a sum of 2 to 12 are:

$$2 \rightarrow 1 \qquad 3 \rightarrow 2 \qquad 4 \rightarrow 3 \qquad 5 \rightarrow 4 \qquad 6 \rightarrow 5$$

$$7 \rightarrow 6 \qquad 8 \rightarrow 5 \qquad 9 \rightarrow 4 \qquad 10 \rightarrow 3 \qquad 11 \rightarrow 2 \qquad 12 \rightarrow 1$$

- How do we compute the probability that a particular sum will occur? Consider the probability of the sum 5 occurring. There are a total of 4 ways out of 36 outcomes that yield the sum of 5 so its exact probability is 4/36=1/9.
- For a pair of fair dice, the exact probability of each total is:

2	3	4	5	6	7	8	9	10	11	12
$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
.03	.06	.08	.11	.14	.17	.14	.11	.08	.06	.03

Note that the probabilities are all positive and the sum of the probabilities is 1, as expected.

Now we compare these to our estimated probabilities. Clearly they don't agree but for large n they are a "good" approximation. Remember that are using a frequency interpretation of probability.

sum $\rightarrow$	2	3	4	5	6	7	8	9	10	11	12
exact	.03	.06	.08	.11	.14	.17	.14	.11	.08	.06	.03
$n = 100$	.02	.05	.07	.09	.15	.12	.15	.11	.15	.07	.02
$n = 1000$	.034	.048	.082	.123	.136	.171	.127	.115	.091	.049	.024
$n = 10000$	.026	.0555	.0867	.1181	.1358	.1711	.1347	.1088	.0822	.0553	.0258

- Our plot of the exact probabilities is an example of a discrete PDF and the corresponding plot of our estimated probabilities is an approximation to the discrete PDF. The PDF assigns a probability $p(x)$ to each outcome X in our set Ω of all possible outcomes. In our example there were 11 outcomes and

$$\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} .$$

- Recall that the discrete PDF describes the relative likelihood for a random variable to occur at a given point.
- What properties should a discrete PDF have? Clearly two obvious requirements for a discrete PDF are
  1. $0 \leq p(x_i)$ for each x_i
  2. $\sum_{i=1}^{n} p(x_i) = 1$ where n is the number of possible outcomes.

Exercise. Suppose you have a die which is not fair in the sense that each side has the following probabilities

$$1 \to \frac{1}{9} \quad 2 \to \frac{2}{9} \quad 3 \to \frac{2}{9} \quad 4 \to \frac{2}{9} \quad 5 \to \frac{1}{9} \quad 6 \to \frac{1}{9}$$

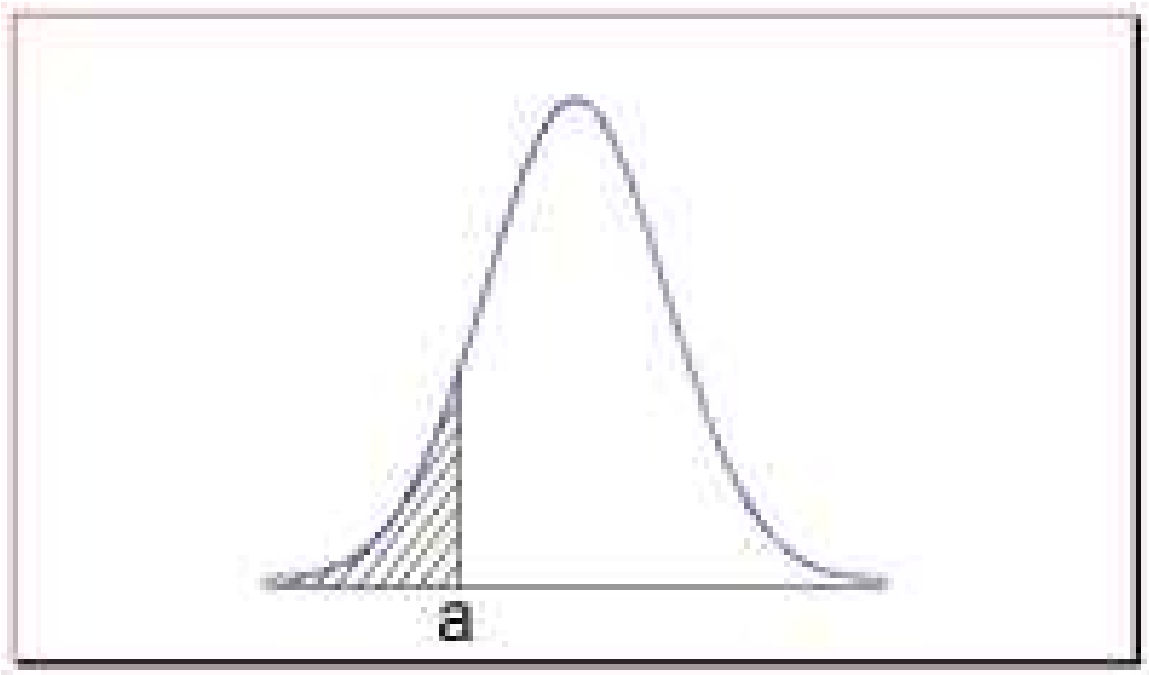
Would you bet on rolling an even or an odd number? Why?

Write a code to simulate rolling one of these loaded dice and estimate the PDF. Graph it.

Continuous Probability Density Functions

- We have seen that the discrete PDF provides us with probabilities for each value in the finite set of outcomes Ω. The continuous PDF provides us with probabilities for a $range$ of values in the set of outcomes.
- Recall that in the discrete case, the sum of all probabilities had to be 1. The analog of that for the continuous case is that the $total\ area$ under the curve is 1.
- If we determine the area under the curve between two points, then it is just the probability that the outcome will be between those values.
- For example, in the plot below the probability that the random variable is $\leq a$ is the integral under the curve from $-\infty$ to a.

$$\int_{-\infty}^{a} f(x)\,dx$$ is the probability that the random variable is $\leq a$.



Mean, Variance and Expected Values

- When we have a large collection of numbers (such as scores from a large lecture exam) we are not usually interested in the individual numbers but rather in certain descriptive quantities such as average and how far from the average a certain percentage of students are. The same is true for a

probability density function.

- Three important characteristics of a PDF are its mean or expected value or average, its variance, and the standard deviation.
- For a discrete PDF, assume that we have a finite set Ω of all possible n outcomes and the ith outcome for the random variable X is denoted by x_i.

  We can calculate the expected value (i.e., the mean) of a random variable X, denoted either μ or $E[X]$, from the formula

$$\mu = \sum_{i=1}^{n} p(x_i) x_i$$

- For a continuous PDF, where the random variables range between a and b we can calculate the mean from the formula

$$\mu = \int_a^b p(x) x \, dx$$

- If you recall the definition of the center of mass of a laminar sheet

$$\frac{\int_D x \rho(x) \, dA}{\int_D \rho \, dA}$$

where ρ is the density you can see an analog of this formula because in our case $\int_a^b p\, dx = 1$.

- To understand the discrete case, let's look at an example.

  Consider the situation where we have the heights in inches of the 12 members of a women's basketball team given by

  $$\{69, 69, 66, 68, 71, 65, 67, 66, 66, 67, 70, 72\}$$

  We know that if we want the average height of the players we simply compute

  $$\frac{69 + 69 + 66 + 68 + 71 + 65 + 67 + 66 + 66 + 67 + 70 + 72}{12} = 67.9$$

  We can also interpret 67.9 as the mean or expected value of a random variable. To see this, consider choosing one player at random and let the random variable X be the player's height. Then the expected value of X is 67.9 which can also be calculated by forming the product of each x_i and its corresponding probability

  $$69\Big(\frac{1}{12}\Big) + 69\Big(\frac{1}{12}\Big) + 66\Big(\frac{1}{12}\Big) + \cdots + 72\Big(\frac{1}{12}\Big)$$

  which is just the formula we gave for the mean/expected value.

Example Suppose we are going to toss a (fair) coin 3 times. Let X, the random variable, be the number of $heads$ which appear. Find the mean or expected value and explain what it means.

We know that there are 2^3 possible outcomes which are

HHH, HHT, HTH, HTT, TTT, TTH, THT, THH

and each has a probability of $1/8$. Now heads can occur no times, 1 time, 2 times or 3 times. The probability that it occurs 0 times is $1/8$, that it occurs 1 time is $3/8$, that it occurs 2 times is $3/8$ and it occurs 3 times is $1/8$. Therefore the mean or expected value is

$$0\frac{1}{8} + 1\frac{3}{8} + 2\frac{3}{8} + 3\frac{1}{8} = \frac{12}{8} = \frac{3}{2}.$$

This says that if we toss the coin three times the number of times we expect heads to appear is $\frac{3}{2}$ which is one-half the total possible times; exactly what we expected! Another way to get this answer is to consider the set $\{3, 2, 2, 1, 0, 1, 1, 2\}$ indicating the number of heads appearing in 3 tosses and average them to get $(3 + 2 + 2 + 1 + 0 + 1 + 1 + 2)/8 = 12/8 = 3/2$.

Statisticians use the variance and standard deviation of a continuous random variable X as a way of measuring its dispersion, or the degree to which it is "scattered."

The formula for the variance $\mathrm{Var}(X)$ of a discrete PDF is

$$\mathrm{Var}(X) = \sum_{i=1}^{n} p(x_i) * (x_i - \mu))^2$$

and for a continuous PDF

$$\mathrm{Var} = \int_a^b p(x)(x-\mu)^2 \, dx$$

where μ is the expected value or mean.

The standard deviation, $\sigma(X)$ is just $\sqrt{\mathrm{Var}(X)}$.

Note that $\mathrm{Var}(X)$ is the mean or expected value of the function $(X-\mu)^2$, which measures the square of the distance of X from its mean. It is for this reason that $\mathrm{Var}(X)$ is sometimes called the *mean square deviation*, and $\sigma(X)$ is called the *root mean square deviation*. $\mathrm{Var}(X)$ will be larger if X tends to wander far

away from its mean, and smaller if the values of X tend to cluster near its mean.

The reason we take the square root in the definition of $\sigma(X)$ is that $\mathrm{Var}(X)$ is the mean/expected value of the square of the deviation from the mean, and thus is measured in square units. Its square root therefore gives us a measure in ordinary units.

Example Compute the mean (expected value), variance and standard deviation for our example where we throw 2 dice and sum the values. Recall that here the random variable X is the sum of the values.

Returning to our probability table

2	3	4	5	6	7	8	9	10	11	12
$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

we can compute the mean or expected value as

$$\mu = 2\frac{1}{36} + 3\frac{2}{36} + 4\frac{3}{36} + \cdots + 11\frac{2}{36} + 12\frac{1}{36} = 7$$

and the variance as

$$\mathrm{Var}(X) = \frac{1}{36} * (2-7)^2 + \frac{2}{36} * (3-7)^2 + \frac{3}{36} * (4-7)^2 + \ldots + \frac{1}{36} * (12-7)^2 = 5.8333$$

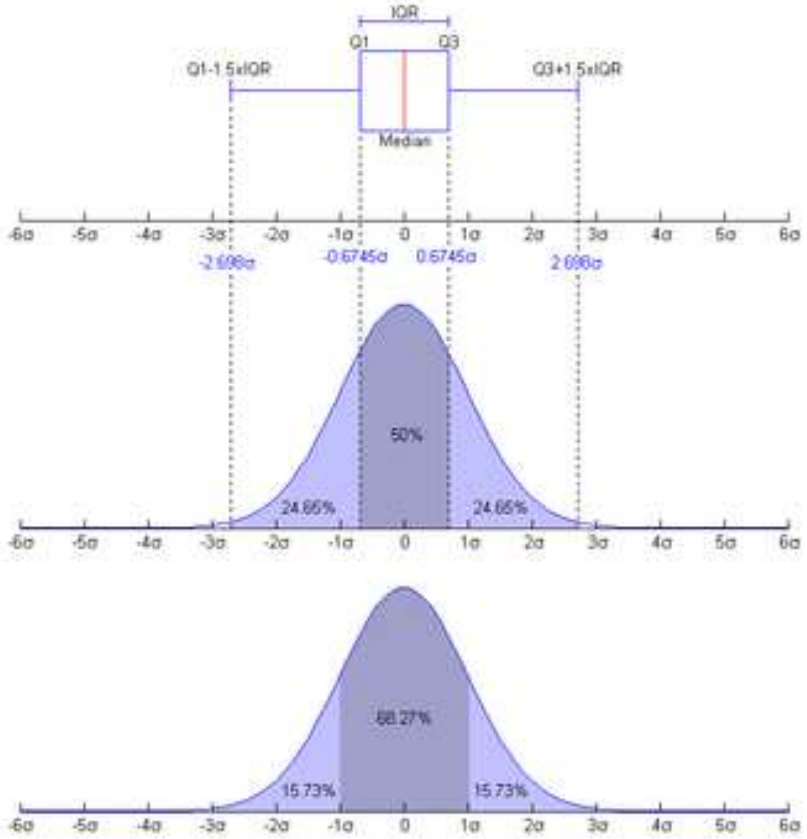
with a standard deviation of $\sigma(X) = \sqrt{5.833} = 2.415$. The mean is what we expected from looking at the PDF. If, for some reason, 7 was impossible to roll, it would still be the expected value.

Normal distribution

In probability theory, the normal (or Gaussian) distribution, is a continuous PDF that is often used as a first approximation to describe real-valued random variables that tend to cluster around a single mean value. The graph of the associated PDF is the well-known bell-shaped curve known as the Gaussian function or bell curve given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where parameter μ is the mean (location of the peak) and σ^2 is the variance (the measure of the width of the distribution). The distribution with $\mu = 0$ and $\sigma^2 = 1$ is called the standard normal distribution.



The normal distribution is considered the most basic continuous PDF and it is used throughout statistics, natural sciences, and social sciences as a simple model for complex phenomena.

randn

Matlab has a built in command `randn` which returns pseudorandom values drawn from the standard normal distribution. We can use it to generate values from a normal distribution with a particular mean and standard deviation; for example, if we want to generate 100 points with mean 1 and standard deviation 2 we use

$$\texttt{r} = 1 + 2. * \texttt{randn}(100, 1);$$

Example Generate 1000 random numbers using `rand` and `randn` and make a histogram of the results. Compute the mean and standard deviation of each using the Matlab commands `mean` and `std`. The command to make a histogram of y values is `hist(y)`.

The Cumulative Density Function

If we have a discrete system with a known PDF, the value of the PDF at (x_i), say $\mathrm{pdf}(x_i)$, is the probability that the outcome x_i will occur.

But suppose we want to know the chances of rolling a 7 or less, using two dice. This is the probability that the outcome is less than or equal to 7; it is so important that it has its own name, the *cumulative density function* or CDF.

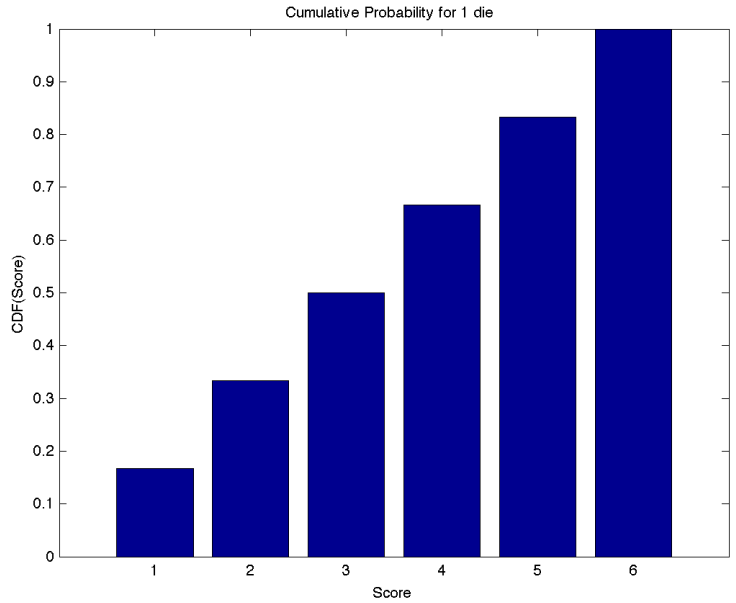
$$\mathrm{cdf}(x) = \text{probability outcome is less than or equal to } x$$

$$= \sum_{y \leq x} \mathrm{pdf}(y)$$

Example If we return to the example when we rolled one die we know that each number was equally probable so each had probability of $1/6$. What is the CDF for each $x_i \in \{1, 2, 3, 4, 5, 6\}$?

If we ask what is $\mathrm{cdf}(6)$ then it should be 1 because we know it is 100% sure that we will roll a number ≤ 6. For the other values we simply sum up the PDFs.

$$\mathrm{cdf}(1) = \frac{1}{6} \qquad \mathrm{cdf}(2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}, \qquad \text{etc.}$$



We see that for our discrete PDF

- cdf(x) is a piecewise constant function defined for all x;
- cdf(x) $= 0$ for x less than the smallest possible outcome;
- cdf(x) $= 1$ for x greater than or equal to the largest outcome;
- cdf(x) is essentially the discrete integral of $\mathrm{pdf}(X)$ over the appropriate interval ;
- cdf(x) is monotonic (and hence invertible);
- the probability that x is between x_1 and x_2 ($x_1 < x \leq x_2$) is

$$\mathrm{cdf}(x_2) - \mathrm{cdf}(x_1)$$

.

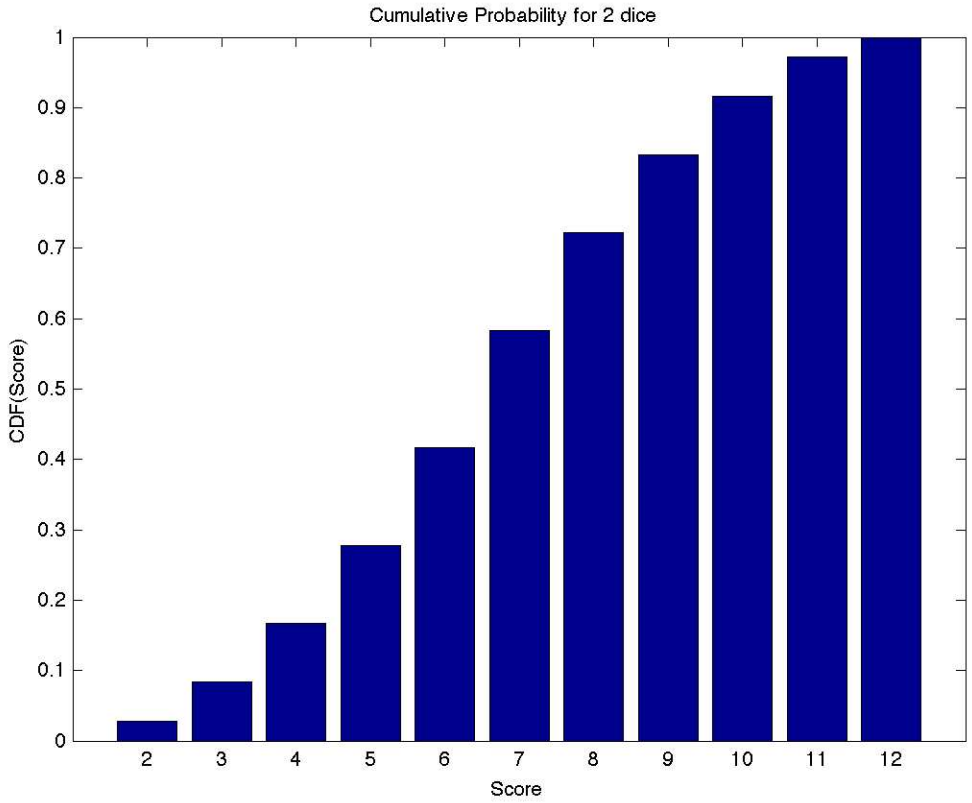
Return again to the problem of rolling two dice and calculating the probability that the sum of the dice is a number between 2 and 12. We found the exact probability of each sum as:

2	3	4	5	6	7	8	9	10	11	12
$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
.03	.06	.08	.11	.14	.17	.14	.11	.08	.06	.03

For a discrete case like this, it's easy to make the corresponding cumulative density function table:

2	3	4	5	6	7	8	9	10	11	12
$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	$\frac{36}{36}$
.03	.08	.16	.28	.42	.58	.72	.83	.92	.97	1.00

The CDF is actually defined for all x, not just the values 2 through 12.
For example, the table tells us that $\mathbf{cdf}(4.3) = 0.16$ and $\mathbf{cdf}(15) = 1$ The latter just tells us that the probability of rolling two dice and getting a sum ≤ 15 is 1.



Here is the plot of the CDF for 2 dice.

Matlab has a built in command `cumsum` which can assist us in calculating the CDF from a discrete PDF. For example, to find the cumulative sum of integers from 1 to 5 we use `cumsum(1:5)`

and the response is `[ 1  3  6  10  15 ]`.

Here is a script to draw the last bar graph for the CDF.

```
x = 2 :  1 2

pdf = [ 1 , 2 , 3 , 4 , 5 , 6 , 5 , 4 , 3 , 2 , 1 ] / 3 6

cdf= cumsum ( pdf )

bar ( x , cdf )

xlabel ( ' Score ' )

ylabel ( 'CDF( Score ) ' )

title ( 'CDF for 2 dice ' )
```

Using the CDF for Simulation

- The CDF can be used to simulate the behavior of a discrete system.
- As an example, consider the problem of simulating rolling two dice and using the CDF to estimate the probabilities where we are given the CDF as

$$\text{cdf} = \{.03, .08, .16, .28, .42, .58, .72, .83, .92, .97, 1.00\}$$

- First recall that the probability that x is between x_1 and x_2 is $\text{cdf}(x_2) - \text{cdf}(x_1)$.
- For our problem we generate a random number r between 0 and 1 for a probability. We determine i such that

$$\text{cdf}(i-1) < r \leq \text{cdf}(i)$$

However, when $i = 1$ this doesn't work so we append our CDF to include 0

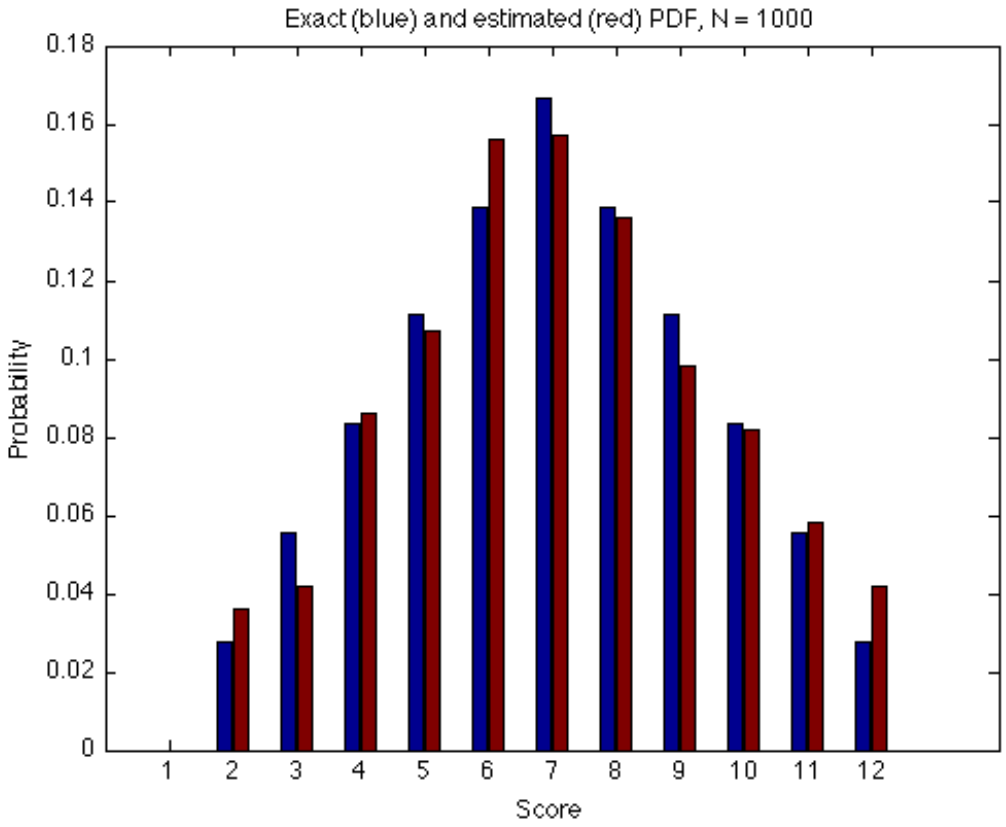
$$\text{cdf} = \{0, .03, .08, .16, .28, .42, .58, .72, .83, .92, .97, 1.00\}$$

For example, if $r = .61$ then $i = 8$.

- We do this lots of times and keep track of how many random numbers were assigned to $i = 1$, $i = 2$, etc.
- Then our estimate for the probability of the sum being say 2 (i.e., corresponding to $i = 2$) is the number of times we have chosen a random number between 0 and 0.03 divided by the total number of random numbers generated.

```
function prob = cd_sim ( n )

cdf = [0 .03 .08 .16 .28 .42 .58 .72 .83 .92 .97 1.00]

count(1:12)=0;

for k = 1:n

    r = rand(1);

    for i = 2:12

      if cdf(i-1) < r & r <= cdf(i)

        count (i) = count(i)+1;

      end

    end

end

freq = count(2:12)/n;
```

```
x=2:12;

bar(x,freq)

end
```



Exact - blue Approximate - red

Exercise. Let's return again to the problem where we have a die which is not fair in the sense that each side has the following probabilities

$$1 \to \frac{1}{9} \quad 2 \to \frac{2}{9} \quad 3 \to \frac{2}{9} \quad 4 \to \frac{2}{9} \quad 5 \to \frac{1}{9} \quad 6 \to \frac{1}{9}$$

Compute the exact CDF. What is the probability that you will roll a 3 or less?

Write a program to use your CDF to approximate the exact probabilities. Compare.

The Monte Carlo Method

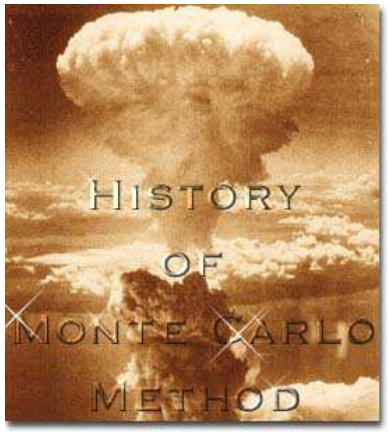
The Monte Carlo Method (MCM) uses random numbers to solve problems. Here we will view MCM as a sampling method.

First we will see how we can sample different geometric shapes and then apply the sampling to different problems.

The MC method is named after the city in the Monaco principality, because of

a roulette, a simple random number generator. The name and the systematic development of Monte Carlo methods dates from about 1944.

The real use of Monte Carlo methods as a research tool stems from work on the atomic bomb during the second world war. This work involved a direct simulation of the probabilistic problems concerned with random neutron diffusion in fissile material; but even at an early stage of these investigations, von Neumann and Ulam refined this particular " Russian roulette" and "splitting" methods. However, the systematic development of these ideas had to await the work of Harris and Herman Kahn in 1948. Other early researchers into the field were Fermi, Metropolis, and Ulam.



- We can think of the output of `rand()`, or any stream of uniform random numbers, as a method of sampling the unit interval.
- We expect that if we plot the points along a line, they would be roughly evenly distributed
- Random numbers are actually a very good way of exploring or sampling many geometrical regions. Since we have a good source of random numbers for the unit interval, it's worth while to think about how we could modify such numbers to sample these other regions.

Sampling the interval $[a, b]$

Suppose we need uniform random numbers, but we want them in the interval $[a, b]$

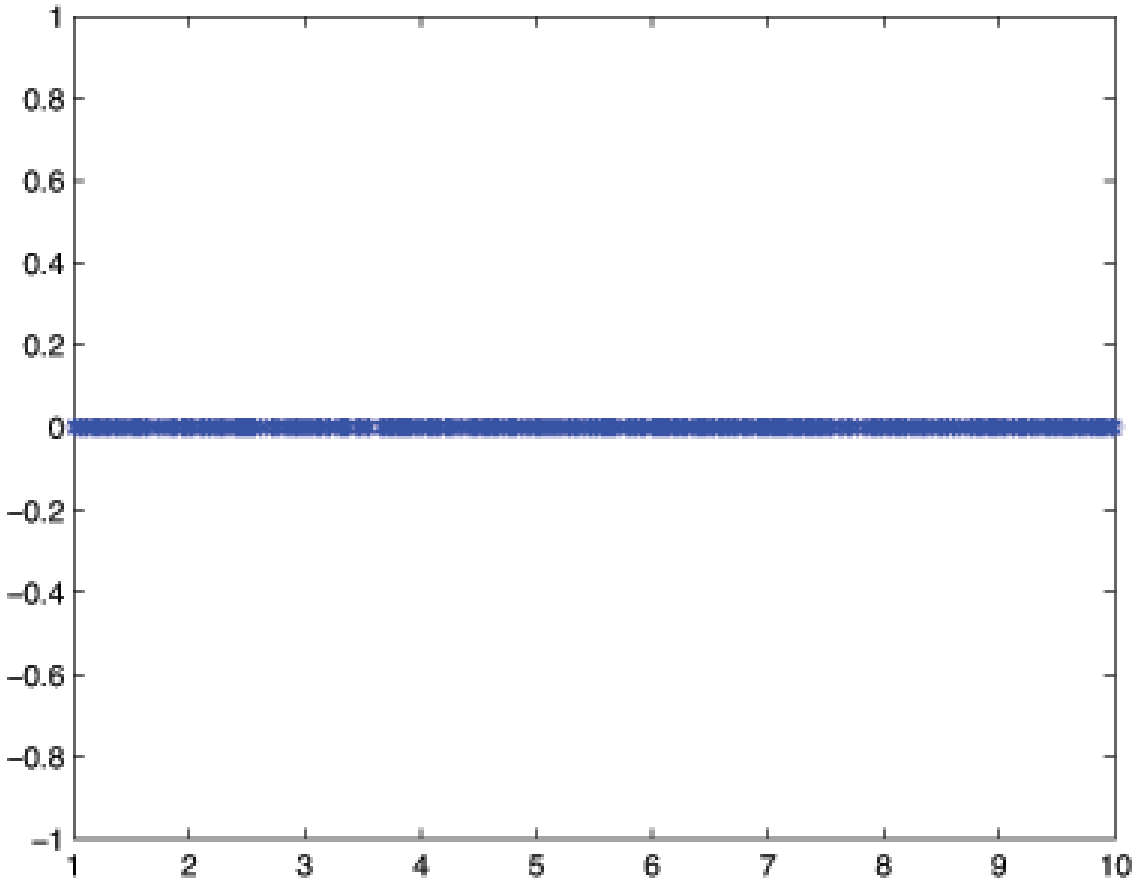
We have seen that we can use `rand` but now we have to shift and scale. The numbers have to be shifted by a and scaled by $b - a$. Why?

$$\begin{aligned} r =& \text{rand}(); \\ s =& a + (b - a) * r \end{aligned}$$

or we can do this for hundreds of values at once:

$$\begin{aligned} r =& \text{rand}(10000, 1); \\ s =& a + (b - a) * r; \end{aligned}$$

1000 uniformly sampled points on $[1, 10]$



Sampling the Unit Square

We might need uniform random numbers in the unit square.

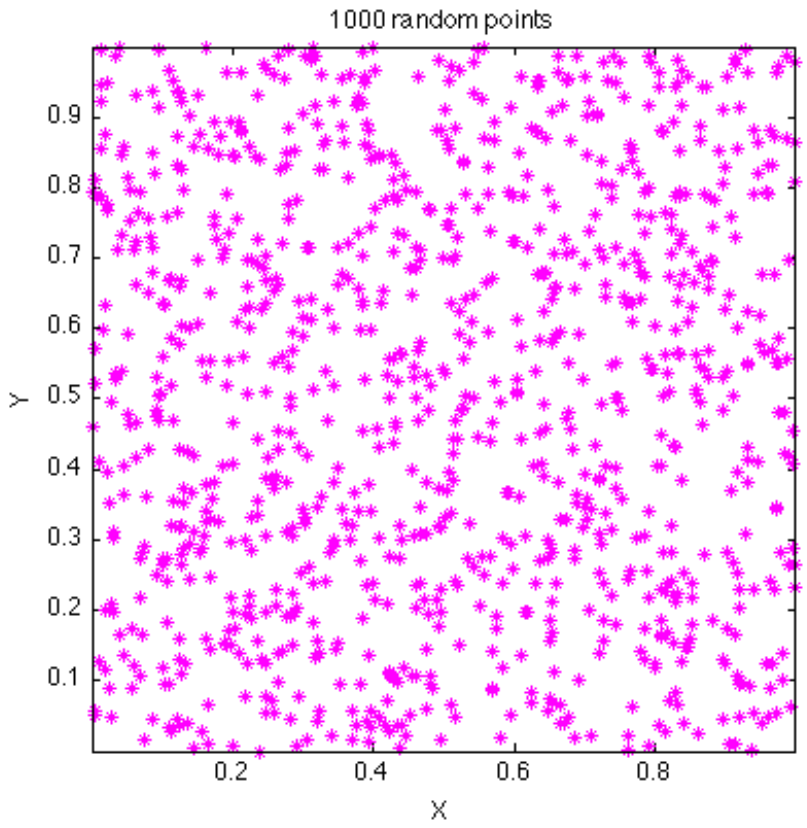
We could compute an x and y value separately for each point

$$x = \text{rand}();$$
$$y = \text{rand}();$$

or we can do this for hundreds of values at once to create a $2 \times n$ array:

$$xy = \text{rand}(2, 10000);$$

Of course if our domain is not the unit square, then we must map the points.



Sampling the Unit Circle

Suppose we need to evenly sample points in the unit circle; i.e., the circle centered at (0,0) with radius 1.

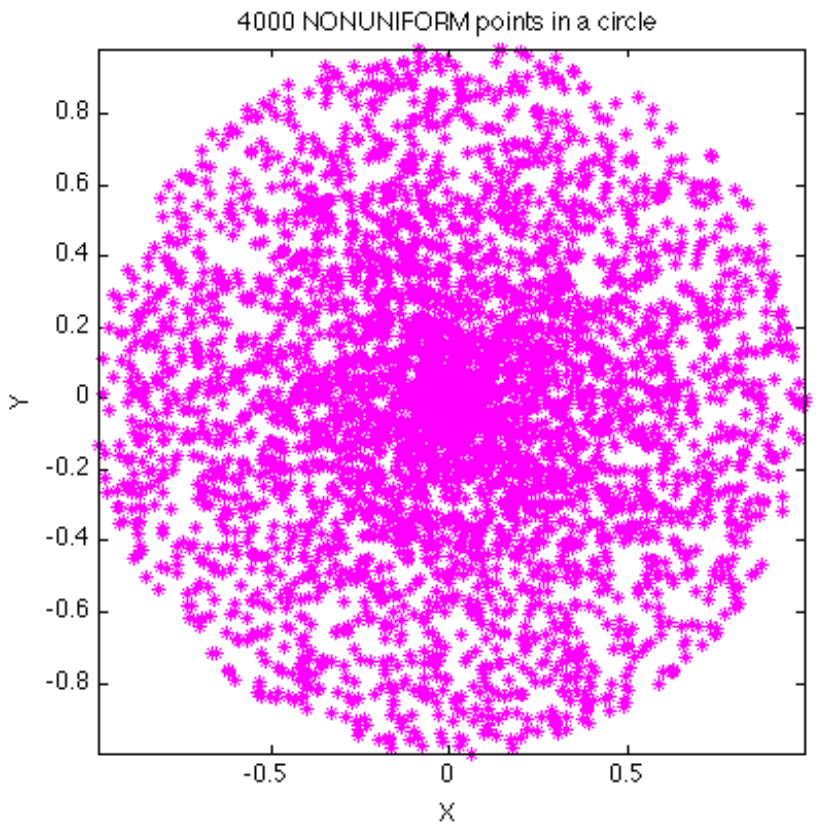
The first thing that might come to mind is to try to choose a radius from 0 to 1, and then choose an angle between 0 and 2π

$$r = \text{rand}();$$
$$t = 2 * \pi * \text{rand}();$$

but although this seems "random" it is not a **uniform** way to sample the circle as the following figure demonstrates. More points in a circle have a big radius than a small one, so choosing the radius uniformly actually is the wrong thing to do!

How can we fix this problem?

The problem is that when we choose the radius to vary uniformly, we're saying there are the same number of points at every value of the radius we choose between 0 and 1. But of course, there aren't. If we double the radius of a circle, the area of the circle increases by a factor of 4. So to measure area uniformly, we



need r^2 to be uniformly random, in other words, we want to set r to the $square\ root$ of a uniform random number.

$$r = \sqrt{\text{rand}()};$$
$$t = 2 * \pi * \text{rand}();$$

Another approach to sampling the circle is to use rejection.

Suppose we sample the square that contains the circle, and then only plot the points that are in the circle?

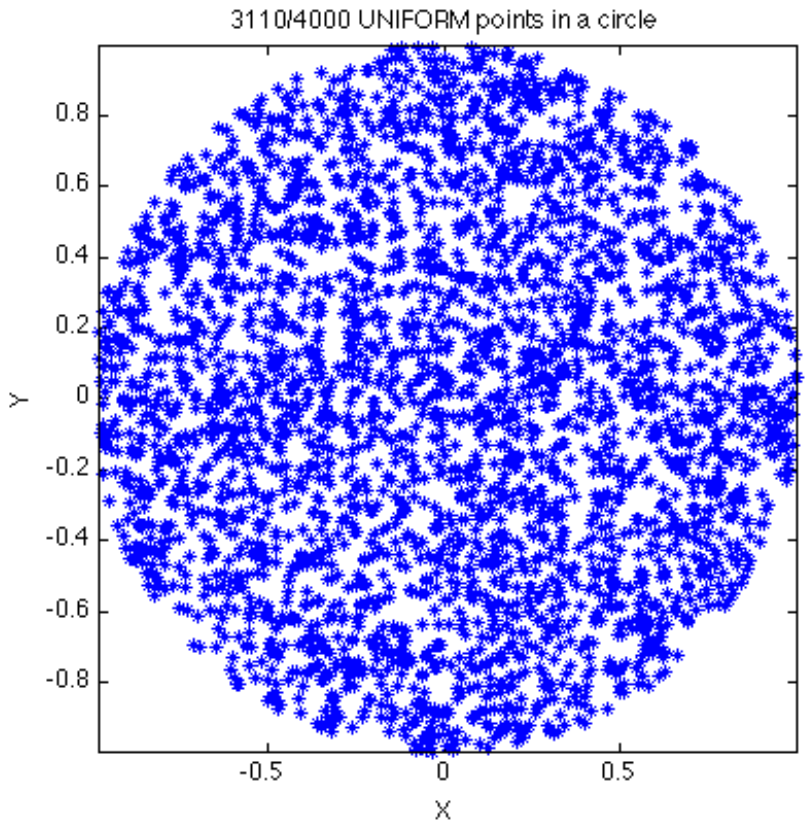
Now we're sure to get uniformly distributed points!

We'll also have to *reject* many points that we generate, which of course is not optimal.

We imbed the unit circle in a square whose vertices are (-1,1), (1,-1), (-1,1), (1,1).

```
x = -1 + 2 * rand ( );
y = -1 + 2 * rand ( );
i = find ( x.^2 + y.^2 < 1 )
plot ( x(i),  y(i), 'b*' )
```

In this example, there were 3110 points in the circle out of 4000 generated. Is there any significance in this ratio?



Sampling using rejection.

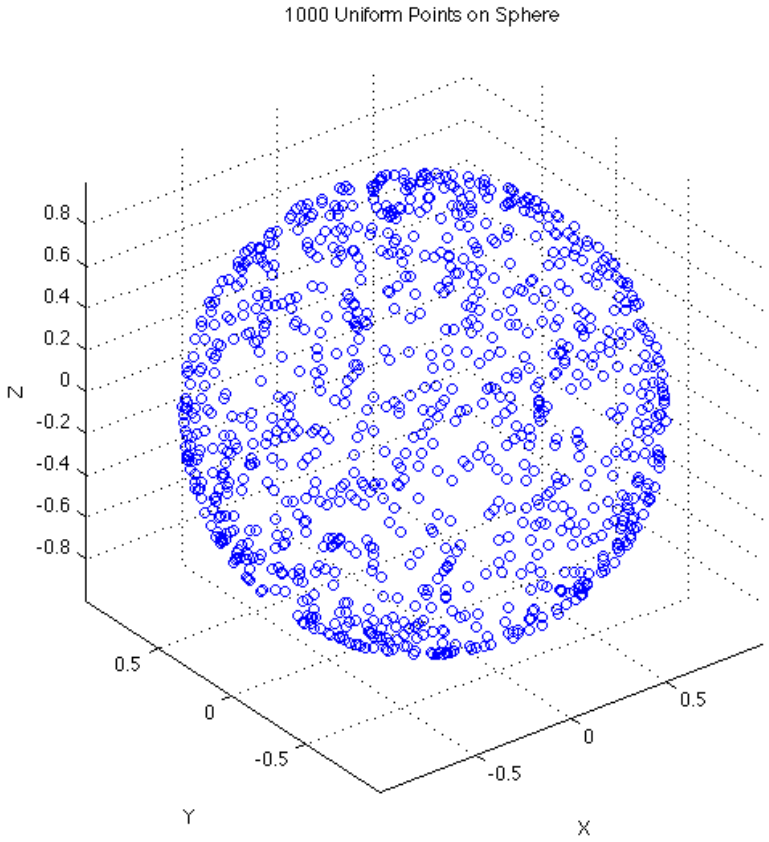
Sampling the Surface of the Unit Sphere

Another important shape to sample is the surface of the unit sphere. We can do this, but we'll have to cheat slightly and use $normal$ random numbers rather than uniform ones. Recall that the command `randn` did this.

It turns out that if you want a point on the surface of the unit sphere, you can just make a vector of 3 normal random numbers, and scale by its length:

```
xyz = randn ( 3, 1000 );
for j = 1 : 1000
  xyz(:,j) = xyz(:,j) ./ norm ( xyz(:,j) );
end
scatter3 ( xyz(1,:), xyz(2,:), xyz(3,:) )
```

This will also work for higher dimensional spheres!



Exercise Use MCM to sample the "L"-shaped region

$0 \leq x \leq 10$ $0 \leq y \leq 4$ for $0 \leq x \leq 2$ and $0 \leq y \leq 2$ for $x > 2$

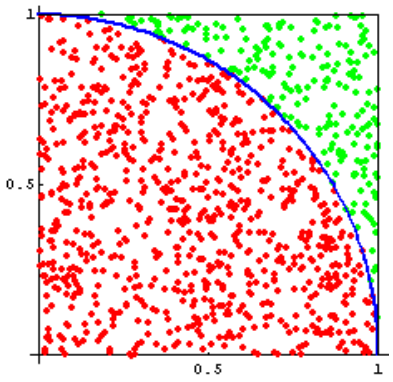
Use 1000 points; plot.

Using MCM to approximate π

- We know that the area of the $unit$ circle (with radius 1) is just π. Suppose we enclose the unit circle in a square say the one centered at the origin with side 2.
- We want to randomly "throw darts" at the square. Our approximation to π will be

$$\frac{\text{the number of darts inside or on the unit circle}}{\text{the total number of darts thrown}}$$

Here is a a plot of the region in the first quadrant.



This method is easily implemented in Matlab. We simply generate a random point (x, y) and decide if it is inside the circle or not. Because `rand` generates random numbers between 0 and 1 we either have to map the numbers between -1 and 1 or use numbers between 0 1 to calculate an approximation to $\pi/4$ (i.e., in the first quadrant) and multiply the result by 4; we choose the later.

Input: number of random points to generate

Output: approximation to π

```
count=0;
for i = 1, n
   x=rand(1); y=rand(1);
   if x^2+y^2 <=  1
       count = count + 1 \par
   end
end
pi = 4.*count / n
```

It is more efficient to generate all the random numbers at once and do vector operations. Note that to square each entry in the vector x (i.e., take dot product) we have to use the appropriate syntax. Here the command `find` finds all occurrences where the condition is satisfied; it returns a vector with these indices so the total number of times this condition was satisfied is the length of this vector.

```
rand('state',12345)
x=rand(1,n) % generate row of random numbers;
y=rand(1,n);
i=find ( x.^2 + y.^2 <= 1); % find locations where this condition is
m=length(i);

pi_approx = 4. * m / n
error = abs ( pi - pi_approx)
```

Here are some approximations and the corresponding errors for some values of n

	10^1	10^2	10^3	10^4	10^5
Approx	3.2	3.16	3.164	3.1292	3.1491
Error	0.0584	0.0184	0.0224	0.0124	0.0075

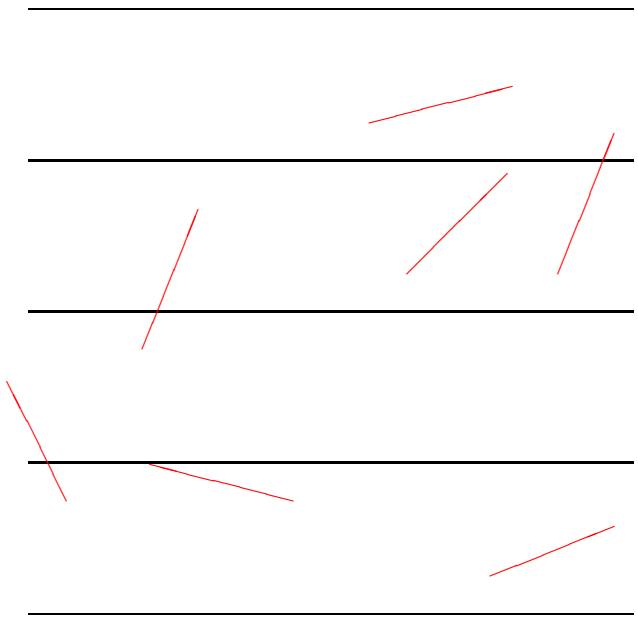
Note that the error is NOT monotonically decreasing as n increases.

Buffon's Needle Problem

An interesting related problem is Buffon's Needle problem which was first proposed in 1777.

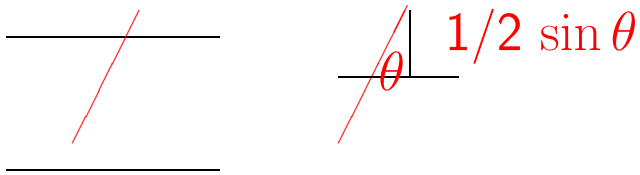
Here's the problem (in a simplified form).

- Suppose you have a table top which you have drawn horizontal lines every 1 inch.
- You now drop a needle of length 1 inch onto the table.
- What is the probability that the needle will be lying across (or touching) one of the lines?
- Actually, one can show that the probability is just $\frac{2}{\pi}$



So if we could simulate this on a computer, then we could have another method for approximating π

- Let's analyze the problem to see how we can implement it.
- Let's assume without loss of generality that the lines are horizontal, they are spaced 1 unit apart and the length of the needle is 1 unit.
- Assume, as in the figure, that we have dropped a needle and that we know the location of the middle of the needle (actually we just need the y-coordinate) and the angle θ the needle makes with the horizon.
- So in the figure we see that the vertical side of the triangle has length $\frac{1}{2}\sin\theta$

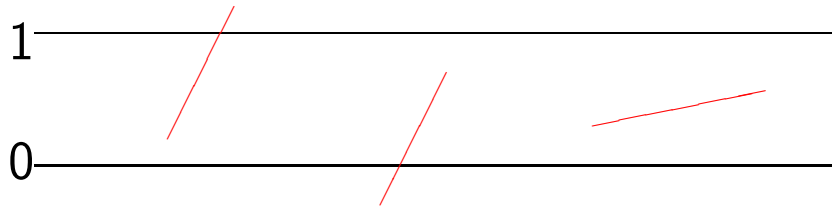


- Since we know the y-coordinate of the middle of the needle we know the y

coordinates of the end of the needle

$$y \pm \frac{1}{2}\sin\theta$$

- Here are the 3 situations we can have



- If the top of the needle has a y-coordinate greater than one, then the needle touches the line, i.e., we have a "hit". If the bottom of the needle has a y-coordinate less than zero, then it touches the other line and we have a "hit".
- Since it is known that the probability that the needle will touch the line is

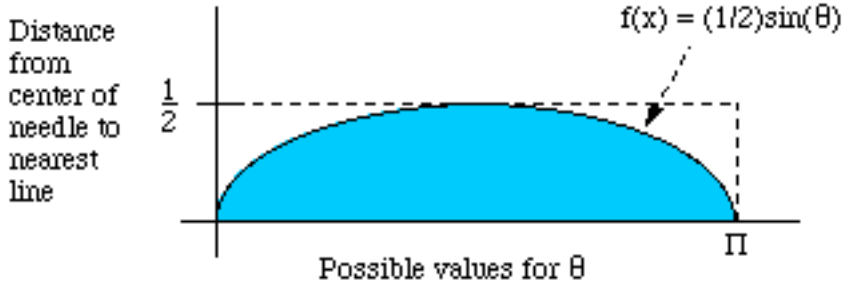
$2/\pi$ then

$$\frac{\texttt{number of hits}}{\texttt{number of points}} \approx \frac{2}{\pi}$$

and thus

$$\pi \approx 2 \times \frac{\texttt{number of points}}{\texttt{number of hits}}$$

One way to see that the probability is $2/\pi$ is to note that the shaded portion in the figure below is found by using the definite integral of (1/2)sin(theta) evaluated from zero to pi. The result is that the shaded portion has a value of 1. The value of the entire rectangle is (1/2)(pi) or pi/2. So, the probability of a hit is 1/(pi/2) or 2/pi. That's approximately .6366197.



Exercise Write pseudo code for an algorithm to approximate π using Buffon's needle problem. Then modify the code for estimating π using the unit circle to solve this problem.

Using Monte Carlo to Approximate an Integral

- Suppose we want to evaluate $\int_a^b f(x)\,dx$
- If $f(x) \geq 0$ for $a \leq x \leq b$ then we know that this integral represents the area under the curve $y = f(x)$ and above the $x-$axis.
- Standard deterministic numerical integration rules approximate this integral by evaluating the integrand $f(x)$ at a set number of points and multiplying by appropriate weights.
  - For example, the midpoint rule is

$$\int_a^b f(x)\,dx \approx f\left(\frac{a+b}{2}\right)(b-a)$$

  - Simpson's rule is

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

- Deterministic numerical integration (or numerical quadrature) formulas have the general form

$$\int_a^b f(x)\, dx \approx \sum_{i=1}^{q} f(x_i) w_i$$

where the points x_i are called quadrature points and the values w_i are quadrature weights

To approximate the integral using Monte Carlo (which is a probablistic approach) we first extend the idea we used to approximate π.

Assume for now that the integrand is greater than or equal to zero in the interval $[a, b]$ then we

- choose a rectangle which includes the area you want to determine; e.g., if $f(x) \leq M$ for all $a \leq x \leq b$ then choose the rectangle with base $b - a$ and height M
  - generate a random point in the rectangle
  - determine if the random point is in desired region
  - take area under curve as a fraction of the area of the rectangle
- First we generate two random points, call them (x, y) and map them to the rectangle enclosing our integral.
- In our method for approximating π we checked to see if $x^2 + y^2 \leq 1$. What do we check in this case?

  At this given x-point we want to see if the random point y is above the

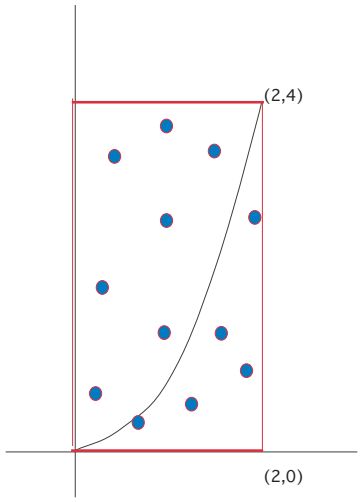
curve or not. It is easy to check to see if

$$y \leq f(x) \quad \text{where } x \text{ is the random point}$$

If this is true, then the point lies in the desired area and our counter is incremented.

- Our approximation to the area, i.e., the value of the integral, is simply the usual fraction times the area of the testing rectangle (whose area is $M(b-a)$).

$$\frac{\texttt{number of hits}}{\texttt{number of points}} \times M(b-a)$$



12 random points generated and
5 in the desired region

Here we are approximating

$$\int_0^2 x^2\,dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3} = 2.67$$

Using 12 points we have that the approximation is

$$\frac{5}{12}(8) = \frac{40}{12} = 3.33$$

where 8 is the area of the box we are sampling in and 5 out of 12 points were in the desired region. Not a good approximation but we only used 12 points!

How do you choose the bounding box? It is not unique.

Example Approximate the integral

$$\int_0^{1/2} \cos(\pi x) = \frac{1}{\pi} \approx 0.3183$$

using MC.

We choose the bounding box to be $[0, .5] \times [0, 1]$. The results are given in the table.

n	approx	error
100	0.3150	0.0033
1000	0.3065	0.0118
10000	0.3160	0.0023
100,000	0.3196	0.0013

In the example of approximating $\int_0^2 x^2\ dx$ we saw that there are a lot of random points generated which do not lie in the region under the curve $f(x) = x^2$.

We will now look at using MCM for approximating an integral from a slightly different viewpoint. This has the advantage that we don't have to take a bounding box and we get a better approximation with the same number of points.

Recall that the Riemann sum for approximating $\int_a^b f(x)\ dx$ with the uniform partition

$$x_0 = a, x_n = b, x_i = x_{i-1} + \Delta x \quad \text{where } \Delta x = \tfrac{b-a}{n}$$

is

$$\int_a^b f(x)\ dx \approx \sum_{i=1}^{n} f(\tilde{x}_i)\Delta x = \sum_{i=1}^{n} f(\tilde{x}_i)\frac{b-a}{n}$$

Here $\tilde{x}_i$ is any point in the interval $[x_{i-1}, x_i]$ so if $\tilde{x}_i$ is the midpoint $(x_{i-1}+x_i)/2$ then we have the midpoint rule. As $n \longrightarrow \infty$ we get the integral.

The MCM approach to this problem can be written as

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\frac{b-a}{n}$$

where now the points x_i are $random$ points in the interval $[a,b]$. This is not a determinate formula because we choose the x_i randomly. For large n, we can get a very good approximation.

Another way to look at this expression is to recall from calculus that the average of the function $f(x)$ over [a,b] is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx \implies \int_a^b f(x)\,dx = (b-a)\bar{f} \approx (b-a) * \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

One can show rigorously that the error is $\mathcal{O}(\frac{1}{\sqrt{n}})$.

Example Consider the example we did before of approximating the integral

$$\int_0^{1/2} \cos(\pi x) = \frac{1}{\pi} \approx 0.3183$$

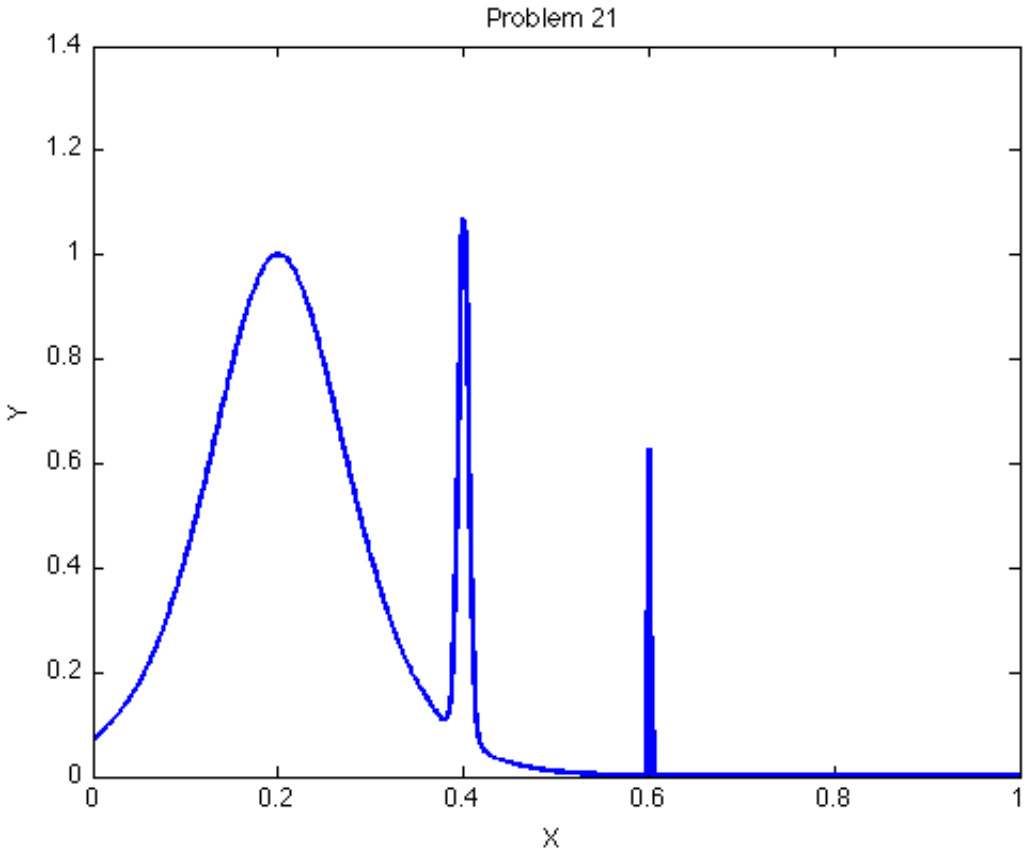
using MC. We now compare the two methods. We have to do the same number of function values in each case.

n	approx	error		
100	0.3150	0.0033	0.3278	0.0095
1000	0.3065	0.0118	0.3157	0.0026
10000	0.3160	0.0023	0.3193	0.0010

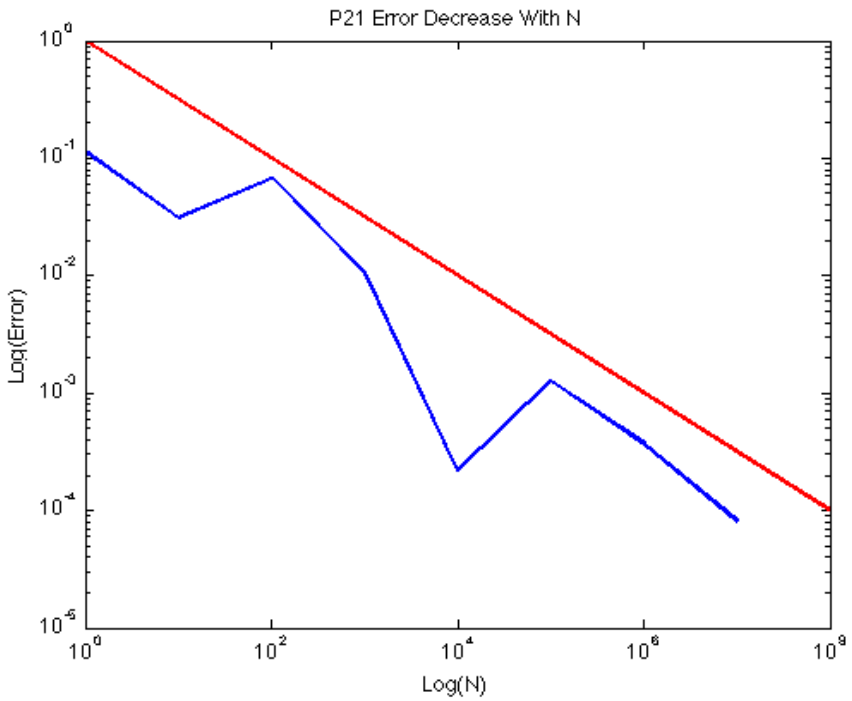
Example Consider the problem of estimating a "hard" integral $\int_0^1 f(x)\ dx$ where

$$\begin{aligned} f(x) =& \text{sech}^2(10.0 * (x - 0.2)) \\ &+ \text{sech}^2(100.0 * (x - 0.4)) \\ &+ \text{sech}^2(1000.0 * (x - 0.6)) \end{aligned}$$

From the plot, it should be clear that this is actually a difficult function to integrate. If we apply the Monte Carlo Method, we get "decent" convergence to the exact value of 0.21080273631018169851



1	0.096400	1.1e-01
10	0.179749	3.1e-02
100	0.278890	6.8e-02
1000	0.221436	1.0e-02
10,000	0.210584	2.1e-04
100,000	0.212075	1.2e-03
1,000,000	0.211172	3.6e-04
10,000,000	0.210721	8.1e-05



Here is a plot of the error on a log-log plot. Note that the slope is approximately -1/2. This is what we expect because we said that the error $= C\frac{1}{\sqrt{n}}$ for a constant C so the log of the error is $\log(n^{-1/2}) = -0.5 \log n$.

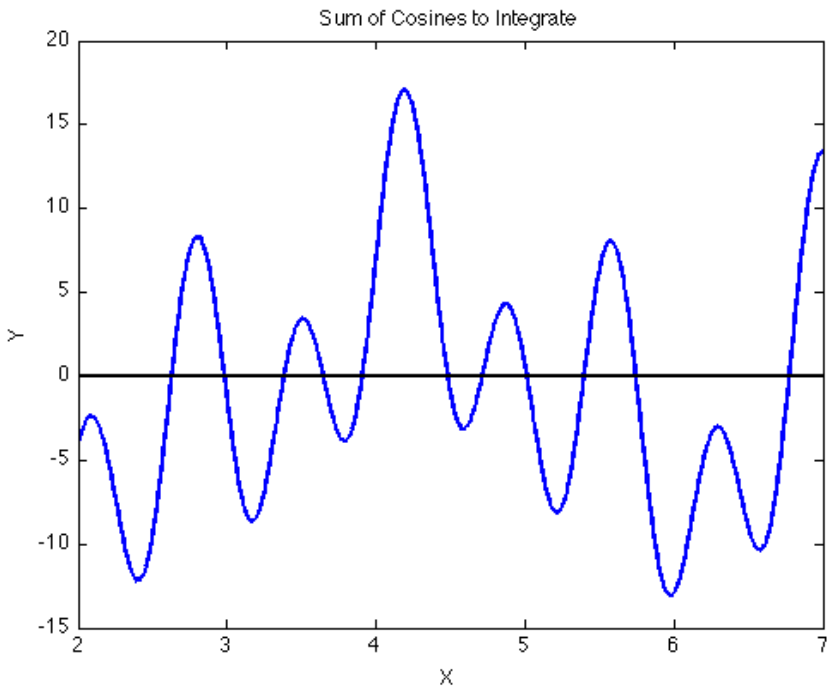
Now let's approximate the integral of a function when the interval is not $[0, 1]$. Specifically we want to integrate

$$\begin{aligned} f(x) = & \cos(x) \\ & +5 * \cos(1.6 * x) \\ & -2 * \cos(2.0 * x) \\ & +5 * \cos(4.5 * x) \\ & +7 * \cos(9.0 * x); \end{aligned}$$

over [2,7]. This is a very "wiggly" function which we you can see from the plot.

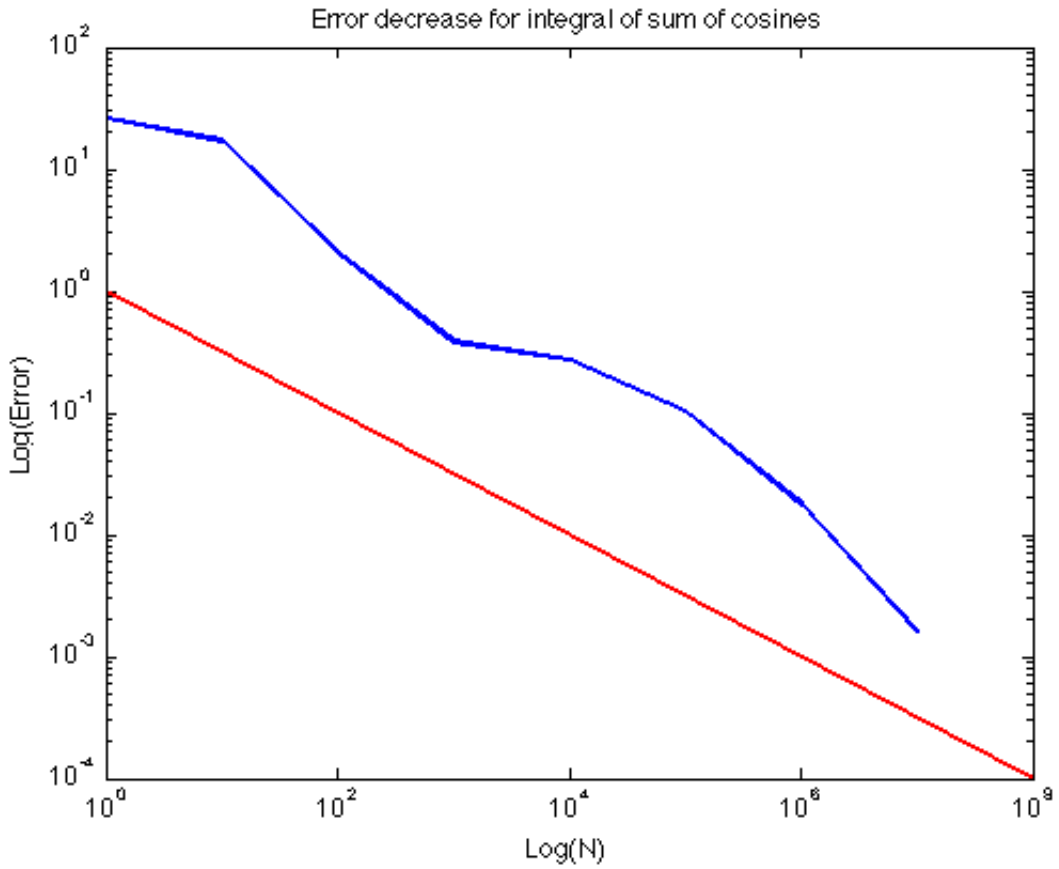
Because the interval is $[2, 7]$ we have to shift the interval from [0,1] to [2,7]. This is easy to do with a linear map; we want $0 \longrightarrow 2$ and $1 \longrightarrow 7$ so the linear function that does this is $y = 2 + 5x$ so a random point x in the interval $[2, 7]$ is just `x = 2.0 + ( 7.0 - 2.0 ) * rand ( n, 1 )`.

Applying the Monte Carlo Method, we get "decent" convergence to the exact solution -4.527569. . .:



1	21.486936	2.6e+01
10	-21.701206	1.7e+01
100	-2.472846	2.0e+00
1000	-4.911594	3.8e-01
10000	-4.253230	2.7e-01
100000	-4.424016	1.0e-01
1000000	-4.509781	1.7e-02
10000000	-4.529185	1.6e-03

Again, we see the approximation to a slope of $-\frac{1}{2}$ on a log-log plot.



- The two functions both seem "hard" to integrate, but for one, we got an error 20 times smaller with the same number of points Why?

- The error depends in part on the variance of the function.
- To get a feeling for the variance of these two functions; remember that the variance is given by the formula $\mathrm{Var} = \int_a^b x(x - \text{mean})^2\, dx$.
  - The first function has an integral of about 0.2 over the interval [0,1], so its average value is 0.2 because the length of the interval is 1. The function ranges between 0 and 1, so the $pointwise$ variance is never more than $(1 - .2)^2 = 0.64$
  - The second function has an integral of about -4.5 over an interval of length 5, for an average value of about -1. Its values vary from -14 to +16 or so, so the pointwise variance can be as much as 200.

Integrals in two dimensions

Since we know how to sample a square, let's try to use the Monte Carlo method to compute the integral of $f(x, y)$ where

$$f(x, y) = |x^2 + y^2 - 0.75|$$

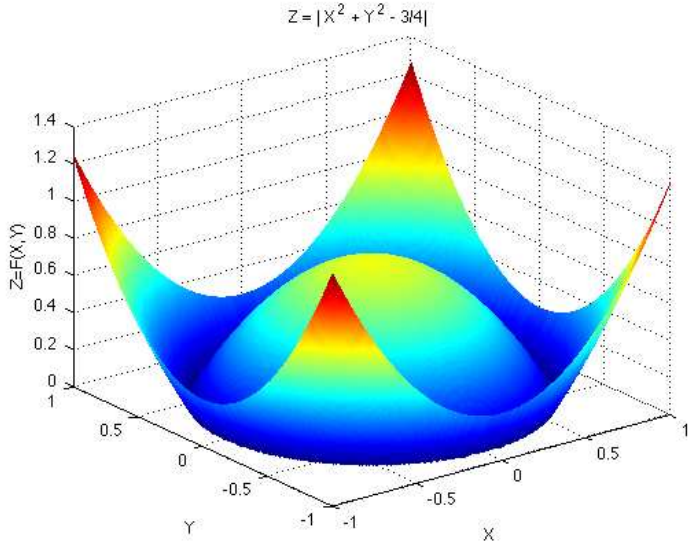
over the square $[-1, +1] \times [-1, +1]$.

Because of the absolute value function, $f(x, y)$ is not differentiable. It would be tricky to try to work out the integral, even though this is a pretty simple function.

Because we are in two dimensions, we need to use pairs of random numbers; when we generate each it will be between [0,1] so we map to $[-1, +1] \times [-1, +1]$ by

```
x= -1.0 + 2.0 * rand ( n, 1 ), y = -1.0 + 2.0 * rand ( n, 1 )
```

Here is a surface plot of our function.



We apply MC in an analogous way

$$\int_c^d \int_a^b f(x,y)\,dx \approx \sum_{i=1}^{n} f(x_i,y_i)\frac{(b-a)(d-c)}{n}$$

Our convergence to the exact solution 1.433812586520645:

1	0.443790	9.900230e-01
10	1.547768	1.139549e-01
100	1.419920	1.389309e-02
1000	1.464754	3.094170e-02
10000	1.430735	3.077735e-03
100000	1.428808	5.004836e-03
1000000	1.432345	1.467762e-03
10000000	1.433867	5.394032e-05

Sometimes we have to integrate a function over a more complicated domain where it is not as easy to sample in but we have a way to determine if a point is in the region; for example, an "E"-shaped region. What can we do in that case?

The answer is to return to the original way we evaluated an integral by putting a bounding box around the region. This approach is usually called MC with rejection.

If we want to integrate over a region in 2-d that is not a box, we enclose that region in a box. We then generate a point in the box and $reject$ it if it is not in the region of integration.

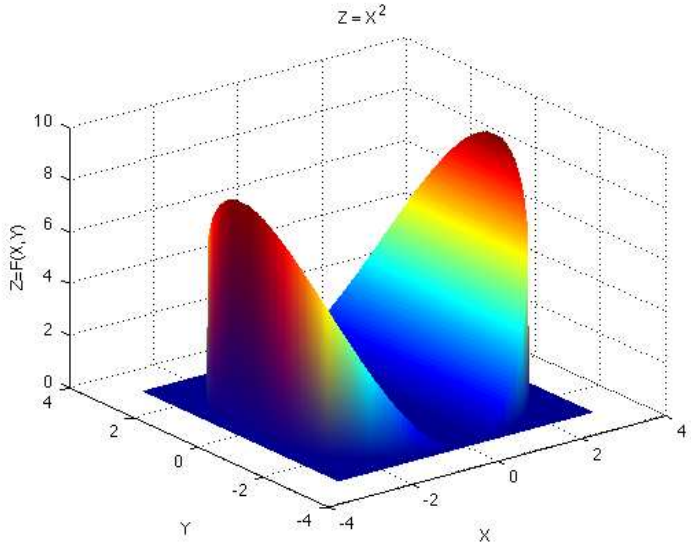
To see how this rejection technique works we look at the example of computing an integral over a circle.

Our integrand will be

$$f(x, y) = x^2$$

and we will integrate over the circle of radius 3 centered at (0,0).

For this problem, the function is smooth, but the region is not a rectangle. It's not hard to work out the exact answer using polar coordinates, because the circle is special. But the Monte Carlo approach with rejection would handle an ellipse, or a region shaped like Mickey Mouse, just as easily.



Now we need to map pairs of random numbers from $[0,1]$ to $[-3,+3] \times [-3,+3]$ and then reject those that lie outside the circle $x^2 + y^2 = 9$. We simply use

```
x = -3.0 + 6.0 * rand ( n, 1 );
```

```
y = -3.0 + 6.0 * rand ( n, 1 );

  if x.^2 + y.^2 <= 9
```

Note that here we are generating n random numbers at once and using the vector operation for exponentiation.

```
  exact = 63.617251235193313079;
  area = pi * 3*3;   % domain we are integrating over

   x = - r + 2 * r * rand ( n, 1 );
   y = - r + 2 * r * rand ( n, 1 );
    i = find ( x .* x + y .* y <= r * r );
    n2 = length ( i );
    result = area * sum (  x(i) .* x(i) ) / n2;  % because our funct
    fprintf ( 1, '  %8d  %f  %e\n', ...
        n2, result, abs ( result - exact ) );

  end
```

If we apply the Monte Carlo Method, we get "decent" convergence:

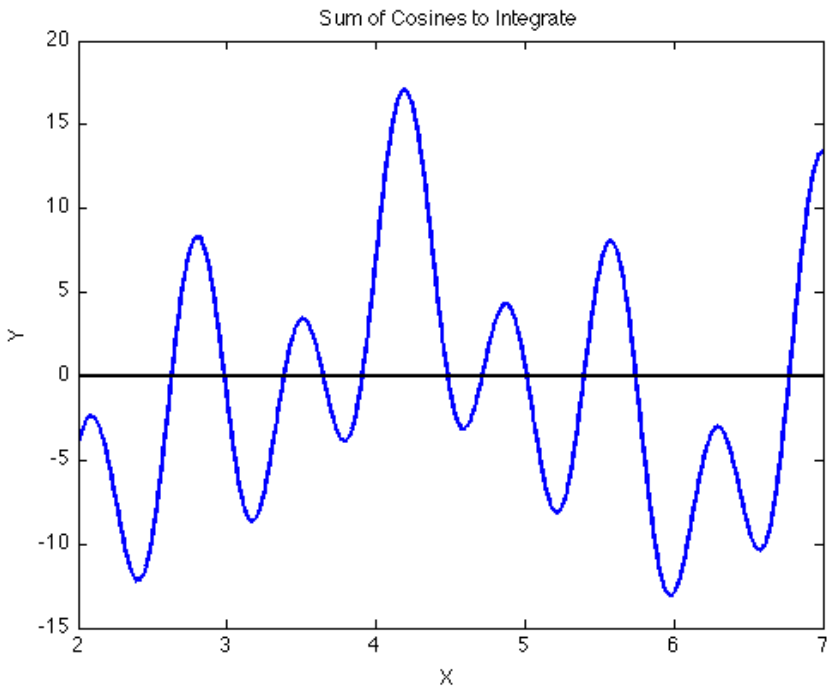
```
      1  74.493641  1.087639e+01
      9  52.240562  1.137669e+01
     78  62.942561  6.746898e-01
    784  61.244910  2.372341e+00
   7816  62.653662  9.635895e-01
  78714  63.405031  2.122204e-01
 785403  63.669047  5.179565e-02
7853972  63.624209  6.958217e-03

Exact    63.617251235193313079
```

Using Monte Carlo Sampling for Optimization

- In the lab you will explore another application of MC sampling – find the extrema of a function.
- A function such as

$$f(x) = \cos(x) + 5 * \cos(1.6 * x) - 2 * \cos(2.0 * x) \\ +5 * \cos(4.5 * x) + 7 * \cos(9.0 * x);$$

  plotted over [2,7] has many *local* extrema. Many minimization routines "get stuck" at these local minima and can't find the *global* minimum.
- You will use MC sampling to estimate where the global minimum is. Note that if we have an algorithm to find a minimum of $f(x)$, we can find its maximum by find the minimum of $-f(x)$.



Exercise Write a script to implement MC for approximating an integral in 1D using the formula

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\frac{b-a}{n}$$

Try out the code on the integral

$$\int_0^1 \cos(4\pi x)\, dx$$

Modify the script to to perform integration in two spatial dimensions. Try your code out on the integral

$$\int_0^3 \int_1^2 (1 + 8xy)\, dydx = 57\,.$$

Simulations using the Monte Carlo Method

- So far we've concentrated on the Monte Carlo method as a means of sampling. This gave us an alternate means of solving integration problems. Of course, there are already other methods for dealing with integration problems.
- We will next consider a Monte Carlo application in which no other method is available, the field of *simulation*.
- We will consider the common features of tossing a coin many times, watching a drop of ink spread into water, determining game strategies, and observing a drunk stagger back and forth along Main Street!
- Using ideas from the Monte Carlo method, we will be able to pose and answer questions about these random processes.

- Computer simulation is intended to model a complex process. The underlying model is made up of simplifications, arbitrary choices, and rough measurements.
- Our hope is that there is still enough relationship to the real world that we can study the results of the simulation and get an idea of how the original process works, predict outputs for given inputs, and how we can improve it by changing some of the parameters.
- A famous applied mathematician once said "the purpose of computations is insight, not numbers". It is good to always keep this in mind!.
- Often it is particularly hard to use rigorous mathematical analysis to prove that a given model will always behave in a certain way (for example, the output won't go to infinity.)
- Sampling tries to answer these questions by practical examples - we try lots of inputs and see what happens.
- We want to consider some examples of simulations that can be done using the MC method and you will look at another in the lab.

A Simple Example from Business

Suppose a friend is starting a business selling cookies and you want to help him succeed (and show off your computational skills).

Assume he buys the cookies from a local bakery at the cost of \$0.40 and sells them for \$1.00; assume that your friend has no overhead so he makes \$0.60 in profit per cookie if he sells all the ones he ordered.

However, if he has some cookies left over at the end of the day, those are given to the homeless shelter and he looses \$0.40 per cookie. He works for four hours each afternoon and feels there is a fairly uniform demand during those hours; he has never sold less than 80 or more than 140 cookies.

Use MC to recommend how many cookies he should order to maximize his profits.

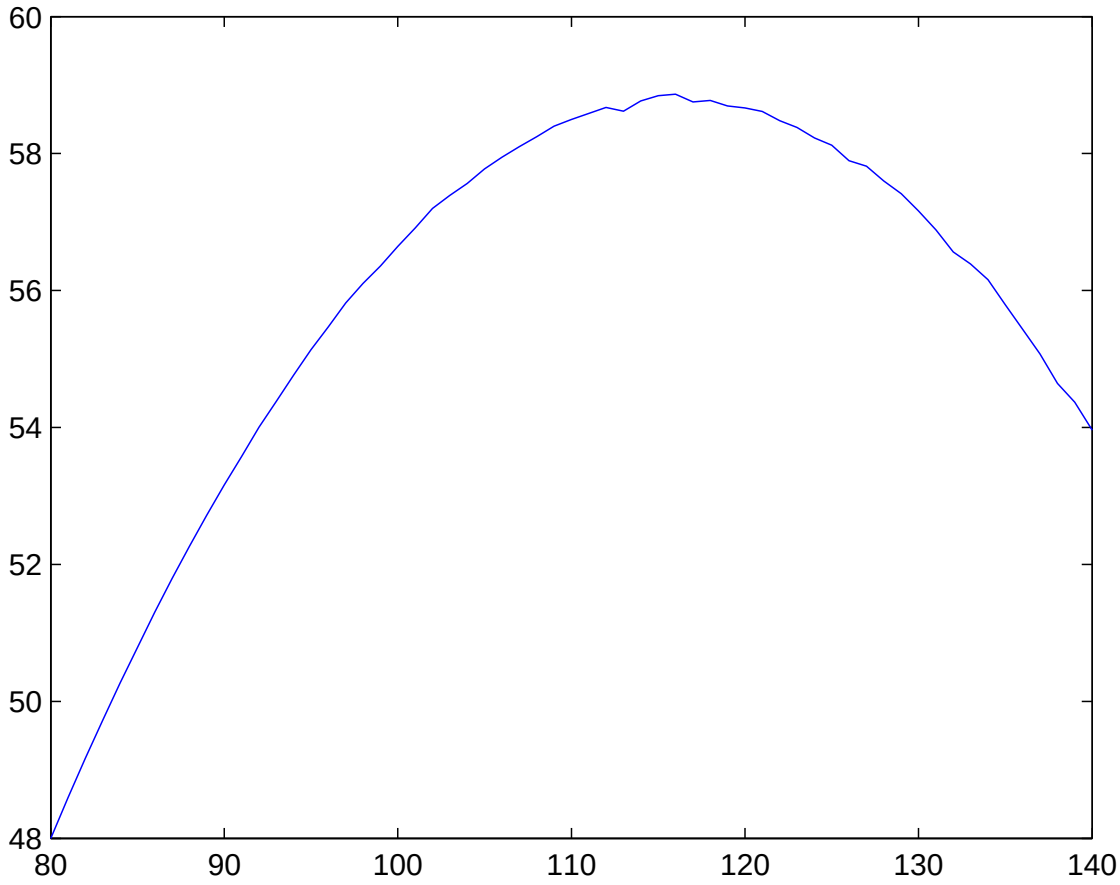
How can we use MC to answer this question?

- Let Q denote the quantity that he orders; D the demand (amount sold)
- Set $Q = 80$
- Generate n replications of D; i.e., generate n random numbers between 80 and 140
- For each replication, compute the daily profit
- After n replications estimate the earnings ordering Q cookies by

$$\text{earnings} = \frac{\sum \text{ daily profit}}{n}$$

- Repeat for integer values of Q between 80 and 140;
- Select the value of Q which yields the best earnings

```
profit = 0.;

for k = 1:n

    d = rand(1);
    d=80 + 60*x;
    if d > q
        profit =  profit + .6*q;
    else
        profit = profit + .6*d-.4*(q-d);
    end

  end

earnings = profit /n

end
```



Profit is maximized by ordering approximately 116 cookies

Brownian Motion

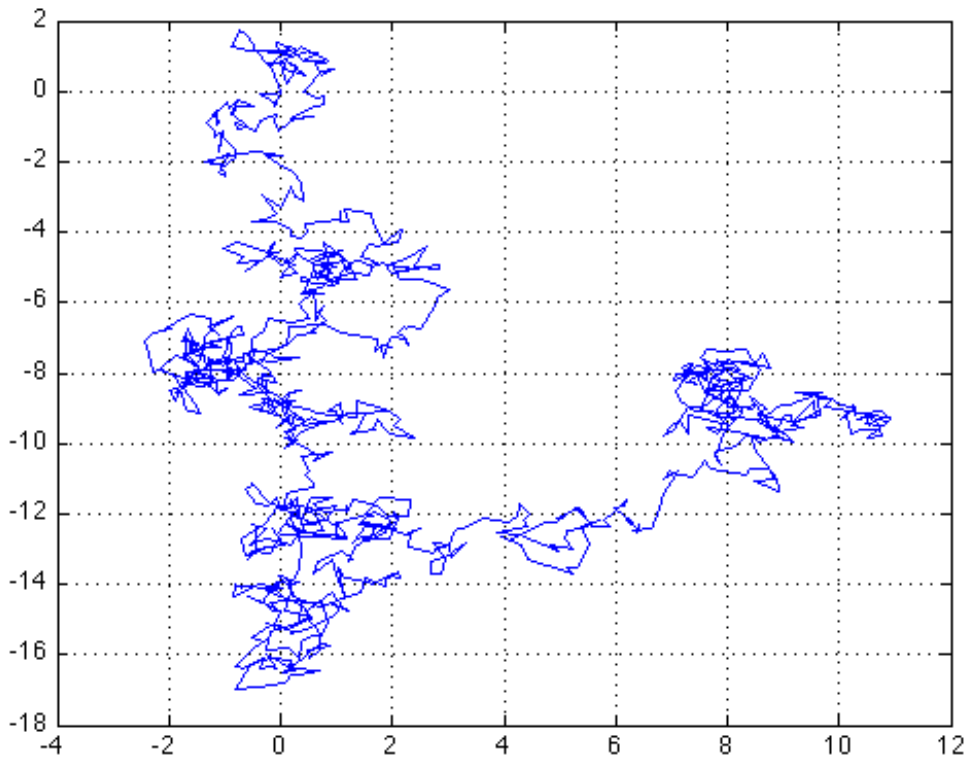
In 1827, Scottish botanist Robert Brown was studying pollen grains which he had mixed in water. Although the water was absolutely still, the pollen grains seemed to quiver and move about randomly. He could not stop the motion, or explain it. He carefully described his observations in a paper.

When other researchers were able to reproduce the same behavior, even using other liquids and other particles, the phenomenon was named Brownian Motion, although no one had a good explanation for what they were observing.

Check out the You Tube video for Brownian Motion at

$$\texttt{http} : //\texttt{www.youtube.com/watch?v} = \texttt{apUl\_baT\_Kc}$$

Here is the result of a simulation of Brownian motion.



In 1905, the same year that he published his paper on special relativity, Albert Einstein wrote a paper explaining Brownian motion. Each pollen grain, he said, was constantly being jostled by the motions of the water molecules on all sides of it. Random imbalances in these forces would cause the pollen grains to twitch and shift.

Moreover, if we observed the particle to be at position (x, y) at time $t = 0$, then its distance from that point at a later time t was a normal random variable with mean 0 and variance t. In other words, its typical distance would grown as $\sqrt{t}$. Recall that the command `randn` used here generates numbers with a normal distribution.

```
T = 10.0;
N = 1000;
h = sqrt ( T / N );
x(1) = 0.0;
y(1) = 0.0;

for i = 1 : N
  x(i+1) = x(i) + h * randn ( );
  y(i+1) = y(i) + h * randn ( );
end
```

We can write this program another way to get rid of the loop.

```
T = 10.0;
```

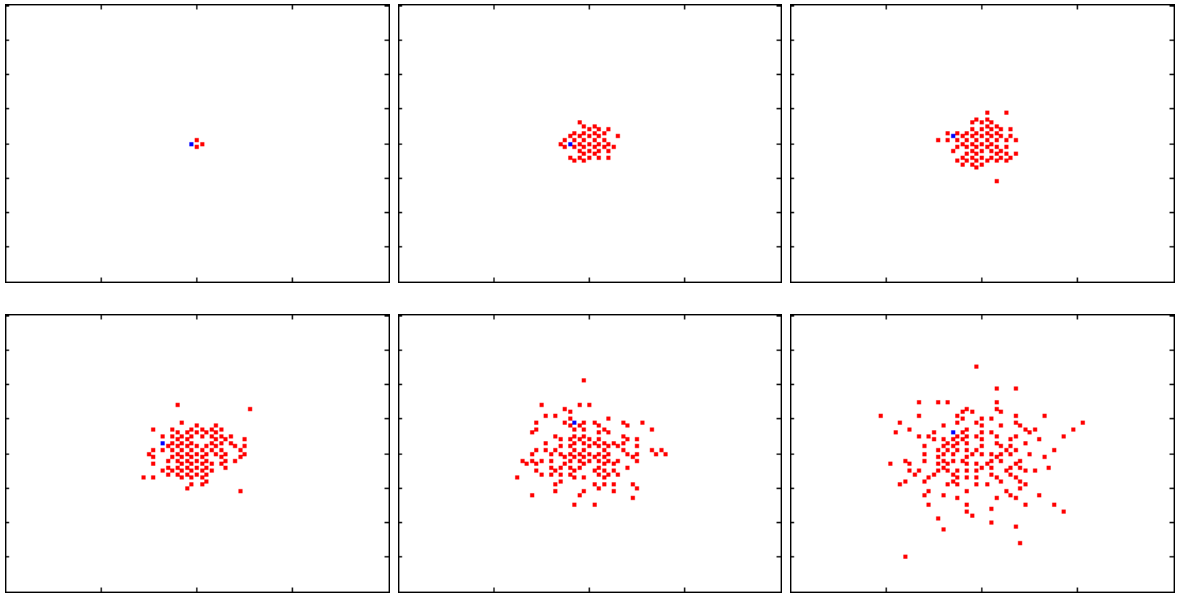
```
N = 1000;
h = sqrt ( T / N );
x(1) = 0.0;
y(1) = 0.0;

x(2:N+1) = h * cumsum ( randn(1:N,1) );
y(2:N+1) = h * cumsum ( randn(1:N,1) );
```

Brownian motion also explained the phenomenon of **diffusion**, in which a drop of ink in water slowly expands and mixes.

As particles of ink randomly collide with water molecules, they spread and mix, without requiring any stirring.

The mixing obeys the $\sqrt{t}$ law, so that, roughly speaking, if the diameter of the ink drop doubles in 10 seconds, it will double again in 40 seconds.

The physical phenomenon of Brownian Motion has been explained by assuming



that a pollen grain was subjected to repeated random impulses. This model was intriguing to physicists and mathematicians, and they soon made a series of abstract, simplified versions of it whose properties they were able to analyze, and which often could be applied to new problems.

The simplest version is known as the Random Walk in 1D.

Random Walks

100 Drunken Sailors

We'll introduce the random walk with a story about a captain whose ship was carrying a load of rum. The ship was tied up at the dock, the captain was asleep, and the 100 sailors of the crew broke into the rum, got drunk, and staggered out onto the dock.

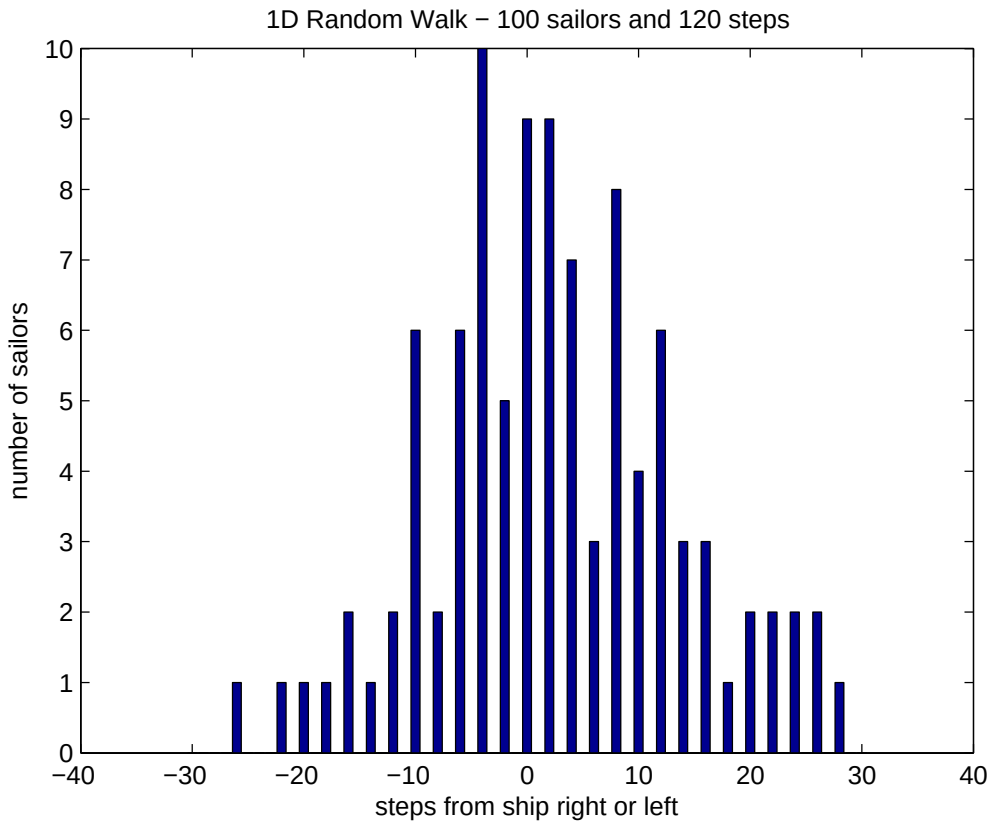
The dock was 100 yards long, and the ship was at the 50 yard mark. The sailors were so drunk that each step they took was in a random direction, to the left or right. (We're ignoring the width of the pier because they can only move to the right or left.) They were only able to manage one step a minute. Two hours later, the captain woke up.

"Oh no!" he said, "There's only 50 steps to the left or right and they fall into the sea! And two hours makes 120 steps! They will all be drowned for sure!"

But he was surprised to see that around 80 of the crew were actually within 11 steps to the left or right, and that **all** of the crew was alive and safe, though in sorry shape.

How can this be explained?

We will use the idea of Random Walk to simulate it. Here's a frequency plot for the simulation.



- We want to simulate this experiment using a random walk.
- Taking n random steps is like adding up random $+1$'s and -1's; the average of such a sum tends to zero, with an error that is roughly $\frac{1}{\sqrt{n}}$ (in our problem $\sqrt{n} \approx 11$)

$$|\text{average} - 0| \approx \frac{1}{\sqrt{n}}$$

then

$$|\frac{\text{sum}}{n}| \approx \frac{1}{\sqrt{n}}$$

and so

$$|\text{sum}| \approx \sqrt{n}.$$

Simulating the Drunken Sailor Random Walk Experiment

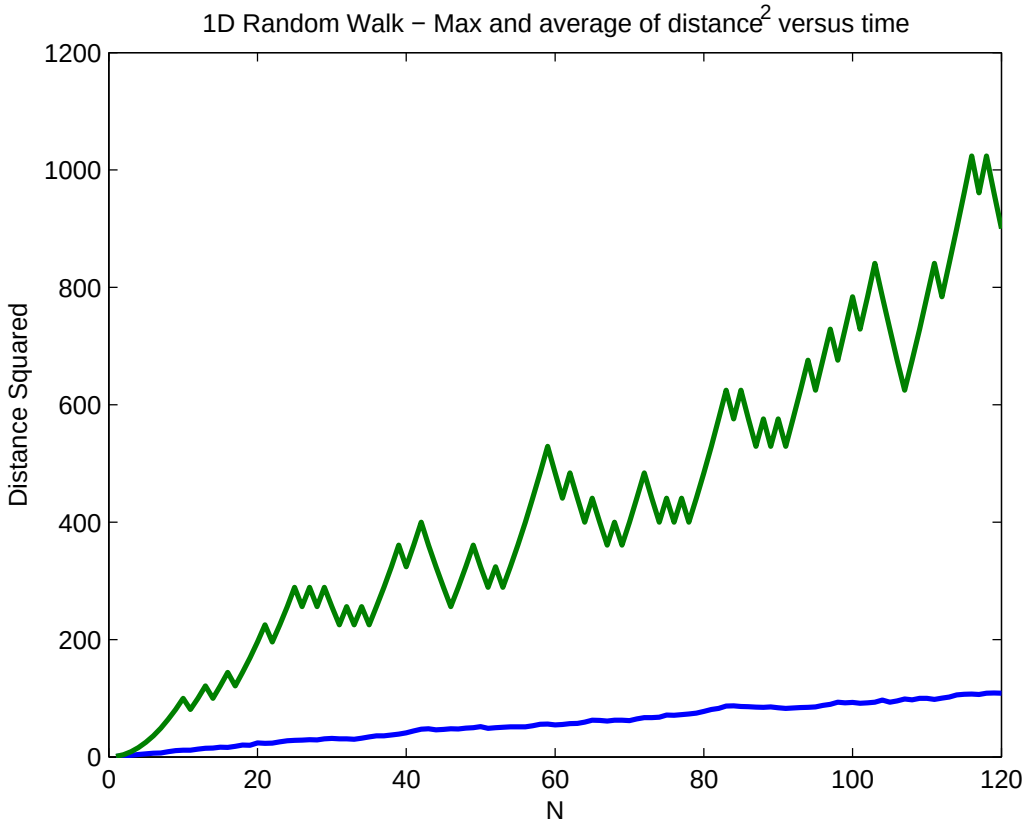
Our model for a random walk in 1D is very simple.

- We let x represent the position of a point on a line.
- We assume that at step $n = 0$ the position is $x = 0$.
- We assume that on each new step, we move one unit left or right, chosen at random.
- From what we just said, we can expect that after n steps, the distance of the point from 0 will on average be about $\sqrt{n}$. If we compare n to the $square$ of the distance, we can hope for a nice straight line.
- First let's look at how we might simulate this experiment. In this code we want to plot the number of steps n on the x-axis and the square of the distance traveled and the square of the maximum distance from the origin each sailor got.
- For each sailor we will take n steps; before starting we set $x = 0$ (the location given in units of steps to right or left) and at each step

- generate a random number r between 0 and 1(we could have mapped r between -1 and 1)
- if $0 \leq r < 0.5$ we move to the left so $x = x - 1$
- if $.5 < r \leq 1$ we move to the right so $x = x + 1$

```
%  Take WALK_NUM walks, each of length STEP_NUM random +1 or -1 step
%
  time=1:step_num;

  for walk = 1 : walk_num

    x = 0;

    for step = 1 : step_num

      r = rand ( );
%
%  Take the step.
%
```

```
      if ( r <= 0.5 )
        x = x - 1;
      else
        x = x + 1;
      end
%
%  Update the average and max.
      x2_ave(step) = x2_ave(step) + x^2;
      x2_max(step) = max ( x2_max(step), x^2 );

    end

    x2_ave(:,:) = x2_ave(:,:) / walk_num;
%
%  Plot the results.
    plot ( time, x2_ave, time, x2_max, 'LineWidth', 2 );
```



Blue - square of average distance Green - square of maximum distance

The square of the average distance behaves linearly but the square of the maximum distance each sailor traveled from the starting point varies a lot.

- In 1D, how many possible random walks of n steps are there?
- To answer this, consider flipping a coin. If we flip it once, there are two choices H and T. If we flip it two times there are $2^2 = 4$ choices

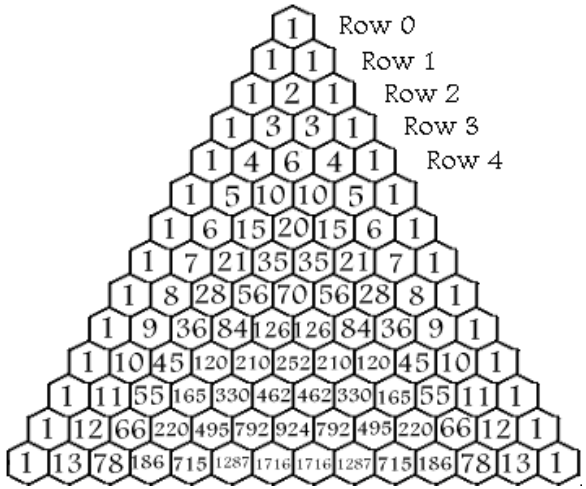
  $$HH \quad HT \quad TT \quad TH$$

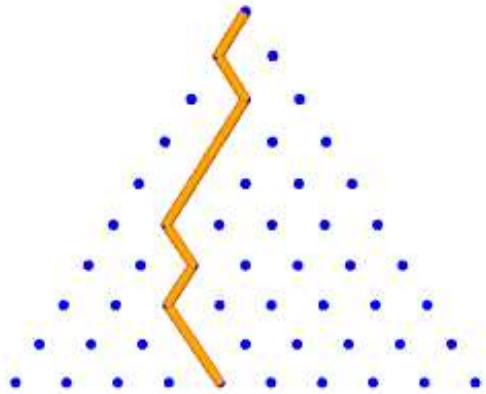
  and if we flip it 3 times there are $2^3 = 8$ choices

  $$HHH \quad HHT \quad HTH \quad HTT \quad TTT \quad THT \quad THH \quad TTH$$

  so in general there are 2^n outcomes for flipping a coin n times.
- Our random walk in 1D is like flipping a coin; there are only two choices – right or left.
- So if we have 120 steps $2^{120} \approx 10^{36}$ possible random walks.
- How many of them can end up at a given position?
- There is only one that can end up at n.
- There are n out of 2^n (i.e., 120 out of 10^{36}) that end at $n-2$; they involve $n-1$ steps of $+1$ and one step of -1; these can occur at n places. For example, one sailor could go left on the first step and right on all remaining ones; another sailor could go right on all steps except the second one, etc.

- One can show that, in general, there are $\binom{n}{k}$ distinct random walks that will end up at $\mathbf{n - 2*k}$.
- That means the probability of ending up at a particular spot is just the corresponding combinatorial coefficient divided by 2^n.
- In other words, row n of Pascal's triangle tells you what a random walk can do. But row n of Pascal's triangle can be thought of as dropping balls in a pachinko game that can randomly go left or right at each level!





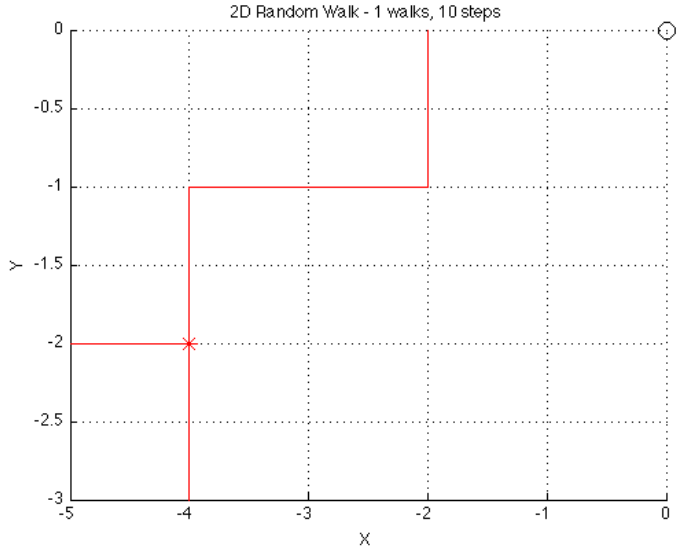
An entry in Pascal's triangle is found by summing the two numbers above it to the left and right. The top 1 is called the zeroth row. An entry in the nth row

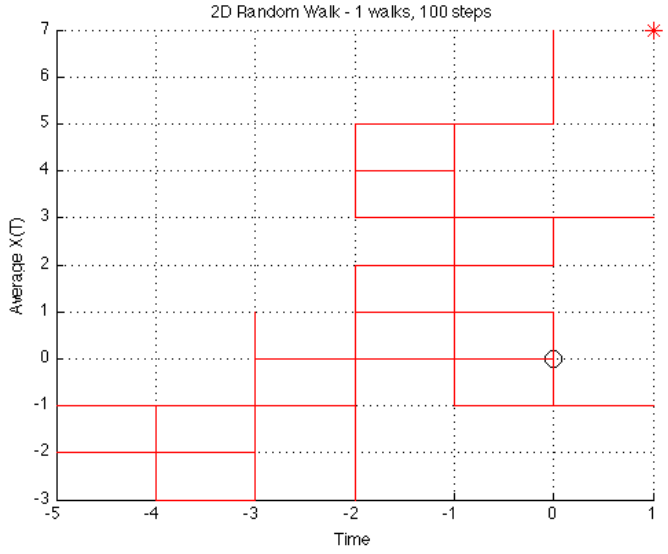
can be found by n `choose` r where r is the element number in that row. For example, in row 3, 1 is the zeroth element, 3 is element number 1, the next three is the 2nd element, and the last 1 is the 3rd element. The formula for n Choose r is

$$\frac{n!}{r!(n-r)!} \qquad \text{where } 0!=1$$

Random Walks in 2D

We can do the same thing in two dimensions. Now we can move to the right or left or to the top or bottom. You will implement this in the lab and use a random walk to solve Laplace's equation in two dimensions.





Exercise Modify the code for using random walks for the drunken sailor problem if each sailor has a 60% chance of moving to the right and a 40% chance of moving to the left. Output your results in a frequency plot where the x axis is the number of steps in each direction from the origin and the y axis is the frequency.

Game Theory

In the 1950's, John von Neumann (who wrote the first computer user manual, for the ENIAC), developed game theory, which sought strategies in multi-player games. The Monte Carlo method was used to simulate the games, and determine the best strategies. We will look at two examples of this – simulating a gambling game and simulating a duel!

The first example is a modification to the Fermat/Pascal point game which makes it much more difficult. Recall in that problem that if one person won, the other person did not lose. Here we assume that one player wins a dollar from the other player.

In particular, assume that there are two gamblers, A and B, who decide to play a game.

A has p dollars and B has q dollars

They decide to flip a coin. If it comes up heads, A wins \$1 from B while tails works the other way.

The game is over when one gambler is bankrupt.

Here are some questions we can ask:

1. What is the probability that A will win?
2. What is the expected value of the game to A, that is, how much will he win or lose, on average?
3. How long will a typical game last?
4. What are the chances that the winner will be ahead the entire game?
5. What happens if we change the amount of money that A and B have?

Here's a typical game, in which A starts with \$3 and B with \$7; assume A wins when a head comes up

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
		H	H	T	T	T	H	T	T	H	T	H	H	T	T	T
A	3	4	5	4	3	2	3	2	1	2	1	2	3	2	1	0
B	7	6	5	6	7	8	7	8	9	8	9	8	7	8	9	10

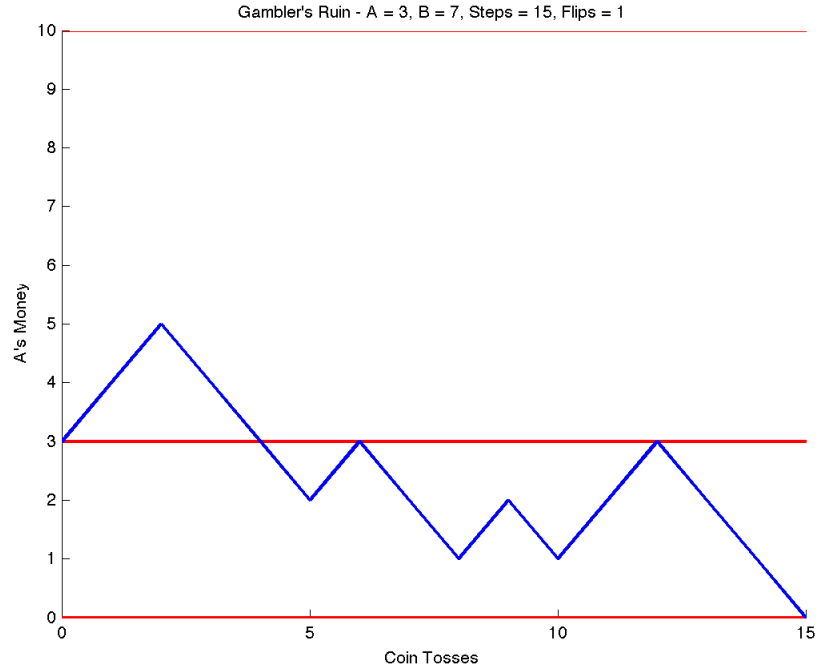
A loses after 15 tosses, having tossed 6 heads and 9 tails.

Clearly the gambler with the most starting money has an advantage, but how much?

It's easy to see that if A starts with \$3 dollars, the game can end in 3 steps. Can we expect that in such a case, a typical game might last 6, or 9 or 12 steps? Does it depend in any way on how much more money B has?

Suppose that A and B start with the same amount of money. In a typical game, is it likely that they will each be in the lead about the same number of tosses?

A single plot can show you an example of how the game works, but it can't tell you all the things that can happen. Here is a plot of the history of A's money for the example game above.



For a simulation, we need to look at a lot of games, and we try different starting bets (the "parameters" of this model).

Let's consider the following cases when we run our simulations:

$A	$B	Remarks
$3	$7	Our starting example.
$30	$70	10 times as long?
$10	$10	Equal stakes, equal odds
$1	$100	Game will be quick.

Structure of the code

- input starting stakes `a_stakes, b_stakes` and number of games to simulate `n_games`
- initialize; set `a_wins=0, b_wins=0, a=a_stakes, b=b_stakes`
- for each game while `a>0` and `b>0`
  - generate a random number to see who wins
  - if a wins, `a = a+ 1; b=b-1`
  - if b wins, `b = b+ 1; a=a-1`
- stop when `a=0` or `b=0`
- after the game is over, increment wins
- after all the games are over, approximate the probability that each player wins

```
  a_wins = 0;
  b_wins = 0;
%
%  Play n_game games.
%
  for game = 1 : n_games

    a = a_stakes;
    b = b_stakes;

    while ( 0 < a & 0 < b )

      r = rand ( );

      if ( r <= 0.5 )
        a = a + 1;
        b = b - 1;
      else
        a = a - 1;
```

```
        b = b + 1;
      end

   end
% increment wins
    if ( b == 0 )
      a_wins = a_wins + 1;
    else
      b_wins = b_wins + 1;
    end

 end
%
%  Average over the number of games.
%
  prob_a_win = a_wins / n_games
  prob_b_win = b_wins / n_games
```

The results of 1000 games for each set of starting stakes.

$A	$B	Length	Max	A Prob
$3	$7	21	135	0.29
$30	$70	2,010	12,992	0.28
$10	$10	101	722	0.49
$1	$100	104	10,472	0.01

From this data you might be able to guess the following:

- the expected number of steps is $A * $B;
- A's chance of winning is `$A / ( $A + $B )` ;

What is the expected value of this game?

A's chance of winning is $\frac{\$A}{\$A + \$B}$

the chance of \$A losing (i.e., B winning) is 1 - this probability, or $\frac{\$B}{\$A + \$B}$.

The expected value is

$$\texttt{value}(\texttt{win}) * \texttt{p}(\texttt{win}) + \texttt{value}(\texttt{loss}) * \texttt{p}(\texttt{loss})$$

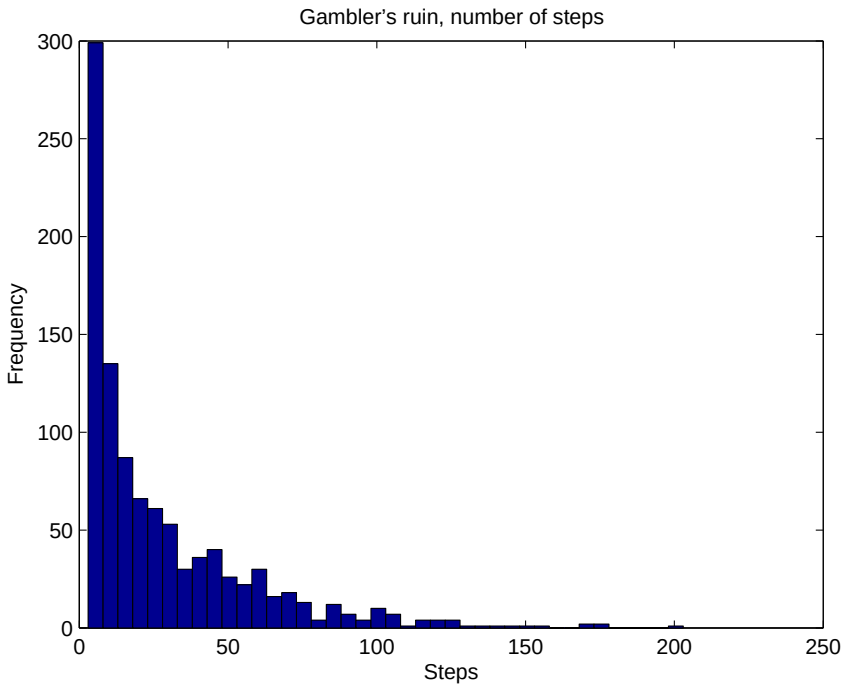
or

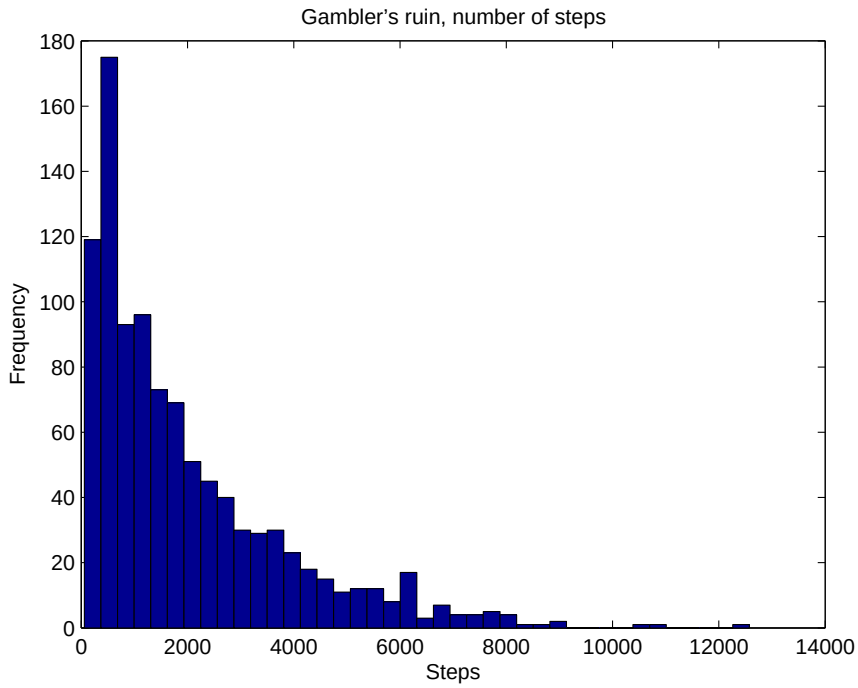
$$\$B\frac{\$A}{\$A + \$B} - \$A\frac{\$B}{\$A + \$B} = 0$$

so even when A has \$1 against B's \$100, it's a fair game.

(small chance of big win) - (big chance of small loss).

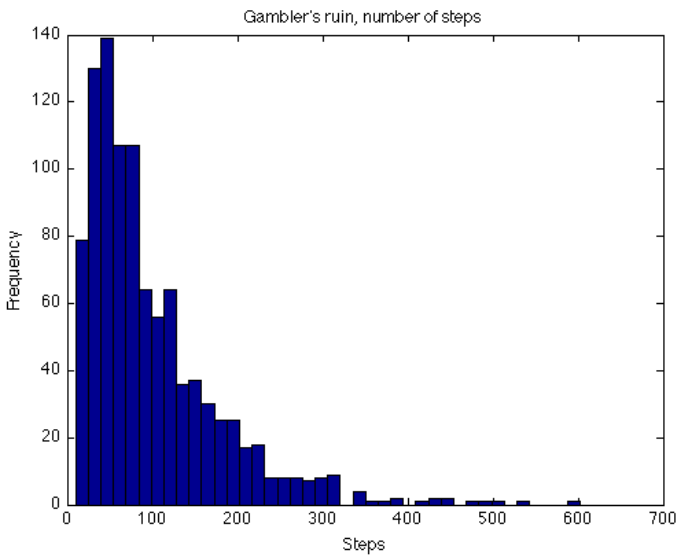
Here are the results of four simulations which mimic our cases above.

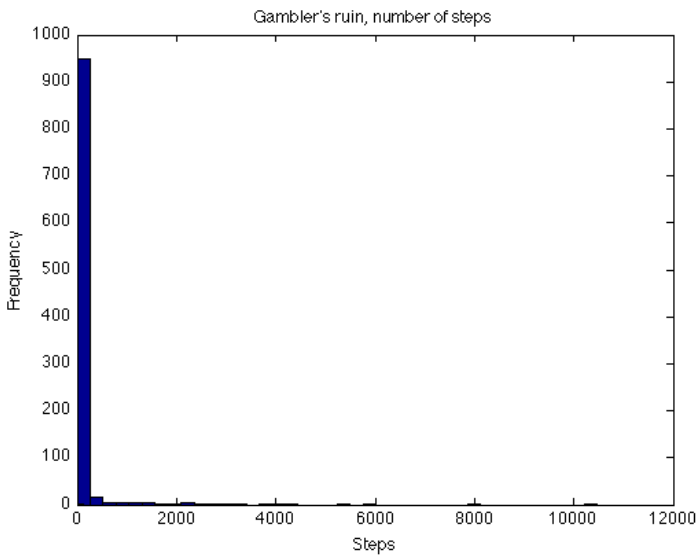




$\mathtt{A} - \$3 \quad \mathtt{B} - \$7$

$\mathtt{A} - \$30 \quad \mathtt{B} - \$70$





A – \$10 B – \$10

A – \$1 B – \$100

Game Theory - Simulating a Duel

Another example in game theory is a duel. Two players $take\ turns$ firing at each other until one is hit. The accuracy, or probability of hitting the opponent on one shot, is known for both players.

In the two person duel there is no strategy (just keep firing), but we want to know the probabilities that either player survives.

If we assume Player A has an accuracy of $\frac{4}{6}$ and Player B an accuracy of $\frac{5}{6}$ and A gets the first shot then Player A can survive if:

1. A hits on the first shot: (probability: 4/6) OR
2. A misses, B misses, A hits (probability: 2/6)*(1/6)*(4/6) OR
3. A misses, B misses, A misses, B misses, A hits:
   (probability: 2/6)*(1/6)*(2/6)*(1/6)*(4/6) OR...

The probability that A survives is thus an infinite sum:

$$\sum_{i=0}^{\infty}((1-\frac{4}{6})(1-\frac{5}{6}))^i\frac{4}{6}$$

This has the form $4/6 * (1 + x + x^2 + x^3 + ...)$ where $0 < x < 1$ so this is equal to $4/6 * 1/(1 - x)$ so the probability that A wins is

$$\frac{p(A)}{p(A)+p(B)-p(A)*p(B)} = \frac{12}{17}$$

while B's chance is

$$\frac{p(B)*(1-p(A))}{p(A)+p(B)-p(A)*p(B)} = \frac{5}{17}$$

We have worked out the exact probabilities in this case however, now we want to simulate duels so we can approximate the probabilities and compare with our exact ones. To *estimate* the probabilities of A and B surviving, we would simulate the duel 100 or 1,000 times, counting the frequency with which each player survived, and dividing by the number of duels. We will also keep track of

the number of turns it takes to complete a duel (i.e., until we get a hit) and then compute the average.

Basically we just need a function which performs the duel and returns the number of shots required and the survivor (either Player 1 or Player 2). The only input that is needed is `p(1:2)` which stores each player's probability. We can then write a script to call this function `n_duel` times (the number of duels we are simulating) and compute the average number of shots required to end the duel. We would also keep a tally of how many duels were won by each player in the calling script so we can estimate probabilities.

How do we decide if the player shooting hits the other player? We generate a random number `r` between 0 and 1; then Player 1 hits his target if `r` $\leq$ `p(1)`; similarly for Player 2. The duel continues until a target is hit.

```
[survivor, turn_num] = function duel (p)
 turn_num = 0;  % count for number of shots required to complete due

 if p(1) <= 0  & p(2) <=0  % make sure input probabilities are valid
```

```
    fprintf ('invalid probabilities so duel fails', /n)
    return
  end

   while ( 1 )
%
%  Player 1 fires.
%
     turn_num = turn_num + 1;
     r = rand ( );
     if ( r <= p(1) ) % Player 1 hits his target
       survivor = 1;
       break  % leave the while loop
     end
%
%  Player 2 fires.
%
     turn_num = turn_num + 1;
     r = rand ( );
```

```
    if ( r <= p(2) ) % Player 2 hits his target
      survivor = 2;
      break
    end

  end
```

Note the use of the `while (1)` here. The expression which must be satisfied in this statement is always true so the only way to escape the loop is through the `break` statement. This is why we added a check at the first of the program to verify that not both probabilities are ≤ 0.

A 3 person duel is more interesting and we will look at it next.

Exercise Download the function `duel.m` which has the code described above implemented. Then write a script which calls this function for each simulation of a duel. The input for your script should be the probabilities of each player

`p(1:2)` and the number of simulations you wish to perform. Your code should (i) compute the average number of shots to complete the duel and (ii) estimate the probabilities and compare with exact. After this is done, make a plot of the average number of shots required versus the exact probability of Player 1.

The Three-Way Duel from "The Good, The Bad, and The Ugly"

In a two person duel there's no strategy (except of course to shoot at your opponent on your turn).

But suppose we move to a three person duel involving A, B, and C. Let's suppose that C is a relatively poor shot.

If A and B are "reasonable", they will probably shoot at each other, since each represents the biggest threat to the other. As soon as one gets hit, it will be C's turn, and that's his best chance, to have the first shot at the remaining opponent. (Remember they take turns shooting.)

But a disaster occurs if, instead of A or B knocking the other out, poor shot C accidentally hits A or B on his turn. Because then C does not get the next shot; rather the survivor gets first shot at C, and we know that whether A or B shoots either is a better shot than C!

Now the coding for this problem is similar to that for the two person duel, except that, on each shot, the player has a choice of whom to shoot at. We said A and B should shoot at each other.

It seems that C should shoot at the better shot, say A, so that if he accidentally hits him, he has to face the weaker player B.

Another possibility: C could shoot at no one (if that's allowed). Then, once A or B is out, C is sure to get the first shot at the survivor.

Coding the Three Person Duel

- The three-way duel is harder. Instead of writing it from scratch let's see what we need to change in the two-way duel code. When we do this, we should see what we would have to revise if we had a four-person duel!
- Now we can't assume that as soon as one person is eliminated, the game is over. So the first thing we have to change is the part of the code that says if the shot is accurate, the game is over and the shooter is the winner.
- Another issue is that now we have a choice of who to aim at. We will take the rational strategy that each player aims at the other player who is the most accurate shot.
- A "brute force" approach to coding this would be to write out what Player 1 will do, then what Player 2 will do and so on. This is straightforward, but cluttered. It might be better to try to abstract what is going on.

- To this end, let's concentrate on one arbitrary player whose index is I (i.e., 1, 2, or 3). If it is I's turn then what should he do?
  - check to see if he is alive; if alive proceed;
  - decide who to aim at (there may be one or two choices) based on their probabilities;
  - generate a random number and see if he hits his target;
  - if he hits his target then check to see if he is the only surviving ;
  - if he is the only survivor then set I as the survivor and return.

Pseudo-code for Player I's turn

```
IF ( I am alive )
: turn_num = turn_num + 1
: IF ( two other players are alive )
: : TARGET = the better one
: ELSE
: : TARGET = the remaining player.
: END
: r = rand ( );
: IF ( r <= p(I) )
: : TARGET is eliminated.
: : IF ( I am the ONLY remaining player now)
: : : survivor = I; break;
: : END
: END
END
```

- To turn our pseudocode into code, we need an efficient way to keep track of who's in the game and who's out.
- One way to do this is to use with the probability array `p`. Once a person has been shot, we could set their `p` value to zero.
- That also makes it easy to choose the target. Always pick the person with the biggest `p` value. But what if that person is yourself!
- One simple thing to do is when it is your turn, temporarily set your probability to zero so you don't choose to shoot yourself. Then reset the value after you've taken your shot (unless you win and it doesn't matter).
- If you have an array `p` then the Matlab command `max(p)` will give you the maximum entry of that array; but if you use `[value,loc]=max(p)` it will give you the maximum entry and its location. So in our case, it will give us the player's number which has the largest probability of hitting his opponent.
- Now there is an easy way to see if everyone has been shot (except you) so that you can be declared the winner. If you hit your target, set the target's probability to 0 (indicating he has been shot). Then all you have to do is sum the probability array `p` and if you get 0, then you are the only survivor.

Remember that you have temporarily set your probability to 0 so you won't shoot yourself.

```
turn_num = turn_num + 1;
p_save = p(i);
p(i) = 0.0;
[ pmax, target ] = max ( p ); % TARGET is index of max.
r = rand ( );
if ( r <= p_save )
  p(target) = 0.0;  % you shot opponent "target"
  if ( sum ( p ) == 0.0 )  % everyone has been shot except you
    survivor = i;
    break;
  end
end
p(i) = p_save;  % reset your probability
```

Conclusion for Three-Way Duel

- This is not the best way, or the only way, but it is a way, and it has some real advantages.
- There are hardly any extra variables (except for `p_save`).
- The code works the same for every player. We never assume that the players were given in a particular order.
- The code works the same if we increase the number of players.
- Your homework will be simulate a three-person duel using this strategy and then to modify your code to change the strategy.