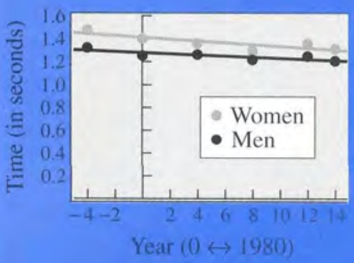
Matrices and Determinants

Chapter 8

The times (in minutes) for the winning men's and women's 1000-meter speed skating events at the winter Olympics are shown below. (In 1994, the winter Olympics occurred only 2 years after the previous winter Olympics.)



Year	Men	Women
1976	1.322	1.474
1980	1.253	1.402
1984	1.263	1.360
1988	1.217	1.294
1992	1.248	1.358
1994	1.207	1.312

By using a graphing utility, you can determine that the best-fitting linear models are

$s = 1.279 - 0.0049t$ Men

$s = 1.411 - 0.0078t$ Women

where s is the time (in minutes) and t is the year, with $t = 0$ representing 1980. According to these two models, the women's times are decreasing a little more rapidly than the men's times. (See Exercises 81 and 82 on page 604.)

Bob Martin / Allsport.

Bonnie Blair won the 1000-meter women's speed skating event in the 1992 and 1994 winter Olympics. These were the first times this event was ever won by an American.

8.1 Matrices and Systems of Equations

Matrices / Elementary Row Operations / Gaussian Elimination with Back-Substitution / Gauss-Jordan Elimination

Matrices

In this section you will study a streamlined technique for solving systems of linear equations. This technique involves the use of a rectangular array of real numbers called a **matrix.**

Note The plural of matrix is *matrices.*

Definition of Matrix

If m and n are positive integers, an $m \times n$ (read "m by n") **matrix** is a rectangular array

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \left.\vphantom{\begin{matrix}a\\a\\a\\a\\a\end{matrix}}\right\} m \text{ rows}$$

in which each **entry** a_{ij} of the matrix is a real number. An $m \times n$ matrix has m **rows** (horizontal lines) and n **columns** (vertical lines).

The entry in the ith row and jth column is denoted by the *double subscript* notation a_{ij}. A matrix having m rows and n columns is said to be of **order** $m \times n$. If $m = n$, the matrix is **square** of order n. For a square matrix, the entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **main diagonal** entries.

EXAMPLE 1 **Examples of Matrices**

a. *Order:* 1 × 1

$$[2]$$

b. *Order:* 1 × 4

$$\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$$

c. *Order:* 2 × 2

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

d. *Order:* 3 × 2

$$\begin{bmatrix} 5 & 0 \\ 2 & -2 \\ -7 & 4 \end{bmatrix}$$

Note A matrix that has only one row is called a **row matrix,** and a matrix that has only one column is called a **column matrix.**

A matrix derived from a system of linear equations (each written in standard form with the constant term on the right) is the **augmented matrix** of the system. Moreover, the matrix derived from the coefficients of the system (but not including the constant terms) is the **coefficient matrix** of the system.

System *Augmented Matrix* *Coefficient Matrix*

$$\begin{array}{rcr} x - 4y + 3z &=& 5 \\ -x + 3y - z &=& -3 \\ 2x \qquad - 4z &=& 6 \end{array} \qquad \left[\begin{array}{rrr:r} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{array}\right] \qquad \begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

Note Note the use of 0 for the missing y-variable in the third equation, and also note the fourth column (of constant terms) in the augmented matrix.

When forming either the coefficient matrix or the augmented matrix of a system, you should begin by vertically aligning the variables in the equations and using 0's for the missing variables.

Given System *Line Up Variables.* *Form Augmented Matrix.*

$$\begin{array}{rcr} x + 3y &=& 9 \\ -y + 4z &=& -2 \\ x - 5z &=& 0 \end{array} \qquad \begin{array}{rrrcr} x + & 3y & & = & 9 \\ & -y & + 4z & = & -2 \\ x & & - 5z & = & 0 \end{array} \qquad \left[\begin{array}{rrr:r} 1 & 3 & 0 & 9 \\ 0 & -1 & 4 & -2 \\ 1 & 0 & -5 & 0 \end{array}\right]$$

Elementary Row Operations

In Section 7.3, you studied three operations that can be used on a system of linear equations to produce an equivalent system.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

In matrix terminology, these three operations correspond to **elementary row operations.** An elementary row operation on an augmented matrix of a given system of linear equations produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are **row-equivalent** if one can be obtained from the other by a sequence of elementary row operations.

Elementary Row Operations

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

Although elementary row operations are simple to perform, they involve a lot of arithmetic. Because it is easy to make a mistake, we suggest that you get in the habit of noting the elementary row operations performed in each step so that you can go back and check your work.

The *Interactive* CD-ROM shows every example with its solution; clicking on the *Try It!* button brings up similar problems. Guided Examples and Integrated Examples show step-by-step solutions to additional examples. Integrated Examples are related to several concepts in the section.

EXAMPLE 2 **Elementary Row Operations**

a. Interchange the first and second rows.

Original Matrix / *New Row-Equivalent Matrix*

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix} \qquad \begin{array}{c} R_2 \\ R_1 \\ \end{array}\begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

b. Multiply the first row by $\frac{1}{2}$.

Original Matrix / *New Row-Equivalent Matrix*

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} \qquad \begin{array}{c} \frac{1}{2}R_1 \rightarrow \\ \\ \end{array}\begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

c. Add -2 times the first row to the third row.

Original Matrix / *New Row-Equivalent Matrix*

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix} \qquad \begin{array}{c} \\ \\ -2R_1 + R_3 \rightarrow \end{array}\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

Note that the elementary row operation is written beside the row that is *changed.*

The *Interactive* CD-ROM offers graphing utility emulators of the *TI-82* and *TI-83*, which can be used with the Examples, Explorations, Technology notes, and Exercises.

Most graphing utilities can perform elementary row operations on matrices. For instance, on a *TI-82* or *TI-83*, you can perform the elementary row operation shown in Example 2(c) as follows.

1. Use the matrix edit feature to enter the matrix as [A].
2. Choose the "* row + (" feature in the matrix math menu.

   * row + (−2, [A], 1, 3) ENTER

The new row-equivalent matrix will be displayed. To do a sequence of row operations, use ANS in place of [A] in each operation. If you want to save this new matrix, you must do this with separate steps.

In Example 2 of Section 7.3, you used Gaussian elimination with back-substitution to solve a system of linear equations. The next example demonstrates the matrix version of Gaussian elimination. The two methods are essentially the same. The basic difference is that with matrices you do not need to keep writing the variables.

EXAMPLE 3 Using Elementary Row Operations

Linear System

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \phantom{{}+5z} &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Associated Augmented Matrix

$$\begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ -1 & 3 & 0 & \vdots & -4 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

Add the first equation to the second equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Add the first row to the second row ($R_1 + R_2$).

$$\begin{array}{r} \\ R_1 + R_2 \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

Add -2 times the first equation to the third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$$

Add -2 times the first row to the third row ($-2R_1 + R_3$).

$$\begin{array}{r} \\ \\ -2R_1 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & -1 & -1 & \vdots & -1 \end{bmatrix}$$

Add the second equation to the third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$$

Add the second row to the third row ($R_2 + R_3$).

$$\begin{array}{r} \\ \\ R_2 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 2 & \vdots & 4 \end{bmatrix}$$

Multiply the third equation by $\frac{1}{2}$.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Multiply the third row by $\frac{1}{2}$.

$$\begin{array}{r} \\ \\ \frac{1}{2}R_3 \rightarrow \end{array} \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

At this point, you can use back-substitution to find that the solution is $x = 1$, $y = -1$, and $z = 2$, as was done in Example 2 of Section 7.3.

Note Remember that you can check a solution by substituting the values of x, y, and z into each equation in the original system.

The last matrix in Example 3 is said to be in **row-echelon form.** The term *echelon* refers to the stair-step pattern formed by the nonzero elements of the matrix. To be in this form, a matrix must have the following properties.

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the following properties.

1. All rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in *row-echelon form* is in **reduced row-echelon form** if every column that has a leading 1 has zeros in every position above and below its leading 1.

Some graphing utilities, such as the *TI-85*, *TI-92*, and *HP-48G*, can automatically transform a matrix to row-echelon form and reduced row-echelon form. Read your user's manual to see if your calculator has this capability. If so, use it to verify the results in this section.

EXAMPLE 4 Row-Echelon Form

The following matrices are in row-echelon form.

a. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The matrices in (b) and (d) also happen to be in *reduced* row-echelon form. The following matrices are not in row-echelon form.

e. $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ **f.** $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

Every matrix has a row-equivalent matrix that is in row-echelon form. For instance, in Example 4, you can change the matrix in part (e) to row-echelon form by multiplying its second row by $\frac{1}{2}$. What elementary row operation could you perform on the matrix in part (f) so that it would be in row-echelon form?

Gaussian Elimination with Back-Substitution

EXAMPLE 5 Gaussian Elimination with Back-Substitution

Solve the system.

$$\begin{aligned} y + z - 2w &= -3 \\ x + 2y - z &= 2 \\ 2x + 4y + z - 3w &= -2 \\ x - 4y - 7z - w &= -19 \end{aligned}$$

> **Study Tip**
>
> Gaussian elimination with back-substitution works well for solving systems of linear equations by hand or with a computer. For this algorithm, the order in which the elementary row operations are performed is important. We suggest operating from *left to right by columns,* using elementary row operations to obtain zeros in all entries directly below the leading 1's.

Solution

$$\begin{array}{r} R_2 \\ R_1 \\ \\ \\ \end{array} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right]$$

First column has leading 1 in upper left corner.

$$\begin{array}{r} \\ \\ -2R_1 + R_3 \rightarrow \\ -R_1 + R_4 \rightarrow \end{array} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{array}\right]$$

First column has zeros below its leading 1.

$$\begin{array}{r} \\ \\ \\ 6R_2 + R_4 \rightarrow \end{array} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{array}\right]$$

Second column has zeros below its leading 1.

$$\begin{array}{r} \\ \\ \frac{1}{3}R_3 \rightarrow \\ \\ \end{array} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -13 & -39 \end{array}\right]$$

Third column has zeros below its leading 1.

$$\begin{array}{r} \\ \\ \\ -\frac{1}{13}R_4 \rightarrow \end{array} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Fourth column has a leading 1.

The matrix is now in row-echelon form, and the corresponding system is

$$\begin{aligned} x + 2y - z &= 2 \\ y + z - 2w &= -3 \\ z - w &= -2 \\ w &= 3. \end{aligned}$$

Using back-substitution, you can determine that the solution is $x = -1$, $y = 2$, $z = 1$, and $w = 3$. Check this in the original system of equations.

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

When solving a system of linear equations, remember that it is possible for the system to have no solution. If, in the elimination process, you obtain a row with zeros except for the last entry, it is unnecessary to continue the elimination process. You can simply conclude that the system is inconsistent.

EXAMPLE 6 A System with No Solution

Solve the system.

$$\begin{aligned} x - y + 2z &= 4 \\ x + z &= 6 \\ 2x - 3y + 5z &= 4 \\ 3x + 2y - z &= 1 \end{aligned}$$

Solution

$$\begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 1 & 0 & 1 & \vdots & 6 \\ 2 & -3 & 5 & \vdots & 4 \\ 3 & 2 & -1 & \vdots & 1 \end{bmatrix} \qquad \begin{matrix} \\ -R_1 + R_2 \rightarrow \\ -2R_1 + R_3 \rightarrow \\ -3R_1 + R_4 \rightarrow \end{matrix} \begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & -1 & 1 & \vdots & -4 \\ 0 & 5 & -7 & \vdots & -11 \end{bmatrix}$$

$$\begin{matrix} \\ \\ R_2 + R_3 \rightarrow \\ \\ \end{matrix} \begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & -2 \\ 0 & 5 & -7 & \vdots & -11 \end{bmatrix}$$

Note that the third row of this matrix consists of zeros except for the last entry. This means that the original system of linear equations is *inconsistent*. You can see why this is true by converting back to a system of linear equations.

$$\begin{aligned} x - y + 2z &= 4 \\ y - z &= 2 \\ 0 &= -2 \\ 5y - 7z &= -11 \end{aligned}$$

Because the third equation is not possible, the system has no solution.

Gauss-Jordan Elimination

With Gaussian elimination, elementary row operations are applied to a matrix to obtain a (row-equivalent) row-echelon form. A second method of elimination, called **Gauss-Jordan elimination,** after Carl Friedrich Gauss (1777–1855) and Wilhelm Jordan (1842–1899), continues the reduction process until a *reduced* row-echelon form is obtained. This procedure is demonstrated in the following example.

For a demonstration of a graphical approach to Gauss-Jordan elimination on a 2 × 3 matrix, see the graphing calculator program for this section in the appendix.

EXAMPLE 7 Gauss-Jordan Elimination

Use Gauss-Jordan elimination to solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Solution

In Example 3, Gaussian elimination was used to obtain the row-echelon form

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

Now, rather than using back-substitution, apply additional elementary row operations until you obtain a matrix in *reduced* row-echelon form. To do this, you must produce zeros above each of the leading 1's, as follows.

$$\begin{array}{r} 2R_2 + R_1 \rightarrow \\ \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Second column has zeros above its leading 1.

$$\begin{array}{r} -9R_3 + R_1 \rightarrow \\ -3R_3 + R_2 \rightarrow \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Third column has zeros above its leading 1.

Now, converting back to a system of linear equations, you have

$$\begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2. \end{aligned}$$

The beauty of Gauss-Jordan elimination is that, from the reduced row-echelon form, you can simply read the solution.

Note *Which technique do you prefer:* Gaussian elimination or Gauss-Jordan elimination?

The elimination procedures described in this section employ an algorithmic approach that is easily adapted to computer use. However, the procedure makes no effort to avoid fractional coefficients. For instance, if the system given in Example 7 had been listed as

$$\begin{aligned} 2x - 5y + 5z &= 17 \\ x - 2y + 3z &= 9 \\ -x + 3y &= -4 \end{aligned}$$

the procedure would have required multiplication of the first row by $\frac{1}{2}$, which would have introduced fractions in the first row. For hand computations, fractions can sometimes be avoided by judiciously choosing the order in which the elementary row operations are applied.

EXAMPLE 8 A System with an Infinite Number of Solutions

Solve the system.

$$\begin{aligned} 2x + 4y - 2z &= 0 \\ 3x + 5y &= 1 \end{aligned}$$

Solution

$$\begin{bmatrix} 2 & 4 & -2 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix} \qquad \tfrac{1}{2}R_1 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix}$$

$$-3R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & -1 & 3 & \vdots & 1 \end{bmatrix}$$

$$-R_2 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

$$-2R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & 5 & \vdots & 2 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned} x + 5z &= 2 \\ y - 3z &= -1. \end{aligned}$$

Solving for x and y in terms of z, you have $x = -5z + 2$ and $y = 3z - 1$. Then, letting $z = a$, the solution set has the form

$$(-5a + 2, 3a - 1, a)$$

where a is a real number. Try substituting values for a to obtain a few solutions. Then check each solution in the original system of equations.

Note You have seen that the row-echelon form of a given matrix *is not* unique; however, the *reduced* row-echelon form of a given matrix *is* unique. Try applying Gauss-Jordan elimination to the row-echelon matrix given at the right to see that you obtain the same reduced row-echelon form as in Example 7.

It is worth noting that the row-echelon form of a matrix is not unique. That is, two different sequences of elementary row operations may yield different row-echelon forms. For instance, the following sequence of elementary row operations on the matrix in Example 3 produces a slightly different row-echelon form.

$$\begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ -1 & 3 & 0 & \vdots & -4 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix} \qquad \begin{array}{r} \curvearrowright R_2 \\ R_1 \\ \end{array} \begin{bmatrix} -1 & 3 & 0 & \vdots & -4 \\ 1 & -2 & 3 & \vdots & 9 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

$$\begin{array}{r} -R_1 \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} 1 & -3 & 0 & \vdots & 4 \\ 1 & -2 & 3 & \vdots & 9 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

$$\begin{array}{r} \\ -R_1 + R_2 \rightarrow \\ -2R_1 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & -3 & 0 & \vdots & 4 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 1 & 5 & \vdots & 9 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ -R_2 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & -3 & 0 & \vdots & 4 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 2 & \vdots & 4 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ \frac{1}{2}R_3 \rightarrow \end{array} \begin{bmatrix} 1 & -3 & 0 & \vdots & 4 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

The corresponding system of linear equations is

$$\begin{aligned} x - 3y \phantom{{}+3z} &= 4 \\ y + 3z &= 5 \\ z &= 2. \end{aligned}$$

Try using back-substitution on this system to see that you obtain the same solution that was obtained in Example 3.

Group Activity Error Analysis

One of your classmates has submitted the following steps for a solution of a system by Gauss-Jordan elimination. Find the error(s) in the solution and discuss how to explain the error(s) to your classmate.

$$\xcancel{\begin{bmatrix} 1 & 1 & \vdots & 4 \\ 2 & 3 & \vdots & 5 \end{bmatrix} \begin{array}{r} \\ -2R_1 + R_2 \rightarrow \\ -1R_2 + R_1 \rightarrow \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 1 & 1 & \vdots & 4 \\ 0 & 1 & \vdots & 5 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & \vdots & 4 \\ 0 & 1 & \vdots & 5 \end{bmatrix} \end{array}}$$

8.1 /// EXERCISES

In Exercises 1–6, determine the order of the matrix.

1. $\begin{bmatrix} 4 & -2 \\ 7 & 0 \\ 0 & 8 \end{bmatrix}$

2. $\begin{bmatrix} 5 & -3 & 8 & 7 \end{bmatrix}$

3. $\begin{bmatrix} 2 \\ 36 \\ 3 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 7 & 15 & 0 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 6 & 7 \end{bmatrix}$

5. $\begin{bmatrix} 33 & 45 \\ -9 & 20 \end{bmatrix}$

6. $\begin{bmatrix} 4 \end{bmatrix}$

In Exercises 7–10, form the augmented matrix for the system of linear equations.

7. $\begin{aligned} 4x - 3y &= -5 \\ -x + 3y &= 12 \end{aligned}$

8. $\begin{aligned} 7x + 4y &= 22 \\ 5x - 9y &= 15 \end{aligned}$

9. $\begin{aligned} x + 10y - 2z &= 2 \\ 5x - 3y + 4z &= 0 \\ 2x + y &= 6 \end{aligned}$

10. $\begin{aligned} 7x - 5y + z &= 13 \\ 19x - 8z &= 10 \end{aligned}$

In Exercises 11–14, write the system of linear equations represented by the augmented matrix. (Use variables x, y, z, and w.)

11. $\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 2 & -3 & 4 \end{array}\right]$

12. $\left[\begin{array}{cc|c} 7 & -5 & 0 \\ 8 & 3 & -2 \end{array}\right]$

13. $\left[\begin{array}{ccc|c} 2 & 0 & 5 & -12 \\ 0 & 1 & -2 & 7 \\ 6 & 3 & 0 & 2 \end{array}\right]$

14. $\left[\begin{array}{cccc|c} 9 & 12 & 3 & 0 & 0 \\ -2 & 18 & 5 & 2 & 10 \\ 1 & 7 & -8 & 0 & -4 \end{array}\right]$

In Exercises 15–18, determine whether the matrix is in row-echelon form. If it is, determine if it is also in reduced row-echelon form.

15. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 2 & 0 & 4 & 0 \\ 0 & -1 & 3 & 6 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & 10 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

In Exercises 19–22, fill in the blanks using elementary row operations to form a row-equivalent matrix.

19. $\begin{bmatrix} 1 & 4 & 3 \\ 2 & 10 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 & 4 & 3 \\ 0 & & -1 \end{bmatrix}$

20. $\begin{bmatrix} 3 & 6 & 8 \\ 4 & -3 & 6 \end{bmatrix}$

$\begin{bmatrix} 1 & & \frac{8}{3} \\ 4 & -3 & 6 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 1 & 4 & -1 \\ 3 & 8 & 10 & 3 \\ -2 & 1 & 12 & 6 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 5 & & \\ 0 & 3 & & \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 1 & -\frac{2}{5} & \frac{6}{5} \\ 0 & 3 & & \end{bmatrix}$

22. $\begin{bmatrix} 2 & 4 & 8 & 3 \\ 1 & -1 & -3 & 2 \\ 2 & 6 & 4 & 9 \end{bmatrix}$

$\begin{bmatrix} 1 & & & \\ 1 & -1 & -3 & 2 \\ 2 & 6 & 4 & 9 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 4 & \frac{3}{2} \\ 0 & & -7 & \frac{1}{2} \\ 0 & 2 & & \end{bmatrix}$

23. Perform the *sequence* of row operations on the matrix. What did the operations accomplish?

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -4 \\ 3 & 1 & -1 \end{bmatrix}$

(a) Add -2 times Row 1 to Row 2.

(b) Add -3 times Row 1 to Row 3.

(c) Add -1 times Row 2 to Row 3.

(d) Multiply Row 2 by $-\frac{1}{5}$.

(e) Add -2 times Row 2 to Row 1.

The *Interactive* CD-ROM contains step-by-step solutions to all odd-numbered Section and Review Exercises. It also provides Tutorial Exercises, which link to Guided Examples for additional help.

24. Perform the *sequence* of row operations on the matrix. What did the operations accomplish?

$$\begin{bmatrix} 7 & 1 \\ 0 & 2 \\ -3 & 4 \\ 4 & 1 \end{bmatrix}$$

(a) Add Row 3 to Row 4.

(b) Interchange Rows 1 and 4.

(c) Add 3 times Row 1 to Row 3.

(d) Add -7 times Row 1 to Row 4.

(e) Multiply Row 2 by $\frac{1}{2}$.

(f) Add the appropriate multiples of Row 2 to Rows 1, 3, and 4.

In Exercises 25–28, write the matrix in row-echelon form. Remember that the row-echelon form of a matrix is not unique.

25. $\begin{bmatrix} 1 & 1 & 0 & 5 \\ -2 & -1 & 2 & -10 \\ 3 & 6 & 7 & 14 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 7 & -5 & 14 \\ -2 & -1 & -3 & 8 \end{bmatrix}$

27. $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 5 & -4 & 1 & 8 \\ -6 & 8 & 18 & 0 \end{bmatrix}$

28. $\begin{bmatrix} 1 & -3 & 0 & -7 \\ -3 & 10 & 1 & 23 \\ 4 & -10 & 2 & -24 \end{bmatrix}$

In Exercises 29–32, use the matrix capabilities of a graphing utility to write the matrix in *reduced* row-echelon form.

29. $\begin{bmatrix} 3 & 3 & 3 \\ -1 & 0 & -4 \\ 2 & 4 & -2 \end{bmatrix}$

30. $\begin{bmatrix} 1 & 3 & 2 \\ 5 & 15 & 9 \\ 2 & 6 & 10 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 2 & 3 & -5 \\ 1 & 2 & 4 & -9 \\ -2 & -4 & -4 & 3 \\ 4 & 8 & 11 & -14 \end{bmatrix}$

32. $\begin{bmatrix} 1 & -3 \\ -1 & 8 \\ 0 & 4 \\ -2 & 10 \end{bmatrix}$

In Exercises 33–36, write the system of linear equations represented by the augmented matrix. Then use back-substitution to find the solution. (Use variables x, y, and z.)

33. $\left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -3 \end{array}\right]$

34. $\left[\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 1 & -1 \end{array}\right]$

35. $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$

36. $\left[\begin{array}{ccc|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & -3 \end{array}\right]$

In Exercises 37–40, an augmented matrix that represents a system of linear equations (in variables x, y, and z) has been reduced using Gauss-Jordan elimination. Write the solution represented by the augmented matrix.

37. $\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -5 \end{array}\right]$

38. $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 4 \end{array}\right]$

39. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 2 \end{array}\right]$

40. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

In Exercises 41–56, solve the system of equations. Use Gaussian elimination with back-substitution or Gauss-Jordan elimination.

41. $\begin{aligned} x + 2y &= 7 \\ 2x + y &= 8 \end{aligned}$

42. $\begin{aligned} 2x + 6y &= 16 \\ 2x + 3y &= 7 \end{aligned}$

43. $\begin{aligned} -3x + 5y &= -22 \\ 3x + 4y &= 4 \\ 4x - 8y &= 32 \end{aligned}$

44. $\begin{aligned} x + 2y &= 0 \\ x + y &= 6 \\ 3x - 2y &= 8 \end{aligned}$

45. $\begin{aligned} 8x - 4y &= 7 \\ 5x + 2y &= 1 \end{aligned}$

46. $\begin{aligned} 2x - y &= -0.1 \\ 3x + 2y &= 1.6 \end{aligned}$

47. $\begin{aligned} -x + 2y &= 1.5 \\ 2x - 4y &= 3 \end{aligned}$

48. $\begin{aligned} x - 3y &= 5 \\ -2x + 6y &= -10 \end{aligned}$

49. $\begin{aligned} x - 3z &= -2 \\ 3x + y - 2z &= 5 \\ 2x + 2y + z &= 4 \end{aligned}$

50. $\begin{aligned} 2x - y + 3z &= 24 \\ 2y - z &= 14 \\ 7x - 5y &= 6 \end{aligned}$

51. $\begin{aligned} x + y - 5z &= 3 \\ x - 2z &= 1 \\ 2x - y - z &= 0 \end{aligned}$

52. $\begin{aligned} 2x + 3z &= 3 \\ 4x - 3y + 7z &= 5 \\ 8x - 9y + 15z &= 9 \end{aligned}$

53. $\begin{aligned} x + 2y + z &= 8 \\ 3x + 7y + 6z &= 26 \end{aligned}$

54. $\begin{aligned} 4x + 12y - 7z - 20w &= 22 \\ 3x + 9y - 5z - 28w &= 30 \end{aligned}$

55. $\begin{aligned} x + 2y &= 0 \\ -x - y &= 0 \end{aligned}$

56. $\begin{aligned} x + 2y &= 0 \\ 2x + 4y &= 0 \end{aligned}$

In Exercises 57–62, use the matrix capabilities of a graphing utility to reduce the augmented matrix and solve the system of equations.

57. $\begin{aligned} 3x + 3y + 12z &= 6 \\ x + y + 4z &= 2 \\ 2x + 5y + 20z &= 10 \\ -x + 2y + 8z &= 4 \end{aligned}$

58. $\begin{aligned} 2x + 10y + 2z &= 6 \\ x + 5y + 2z &= 6 \\ x + 5y + z &= 3 \\ -3x - 15y - 3z &= -9 \end{aligned}$

59. $\begin{aligned} 2x + y - z + 2w &= -6 \\ 3x + 4y + w &= 1 \\ x + 5y + 2z + 6w &= -3 \\ 5x + 2y - z - w &= 3 \end{aligned}$

60. $\begin{aligned} x + 2y + 2z + 4w &= 11 \\ 3x + 6y + 5z + 12w &= 30 \end{aligned}$

61. $\begin{aligned} x + y + z &= 0 \\ 2x + 3y + z &= 0 \\ 3x + 5y + z &= 0 \end{aligned}$

62. $\begin{aligned} x + 2y + z + 3w &= 0 \\ x - y + w &= 0 \\ y - z + 2w &= 0 \end{aligned}$

63. *Think About It* The augmented matrix represents a system of linear equations (in variables x, y, and z) that has been reduced using Gauss-Jordan elimination. Write a system of equations with nonzero coefficients that is represented by the reduced matrix. (The answer is not unique.)

$$\left[\begin{array}{ccc:c} 1 & 0 & 3 & -2 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

64. *Think About It*

(a) Describe the row-echelon form of an augmented matrix that corresponds to a system of linear equations that is inconsistent.

(b) Describe the row-echelon form of an augmented matrix that corresponds to a system of linear equations that has an infinite number of solutions.

65. *Borrowing Money* A small corporation borrowed \$1,500,000 to expand its product line. Some of the money was borrowed at 8%, some at 9%, and some at 12%. How much was borrowed at each rate if the annual interest was \$133,000 and the amount borrowed at 8% was 4 times the amount borrowed at 12%?

66. *Borrowing Money* A small corporation borrowed \$500,000 to expand its product line. Some of the money was borrowed at 9%, some at 10%, and some at 12%. How much was borrowed at each rate if the annual interest was \$52,000 and the amount borrowed at 10% was $2\frac{1}{2}$ times the amount borrowed at 9%?

67. *Partial Fractions* Write the partial fraction decomposition for $(4x^2)/[(x + 1)^2(x - 1)]$.

68. *Electrical Network* The currents in an electrical network are given by the solution of the system

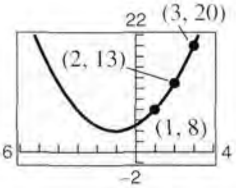
$$\begin{aligned} I_1 - I_2 + I_3 &= 0 \\ 2I_1 + 2I_2 \qquad &= 7 \\ 2I_2 + 4I_3 &= 8 \end{aligned}$$

where I_1, I_2, and I_3 are measured in amperes. Solve the system of equations.

In Exercises 69–74, find the specified equation that passes through the points. Use a graphing utility to verify your result.

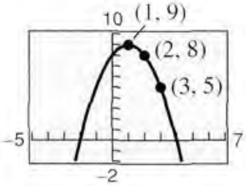
69. Parabola:

$y = ax^2 + bx + c$



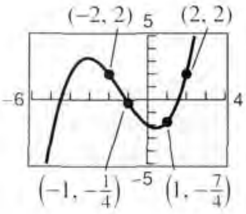
70. Parabola:

$y = ax^2 + bx + c$



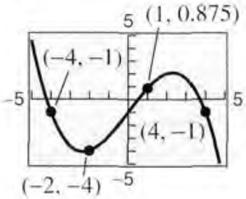
71. Cubic:

$y = ax^3 + bx^2 + cx + d$



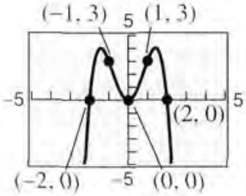
72. Cubic:

$y = ax^3 + bx^2 + cx + d$



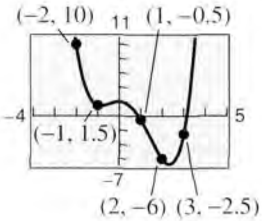
73. Quartic:

$y = ax^4 + \cdots + dx + e$



74. Quartic:

$y = ax^4 + \cdots + dx + e$

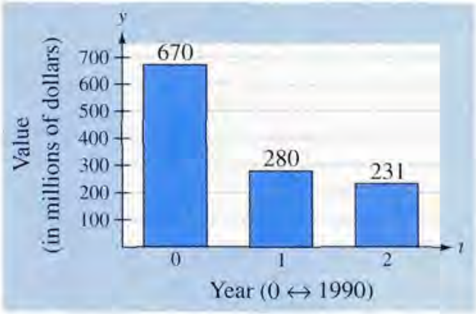


75. *Reading a Graph* The bar graph gives the value y, in millions of dollars, for new orders of civil jet transport aircraft built by U.S. companies in the years 1990 through 1992. (Source: Aerospace Industries Association of America)

(a) Find the equation of the parabola that passes through the points. Let $t = 0$ represent 1990.

(b) Use a graphing utility to graph the parabola.

(c) Use the equation in part (a) to estimate y in 1993.

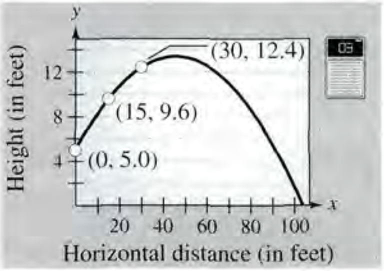


76. *Mathematical Modeling* After the path of a ball thrown by a baseball player is videotaped, it is analyzed on a television set with a grid covering the screen. The tape is paused three times, and the position of the ball is measured each time. The coordinates are approximately (0, 5.0), (15, 9.6), and (30, 12.4). (The x-coordinate measures the horizontal distance from the player in feet, and the y-coordinate is the height of the ball in feet.)

(a) Find the equation of the parabola $y = ax^2 + bx + c$ that passes through the three points.

(b) Use a graphing utility to graph the parabola. Approximate the maximum height of the ball and the point at which the ball strikes the ground.

(c) Find analytically the maximum height of the ball and the point at which it strikes the ground.



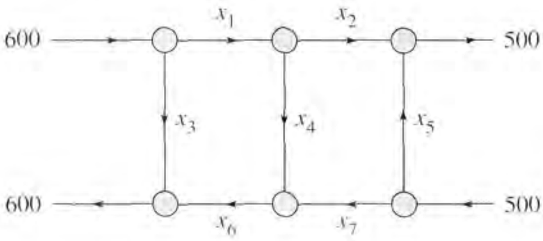
Network Analysis **In Exercises 77–80, answer the questions about the specified network. (In a network it is assumed that the total flow into each junction is equal to the total flow out of the junction.)**

77. Water is flowing through a network of pipes (in thousands of cubic meters per hour). (See figure.)

(a) Solve this system for the water flow represented by x_i, $i = 1, 2, 3, 4, 5, 6, 7$.

(b) Find the network flow pattern when $x_6 = x_7 = 0$.

(c) Find the network flow pattern when $x_5 = 1000$ and $x_6 = 0$.

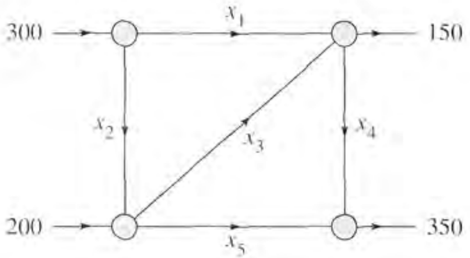


78. The flow of traffic (in vehicles per hour) through a network of streets is shown in the figure.

(a) Solve this system for the traffic flow represented by x_i, $i = 1, 2, 3, 4, 5$.

(b) Find the traffic flow when $x_2 = 200$ and $x_3 = 50$.

(c) Find the traffic flow when $x_2 = 150$ and $x_3 = 0$.



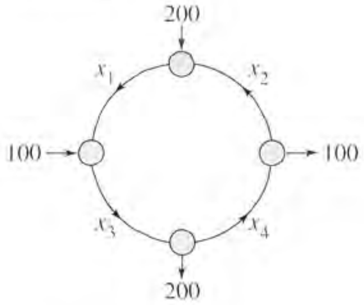
79. The flow of traffic (in vehicles per hour) through a network of streets is shown in the figure.

(a) Solve this system for the traffic flow represented by x_i, $i = 1, 2, 3, 4$.

(b) Find the traffic flow when $x_4 = 0$.

(c) Find the traffic flow when $x_4 = 100$.

Figure for 79

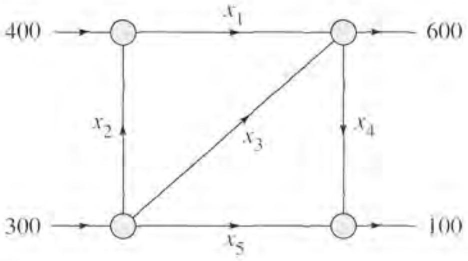


80. The flow of traffic (in vehicles per hour) through a network of streets is shown in the figure.

(a) Solve this system for the traffic flow represented by x_i, $i = 1, 2, 3, 4, 5$.

(b) Find the traffic flow when $x_3 = 0$ and $x_5 = 100$.

(c) Find the traffic flow when $x_3 = x_5 = 100$.



81. *Chapter Opener* Use the models on page 589 to estimate the men's and women's winning times in the 1000-meter speed skating events in the year 2002.

82. *Chapter Opener* If the models on page 589 continue to represent the winning times in the 1000-meter speed skating events, in which winter Olympics will the women's time be less than the men's time?

Review **Solve Exercises 83–86 as a review of the skills and problem-solving techniques you learned in previous sections. Graph the function, and check each graph with a graphing utility.**

83. $f(x) = 2^{x-1}$

84. $g(x) = 3^{-x/2}$

85. $h(x) = \log_2(x - 1)$

86. $f(x) = 3 + \ln x$

8.2 Operations with Matrices

Equality of Matrices / Matrix Addition and Scalar Multiplication / Matrix Multiplication / Applications

Equality of Matrices

In Section 8.1, you used matrices to solve systems of linear equations. Matrices, however, can do much more than this. There is a rich mathematical theory of matrices, and its applications are numerous. This section and the next two introduce some fundamentals of matrix theory. It is standard mathematical convention to represent matrices in any of the following three ways.

1. A matrix can be denoted by an uppercase letter such as A, B, or C.
2. A matrix can be denoted by a representative element enclosed in brackets, such as $[a_{ij}]$, $[b_{ij}]$, or $[c_{ij}]$.
3. A matrix can be denoted by a rectangular array of numbers such as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}.$$

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if they have the same order $(m \times n)$ and $a_{ij} = b_{ij}$ for $1 \le i \le m$ and $1 \le j \le n$. In other words, two matrices are equal if their corresponding entries are equal.

A British mathematician, Arthur Cayley, invented matrices around 1858. Cayley was a Cambridge University graduate and a lawyer by profession. His ground-breaking work on matrices was begun as he studied the theory of transformations. Cayley also was instrumental in the development of determinants. Cayley and two American mathematicians, Benjamin Peirce (1809–1880) and his son Charles S. Peirce (1839–1914), are credited with developing "matrix algebra."

EXAMPLE 1 Equality of Matrices

Solve for a_{11}, a_{12}, a_{21}, and a_{22} in the following matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$$

Solution

Because two matrices are equal only if their corresponding entries are equal, you can conclude that

$$a_{11} = 2, \quad a_{12} = -1, \quad a_{21} = -3, \quad \text{and} \quad a_{22} = 0.$$

Matrix Addition and Scalar Multiplication

You can **add** two matrices (of the same order) by adding their corresponding entries.

Definition of Matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of order $m \times n$, their **sum** is the $m \times n$ matrix given by

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices of different orders is undefined.

Most graphing utilities can perform matrix addition and scalar multiplication. If you have such a graphing utility, duplicate the matrix operations in Examples 2 and 3. Try adding two matrices of different orders such as

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and}$$

$$B = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

What error message does your utility display?

EXAMPLE 2 Addition of Matrices

a. $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

d. The sum of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$$

is undefined.

In work with matrices, numbers are usually referred to as **scalars.** In this text, scalars will always be real numbers. You can multiply a matrix A by a scalar c by multiplying each entry in A by c.

Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, the **scalar multiple** of A by c is the $m \times n$ matrix given by

$$cA = [ca_{ij}].$$

The symbol $-A$ represents the scalar product $(-1)A$. Moreover, if A and B are of the same order, $A - B$ represents the sum of A and $(-1)B$. That is,

$$A - B = A + (-1)B. \qquad \text{Subtraction of matrices}$$

EXAMPLE 3 Scalar Multiplication and Matrix Subtraction

For the following matrices, find (a) $3A$, (b) $-B$, and (c) $3A - B$.

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Solution

a. $$3A = 3\begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \qquad \text{Scalar multiplication}$$

$$= \begin{bmatrix} 3(2) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} \qquad \text{Multiply each entry by 3.}$$

$$= \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} \qquad \text{Simplify.}$$

b. $$-B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} \qquad \text{Definition of negation}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix} \qquad \text{Multiply each entry by } -1.$$

c. $$3A - B = \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} \qquad \text{Matrix subtraction}$$

$$= \begin{bmatrix} 4 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix} \qquad \text{Subtract corresponding entries.}$$

EXPLORATION

Select two 3×2 matrices A and B. Enter them into your graphing utility and calculate $A + B$ and $B + A$. What do you observe?

Now select a real number c and calculate $c(A + B)$ and $cA + cB$. What do you observe?

It is often convenient to rewrite the scalar multiple cA by factoring c out of every entry in the matrix. For instance, in the following example, the scalar $\frac{1}{2}$ has been factored out of the matrix.

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{5}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ 5 & 1 \end{bmatrix}$$

The properties of matrix addition and scalar multiplication are similar to those of addition and multiplication of real numbers.

Properties of Matrix Addition and Scalar Multiplication

Let A, B, and C be $m \times n$ matrices and let c and d be scalars.

1. $A + B = B + A$	Commutative Property of Matrix Addition
2. $A + (B + C) = (A + B) + C$	Associative Property of Matrix Addition
3. $(cd)A = c(dA)$	Associative Property of Scalar Multiplication
4. $1A = A$	Scalar Identity
5. $c(A + B) = cA + cB$	Distributive Property
6. $(c + d)A = cA + dA$	Distributive Property

Note that the Associative Property of Matrix Addition allows you to write expressions such as $A + B + C$ without ambiguity because the same sum occurs no matter how the matrices are grouped. In other words, you obtain the same sum whether you group $A + B + C$ as $(A + B) + C$ or as $A + (B + C)$. This same reasoning applies to sums of four or more matrices.

EXAMPLE 4 Addition of More Than Two Matrices

By adding corresponding entries, you obtain the following sum of four matrices.

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Most graphing utilities can add and subtract matrices and multiply matrices by scalars. For instance, on a *TI-82* or *TI-83*, you can find the sum of

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 4 \\ 2 & -5 \end{bmatrix}$$

by entering the matrices and then using the following keystrokes.

[*A*] + [*B*] ENTER

One important property of addition of real numbers is that the number 0 is the additive identity. That is, $c + 0 = c$ for any real number c. For matrices, a similar property holds. That is, if A is an $m \times n$ matrix and O is the $m \times n$ **zero matrix** consisting entirely of zeros, then $A + O = A$.

In other words, O is the **additive identity** for the set of all $m \times n$ matrices. For example, the following matrices are the additive identities for the set of all 2×3 and 2×2 matrices.

$$O = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Zero } 2 \times 3 \text{ matrix}} \quad \text{and} \quad O = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{Zero } 2 \times 2 \text{ matrix}}$$

Note The algebra of real numbers and the algebra of matrices also have important differences, which will be discussed later.

The algebra of real numbers and the algebra of matrices have many similarities. For example, compare the following solutions.

Real Numbers *(Solve for x.)*	*$m \times n$ Matrices* *(Solve for X.)*
$x + a = b$	$X + A = B$
$x + a + (-a) = b + (-a)$	$X + A + (-A) = B + (-A)$
$x + 0 = b - a$	$X + O = B - A$
$x = b - a$	$X = B - A$

EXAMPLE 5 Solving a Matrix Equation

Solve for X in the equation $3X + A = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

Solution

Begin by solving the equation for X to obtain

$$3X = B - A \quad \Longrightarrow \quad X = \frac{1}{3}(B - A).$$

Now, using the matrices A and B, you have

$$\begin{aligned} X &= \frac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right) \\ &= \frac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}. \end{aligned}$$

Matrix Multiplication

The third basic matrix operation is **matrix multiplication.** At first glance, the following definition may seem unusual. You will see later, however, that this definition of the product of two matrices has many practical applications.

Note The definition of matrix multiplication indicates a *row-by-column* multiplication, where the entry in the ith row and jth column of the product AB is obtained by multiplying the entries in the ith row of A by the corresponding entries in the jth column of B and then adding the results. Example 6 illustrates this process.

Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, the **product** AB is an $m \times p$ matrix

$$AB = [c_{ij}]$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdot \cdot \cdot + a_{in}b_{nj}$.

EXAMPLE 6 Finding the Product of Two Matrices

Find the product AB where

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}.$$

Solution

First, note that the product AB is defined because the number of columns of A is equal to the number of rows of B. Moreover, the product AB has order 3×2, and is of the form

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

To find the entries of the product, multiply each row of A by each column of B, as follows. Use a graphing utility to check this result.

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(-3) + (3)(-4) & (-1)(2) + (3)(1) \\ (4)(-3) + (-2)(-4) & (4)(2) + (-2)(1) \\ (5)(-3) + (0)(-4) & (5)(2) + (0)(1) \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

Some graphing utilities, such as the *TI-82* and *TI-83*, are able to add, subtract, and multiply matrices. If you have such a graphing utility, enter the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -5 & 1 \end{bmatrix} \quad \text{and}$$

$$B = \begin{bmatrix} -3 & 2 & 1 \\ 4 & -2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

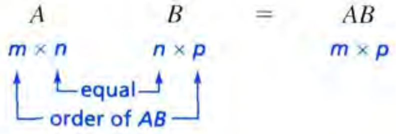
and use the following keystrokes to find the product of the matrices.

[A] ☐× [B] ☐ENTER

You should get:

$$\begin{bmatrix} 8 & 4 & 10 \\ -25 & 16 & 5 \end{bmatrix}.$$

Be sure you understand that for the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix. That is, the middle two indices must be the same and the outside two indices give the order of the product, as shown in the following diagram.



EXPLORATION

Use a graphing utility to multiply the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and }$$

$$B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}.$$

Do you obtain the same result for the product AB as for the product BA? What does this tell you about matrix multiplication and commutativity?

EXAMPLE 7 Matrix Multiplication

a. $$\underset{2 \times 3}{\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}} \underset{3 \times 3}{\begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}} = \underset{2 \times 3}{\begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}}$$

b. $$\underset{2 \times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}} \underset{2 \times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{2 \times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}}$$

c. $$\underset{2 \times 2}{\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}} \underset{2 \times 2}{\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}} = \underset{2 \times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

d. $$\underset{1 \times 3}{\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}} \underset{3 \times 1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} = \underset{1 \times 1}{\begin{bmatrix} 1 \end{bmatrix}}$$

e. $$\underset{3 \times 1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} \underset{1 \times 3}{\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}} = \underset{3 \times 3}{\begin{bmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}}$$

f. The product AB for the following matrices is not defined.

$$A = \underset{3 \times 2}{\begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 1 & 4 \end{bmatrix}} \quad \text{and} \quad B = \underset{3 \times 4}{\begin{bmatrix} -2 & 3 & 1 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -1 & 0 & 1 \end{bmatrix}}$$

Note In parts (d) and (e) of Example 7, note that the two products are different. Matrix multiplication is not, in general, commutative. That is, for most matrices, $AB \neq BA$.

The general pattern for matrix multiplication is as follows. To obtain the entry in the ith row and the jth column of the product AB, use the ith row of A and the jth column of B.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3j} & \cdots & b_{3p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

$$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = c_{ij}$$

Properties of Matrix Multiplication

Let A, B, and C be matrices and let c be a scalar.

1. $A(BC) = (AB)C$ Associative Property of Matrix Multiplication

2. $A(B + C) = AB + AC$ Distributive Property

3. $(A + B)C = AC + BC$ Distributive Property

4. $c(AB) = (cA)B = A(cB)$ Associative Property of Scalar Multiplication

The $n \times n$ matrix that consists of 1's on its main diagonal and 0's elsewhere is called the **identity matrix of order *n*** and is denoted by

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \qquad \text{Identity matrix}$$

Note that an identity matrix must be *square*. When the order is understood to be n, you can denote I_n simply by I. If A is an $n \times n$ matrix, the identity matrix has the property that $AI_n = A$ and $I_nA = A$. For example,

$$\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix}.$$

Applications

One application of matrix multiplication is representation of a system of linear equations. Note how the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

can be written as the matrix equation $AX = B$, where A is the *coefficient matrix* of the system, and X and B are column matrices.

Note The column matrix B is also called a *constant* matrix. Its entries are the constant terms in the system of equations.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \times \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{B}$$

EXAMPLE 8 Solving a System of Linear Equations

Solve the matrix equation $AX = B$ for X, where

$$A = \overset{\text{Coefficient matrix}}{\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix}} \quad \text{and} \quad B = \overset{\text{Column matrix}}{\begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}}.$$

Solution

As a system of linear equations, $AX = B$ is as follows.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= -4 \\ x_2 + 2x_3 &= 4 \\ 2x_1 + 3x_2 - 2x_3 &= 2 \end{aligned}$$

Using Gauss-Jordan elimination on the augmented matrix of this system, you obtain the following reduced row-echelon matrix.

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus, the solution of the system of linear equations is $x_1 = -1$, $x_2 = 2$, and $x_3 = 1$, and the solution of the matrix equation is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Use a graphing utility to verify that $AX = B$.

Real Life

EXAMPLE 9 Softball Team Expenses

Two softball teams submit equipment lists to their sponsors.

	Women's Team	*Men's Team*
Bats	12	15
Balls	45	38
Gloves	15	17

Each bat costs \$48, each ball costs \$4, and each glove costs \$42. Use matrices to find the total cost of equipment for each team.

Solution

The equipment lists and the costs per item can be written in matrix form as

$$E = \begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix} \quad \text{and} \quad C = [48 \quad 4 \quad 42].$$

The total cost of equipment for each team is given by the product

$$CE = [48 \quad 4 \quad 42]\begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix} = [1386 \;\; 1586].$$

Thus, the total cost of equipment for the women's team is \$1386, and the total cost of equipment for the men's team is \$1586.

Group Activity

Problem Posing

Write a matrix multiplication application problem that uses the matrix

$$A = \begin{bmatrix} 20 & 42 & 33 \\ 17 & 30 & 50 \end{bmatrix}.$$

Exchange problems with another student in your class. Form the matrices that represent the problem, and solve the problem. Interpret your solution in the context of the problem. Check with the creator of the problem to see if you are correct. Discuss other ways to represent and/or approach the problem.

8.2 /// EXERCISES

In Exercises 1–4, find x and y.

1. $\begin{bmatrix} x & -2 \\ 7 & y \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 7 & 22 \end{bmatrix}$

2. $\begin{bmatrix} -5 & x \\ y & 8 \end{bmatrix} = \begin{bmatrix} -5 & 13 \\ 12 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 16 & 4 & 5 & 4 \\ -3 & 13 & 15 & 6 \\ 0 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 4 & 2x+1 & 4 \\ -3 & 13 & 15 & 3x \\ 0 & 2 & 3y-5 & 0 \end{bmatrix}$

4. $\begin{bmatrix} x+2 & 8 & -3 \\ 1 & 2y & 2x \\ 7 & -2 & y+2 \end{bmatrix} = \begin{bmatrix} 2x+6 & 8 & -3 \\ 1 & 18 & -8 \\ 7 & -2 & 11 \end{bmatrix}$

In Exercises 5–10, find (a) $A + B$, (b) $A - B$, (c) $3A$, and (d) $3A - 2B$.

5. $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix}$

7. $A = \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix}$

8. $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix}$

9. $A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{bmatrix},$

$B = \begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ -3 & 4 & 9 & -6 & -7 \end{bmatrix}$

10. $A = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \quad B = \begin{bmatrix} -4 \\ 6 \\ 2 \end{bmatrix}$

In Exercises 11–14, solve for X given

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}.$$

11. $X = 3A - 2B$

12. $2X = 2A - B$

13. $2X + 3A = B$

14. $2A + 4B = -2X$

In Exercises 15–20, find (a) AB, (b) BA, and, if possible, (c) A^2. (*Note:* $A^2 = AA$.)

15. $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

16. $A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 3 & -3 \end{bmatrix}$

17. $A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$

18. $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$

19. $A = \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

20. $A = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

In Exercises 21–28, find AB, if possible.

21. $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}$

22. $A = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{bmatrix}$

23. $A = \begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$

24. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

25. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

26. $A = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -2 & 1 & 6 \end{bmatrix}$

27. $A = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & -3 \\ 0 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -11 & 4 \\ 8 & 16 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

28. $A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 6 & 13 & 8 & -17 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix}$

In Exercises 29–34, use the matrix capabilities of a graphing utility to find *AB*.

29. $A = \begin{bmatrix} 5 & 6 & -3 \\ -2 & 5 & 1 \\ 10 & -5 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 2 \\ 8 & 1 & 4 \\ 4 & -2 & 9 \end{bmatrix}$

30. $A = \begin{bmatrix} 11 & -12 & 4 \\ 14 & 10 & 12 \\ 6 & -2 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 12 & 10 \\ -5 & 12 \\ 15 & 16 \end{bmatrix}$

31. $A = \begin{bmatrix} -3 & 8 & -6 & 8 \\ -12 & 15 & 9 & 6 \\ 5 & -1 & 1 & 5 \end{bmatrix}$,

$B = \begin{bmatrix} 3 & 1 & 6 \\ 24 & 15 & 14 \\ 16 & 10 & 21 \\ 8 & -4 & 10 \end{bmatrix}$

32. $A = \begin{bmatrix} -2 & 4 & 8 \\ 21 & 5 & 6 \\ 13 & 2 & 6 \end{bmatrix}$,

$B = \begin{bmatrix} 2 & 0 \\ -7 & 15 \\ 32 & 14 \\ 0.5 & 1.6 \end{bmatrix}$

33. $A = \begin{bmatrix} 9 & 10 & -38 & 18 \\ 100 & -50 & 250 & 75 \end{bmatrix}$,

$B = \begin{bmatrix} 52 & -85 & 27 & 45 \\ 40 & -35 & 60 & 82 \end{bmatrix}$

34. $A = \begin{bmatrix} 15 & -18 \\ -4 & 12 \\ -8 & 22 \end{bmatrix}$,

$B = \begin{bmatrix} -7 & 22 & 1 \\ 8 & 16 & 24 \end{bmatrix}$

In Exercises 35–38, find matrices *A*, *X*, and *B* such that the system of linear equations can be written as the matrix equation *AX* = *B*. Solve the system of equations. Use a graphing utility to check your result.

35. $\begin{aligned} -x + y &= 4 \\ -2x + y &= 0 \end{aligned}$

36. $\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y - z &= -6 \\ 2x - 5y + 5z &= 17 \end{aligned}$

37. $\begin{aligned} 2x + 3y &= 5 \\ x + 4y &= 10 \end{aligned}$

38. $\begin{aligned} x + y - 3z &= -1 \\ -x + 2y \phantom{{}-3z} &= 1 \\ -y + z &= 0 \end{aligned}$

In Exercises 39–42, use the matrix capabilities of a graphing utility to find

$$f(A) = a_0 I_n + a_1 A + a_2 A^2 + \cdots + a_n A^n.$$

39. $f(x) = x^2 - 5x + 2$, $A = \begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix}$

40. $f(x) = x^2 - 7x + 6$, $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

41. $f(x) = x^3 - 10x^2 + 31x - 30$, $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

42. $f(x) = x^2 - 10x + 24$, $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

43. *Think About It* If a, b, and c are real numbers such that $c \neq 0$ and $ac = bc$, then $a = b$. However, if A, B, and C are nonzero matrices such that $AC = BC$, then A is *not necessarily* equal to B. Illustrate this using the following matrices.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$

44. *Think About It* If a and b are real numbers such that $ab = 0$, then $a = 0$ or $b = 0$. However, if A and B are matrices such that $AB = 0$, it is *not necessarily* true that $A = 0$ or $B = 0$. Illustrate this using the following matrices.

$A = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Think About It In Exercises 45–54, use matrices A and B each of order 2×3, C of order 3×2, and D of order 2×2. Determine whether the matrices are of proper order to perform the operation(s). If so, give the order of the answer.

45. $A + 2C$

46. $B - 3C$

47. AB

48. BC

49. $BC - D$

50. $CB - D$

51. $(CA)D$

52. $(BC)D$

53. $D(A - 3B)$

54. $(BC - D)A$

55. *Factory Production* A certain corporation has three factories, each of which manufactures two products. The number of units of product i produced at factory j in one day is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 60 & 40 & 20 \\ 30 & 90 & 60 \end{bmatrix}.$$

Find the production levels if production is increased by 20%. (*Hint:* Because an increase of 20% corresponds to 100% + 20%, multiply the given matrix by 1.2.)

56. *Factory Production* A certain corporation has four factories, each of which manufactures two products. The number of units of product i produced at factory j in one day is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 100 & 90 & 70 & 30 \\ 40 & 20 & 60 & 60 \end{bmatrix}.$$

Find the production levels if production is increased by 10%.

57. *Crop Production* A fruit grower raises two crops, which are shipped to three outlets. The number of units of crop i that are shipped to outlet j is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 100 & 75 & 75 \\ 125 & 150 & 100 \end{bmatrix}.$$

The profit per unit is represented by the matrix

$$B = [\$3.75 \quad \$7.00].$$

Find the product BA, and state what each entry of the product represents.

58. *Revenue* A manufacturer produces three models of a product, which are shipped to two warehouses. The number of units of model i that are shipped to warehouse j is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 5{,}000 & 4{,}000 \\ 6{,}000 & 10{,}000 \\ 8{,}000 & 5{,}000 \end{bmatrix}.$$

The price per unit is represented by the matrix

$$B = [\$20.50 \quad \$26.50 \quad \$29.50].$$

Compute BA and interpret the result.

Exploration In Exercises 59 and 60, let $i = \sqrt{-1}$.

59. Consider the matrix

$$A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}.$$

Find A^2, A^3, and A^4. Identify any similarities with i^2, i^3, and i^4.

60. Find and identify A^2 for the matrix

$$A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

61. *Inventory Levels* A company sells five models of computers through three retail outlets. The inventories are given by S.

$$S = \begin{array}{c} \overbrace{\begin{array}{ccccc} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} \end{array}}^{\text{Model}} \\ \begin{bmatrix} 3 & 2 & 2 & 3 & 0 \\ 0 & 2 & 3 & 4 & 3 \\ 4 & 2 & 1 & 3 & 2 \end{bmatrix} \end{array} \left.\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\right\} \text{Outlet}$$

The wholesale and retail prices are given by T.

$$T = \begin{array}{c} \overbrace{\begin{array}{cc} \text{Wholesale} & \text{Retail} \end{array}}^{\text{Price}} \\ \begin{bmatrix} \$840 & \$1100 \\ \$1200 & \$1350 \\ \$1450 & \$1650 \\ \$2650 & \$3000 \\ \$3050 & \$3200 \end{bmatrix} \end{array} \left.\begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{array}\right\} \text{Model}$$

Compute ST and interpret the result.

8.3 The Inverse of a Square Matrix

The Inverse of a Matrix / Finding Inverse Matrices / The Inverse of a 2 × 2 Matrix / Systems of Linear Equations

The Inverse of a Matrix

This section further develops the algebra of matrices. To begin, consider the real number equation $ax = b$. To solve this equation for x, multiply both sides of the equation by a^{-1} (provided that $a \neq 0$).

$$\begin{aligned} ax &= b \\ (a^{-1}a)x &= a^{-1}b \\ (1)x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

The number a^{-1} is called the *multiplicative inverse of a* because $a^{-1}a = 1$. The definition of the multiplicative inverse of a matrix is similar.

Note The symbol A^{-1} is read "A inverse."

Definition of the Inverse of a Square Matrix

Let A be an $n \times n$ matrix. If there exists a matrix A^{-1} such that

$$AA^{-1} = I_n = A^{-1}A$$

A^{-1} is called the **inverse** of A.

Note Recall that it is not always true that $AB = BA$, even if both products are defined. However, if A and B are both square matrices and $AB = I_n$, it can be shown that $BA = I_n$. Hence, in Example 1, you need only to check that $AB = I_2$.

EXAMPLE 1 The Inverse of a Matrix

Show that B is the inverse of A, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

Solution

To show that B is the inverse of A, show that $AB = I = BA$, as follows.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 + 2 & 2 - 2 \\ -1 + 1 & 2 - 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 + 2 & 2 - 2 \\ -1 + 1 & 2 - 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If a matrix A has an inverse, A is called **invertible** (or **nonsingular**); otherwise, A is called **singular.** A nonsquare matrix cannot have an inverse. To see this, note that if A is of order $m \times n$ and B is of order $n \times m$ (where $m \neq n$), the products AB and BA are of different orders and therefore cannot be equal to each other. Not all square matrices possess inverses (see the matrix at the bottom of page 622). If, however, a matrix does have an inverse, that inverse is unique. The following example shows how to use systems of equations to find the inverse of a matrix.

EXAMPLE 2 Finding the Inverse of a Matrix

Find the inverse of

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}.$$

Solution

To find the inverse of A, try to solve the matrix equation $AX = I$ for X.

$$\overset{A}{\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}}\overset{X}{\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating corresponding entries, you obtain the following two systems of linear equations.

$$\begin{aligned} x_{11} + 4x_{21} &= 1 \\ -x_{11} - 3x_{21} &= 0 \end{aligned} \qquad \begin{aligned} x_{12} + 4x_{22} &= 0 \\ -x_{12} - 3x_{22} &= 1 \end{aligned}$$

From the first system you can determine that $x_{11} = -3$ and $x_{21} = 1$, and from the second system you can determine that $x_{12} = -4$ and $x_{22} = 1$. Therefore, the inverse of A is

$$X = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}.$$

You can use matrix multiplication to check this result.

Check

$$AA^{-1} = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$A^{-1}A = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Most graphing utilities have the capability of finding the inverse of a square matrix. For instance, to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 2 & -1 \\ -2 & 0 & 1 \end{bmatrix}$$

on a *TI-82* or *TI-83*, enter the matrix. Then use the following keystrokes

$$[\text{A}]\ \boxed{x^{-1}}\ \boxed{\text{ENTER}}$$

After you find A^{-1}, store it as [B] and use the graphing utility to find [A] × [B] and [B] × [A]. What can you conclude?

Finding Inverse Matrices

In Example 2, note that the two systems of linear equations have the *same coefficient matrix* A. Rather than solve the two systems represented by

$$\left[\begin{array}{rrcr}1&4&\vdots&1\\-1&-3&\vdots&0\end{array}\right] \text{ and } \left[\begin{array}{rrcr}1&4&\vdots&0\\-1&-3&\vdots&1\end{array}\right]$$

separately, you can solve them *simultaneously* by **adjoining** the identity matrix to the coefficient matrix to obtain

$$\begin{array}{rrcrr}\multicolumn{2}{c}{A} & & \multicolumn{2}{c}{I}\\ \left[\;1\right.&4&\vdots&1&\left.0\;\right]\\ \left[-1\right.&-3&\vdots&0&\left.1\;\right]\end{array}.$$

Then, applying Gauss-Jordan elimination to this matrix, you can solve *both* systems with a single elimination process, as follows.

$$\begin{aligned} &\left[\begin{array}{rrcrr}1&4&\vdots&1&0\\-1&-3&\vdots&0&1\end{array}\right]\\ R_1+R_2\rightarrow&\left[\begin{array}{rrcrr}1&4&\vdots&1&0\\0&1&\vdots&1&1\end{array}\right]\\ -4R_2+R_1\rightarrow&\left[\begin{array}{rrcrr}1&0&\vdots&-3&-4\\0&1&\vdots&1&1\end{array}\right]\end{aligned}$$

Thus, from the "doubly augmented" matrix $[A \;\vdots\; I]$, you obtained the matrix $[I \;\vdots\; A^{-1}]$.

$$\begin{array}{rrcrr}\multicolumn{2}{c}{A} & & \multicolumn{2}{c}{I}\\ 1&4&\vdots&1&0\\-1&-3&\vdots&0&1\end{array} \;\Rightarrow\; \begin{array}{rrcrr}\multicolumn{2}{c}{I} & & \multicolumn{2}{c}{A^{-1}}\\ 1&0&\vdots&-3&-4\\0&1&\vdots&1&1\end{array}$$

This procedure (or algorithm) works for any square matrix that has an inverse.

EXPLORATION

Select two 2×2 matrices A and B that have inverses. Enter them into your graphing utility and calculate $(AB)^{-1}$. Then calculate $B^{-1}A^{-1}$ and $A^{-1}B^{-1}$. Make a conjecture about the inverse of a product of two invertible matrices.

Finding an Inverse Matrix

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \;\vdots\; I]$. Note that we separate the matrices A and I by a dotted line. We call this process **adjoining** the matrices A and I.
2. If possible, row reduce A to I using elementary row operations on the *entire* matrix $[A \;\vdots\; I]$. The result will be the matrix $[I \;\vdots\; A^{-1}]$. If this is not possible, A is not invertible.
3. Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.

EXAMPLE 3 **Finding the Inverse of a Matrix**

Find the inverse of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$.

Solution

Begin by adjoining the identity matrix to A to form the matrix

$$[A \;\vdots\; I] = \begin{bmatrix} 1 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 1 & 0 & -1 & \vdots & 0 & 1 & 0 \\ 6 & -2 & -3 & \vdots & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations to obtain the form $[I \;\vdots\; A^{-1}]$ results in

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & -2 & -3 & 1 \\ 0 & 1 & 0 & \vdots & -3 & -3 & 1 \\ 0 & 0 & 1 & \vdots & -2 & -4 & 1 \end{bmatrix}.$$

Therefore, the matrix A is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}.$$

Try using a graphing utility to confirm this result by multiplying A by A^{-1} to obtain I.

Verify the computations in Example 3 with your graphing utility. Enter the 3×6 matrix $[A \;\vdots\; I]$ and row reduce it to the matrix $[I \;\vdots\; A^{-1}]$, as follows.

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{bmatrix}$$

What happens if you try this method to find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}?$$

The process shown in Example 3 applies to any $n \times n$ matrix A. If A has an inverse, this process will find it. If A does not have an inverse, the process will tell us so. For instance, the following matrix has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

To confirm that matrix A above has no inverse, begin by adjoining the identity matrix to A to form

$$[A \;\vdots\; I] = \begin{bmatrix} 1 & 2 & 0 & \vdots & 1 & 0 & 0 \\ 3 & -1 & 2 & \vdots & 0 & 1 & 0 \\ -2 & 3 & -2 & \vdots & 0 & 0 & 1 \end{bmatrix}.$$

Then use elementary row operations to obtain

$$\begin{bmatrix} 1 & 2 & 0 & \vdots & 1 & 0 & 0 \\ 0 & -7 & 2 & \vdots & -3 & 1 & 0 \\ 0 & 0 & 0 & \vdots & -1 & 1 & 1 \end{bmatrix}.$$

At this point in the elimination process you can see that it is impossible to obtain the identity matrix I on the left. Therefore, A is not invertible.

The Inverse of a 2×2 Matrix

Using Gauss-Jordan elimination to find the inverse of a matrix works well (even as a computer technique) for matrices of order 3×3 or greater. For 2×2 matrices, however, many people prefer to use a formula for the inverse rather than Gauss-Jordan elimination. This simple formula, which works *only* for 2×2 matrices, is explained as follows. If A is a 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. If $ad - bc \neq 0$, the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Try verifying this inverse by multiplication.

Note The denominator $ad - bc$ is called the **determinant** of the 2×2 matrix A. You will study determinants in the next section.

EXPLORATION

Use a graphing utility with matrix operations to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}.$$

What message appears on the screen? Why does the graphing utility display this message?

EXAMPLE 4 Finding the Inverse of a 2×2 Matrix

If possible, find the inverse of the matrix.

a. $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$

b. $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

Solution

a. For the matrix A, apply the formula for the inverse of a 2×2 matrix to obtain

$$ad - bc = (3)(2) - (-1)(-2) = 4.$$

Because this quantity is not zero, the inverse is formed by interchanging the entries on the main diagonal, changing the signs of the other two entries, and multiplying by the scalar $\frac{1}{4}$, as follows.

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

b. For the matrix B, you have

$$ad - bc = (3)(2) - (-1)(-6) = 0$$

which means that B is not invertible.

Systems of Linear Equations

You know that a system of linear equations can have exactly one solution, infinitely many solutions, or no solution. If the coefficient matrix A of a *square* system (a system that has the same number of equations as variables) is invertible, the system has a unique solution, which is given as follows.

The formula $X = A^{-1}B$ is used on most graphing utilities to solve linear systems that have invertible coefficient matrices. That is, you enter the $n \times n$ coefficient matrix [A] and the $n \times 1$ column matrix [B]. The solution X is given by $[A]^{-1}[B]$.

A System of Equations with a Unique Solution

If A is an invertible matrix, the system of linear equations represented by $AX = B$ has a unique solution given by

$$X = A^{-1}B.$$

Note Use Gauss-Jordan elimination or a graphing utility to verify A^{-1} for the system of equations in Example 5.

EXAMPLE 5 Solving a System of Equations Using an Inverse

Use an inverse matrix to solve the system.

$$\begin{aligned} 2x + 3y + z &= -1 \\ 3x + 3y + z &= 1 \\ 2x + 4y + z &= -2 \end{aligned}$$

Solution

$$X = A^{-1}B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

Thus, the solution is $x = 2$, $y = -1$, and $z = -2$.

Group Activity

Finding an Inverse Matrix

Use a graphing utility to decide which of the following matrices is (are) invertible.

a. $A = \begin{bmatrix} -3 & 2 \\ 7 & 4 \end{bmatrix}$

b. $B = \begin{bmatrix} 1 & -4 & 2 \\ 2 & -9 & 5 \\ 1 & -5 & 4 \end{bmatrix}$

c. $C = \begin{bmatrix} -4 & 6 & -4 \\ 2 & 4 & 0 \\ 6 & -2 & 4 \end{bmatrix}$

8.3 /// EXERCISES

In Exercises 1–8, show that B is the inverse of A.

1. $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$

5. $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \frac{1}{3}\begin{bmatrix} -4 & -5 & 3 \\ -4 & -8 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$

7. $A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 1 \\ -1 & 1 & -2 & 1 \\ 4 & -1 & 1 & 0 \end{bmatrix}$,

$B = \begin{bmatrix} -1 & 2 & -1 & -1 \\ -4 & 9 & -5 & -6 \\ 0 & 1 & -1 & -1 \\ 3 & -5 & 3 & 3 \end{bmatrix}$

8. $A = \begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}$,

$B = \frac{1}{3}\begin{bmatrix} -3 & 1 & 1 & -3 \\ -3 & -1 & 2 & -3 \\ 0 & 1 & 1 & 0 \\ -3 & -2 & 1 & 0 \end{bmatrix}$

In Exercises 9–24, find the inverse of the matrix (if it exists).

9. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

11. $\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

12. $\begin{bmatrix} -7 & 33 \\ 4 & -19 \end{bmatrix}$

13. $\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 11 & 1 \\ -1 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$

16. $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

17. $\begin{bmatrix} 2 & 7 & 1 \\ -3 & -9 & 2 \end{bmatrix}$

18. $\begin{bmatrix} -2 & 5 \\ 6 & -15 \\ 0 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

23. $\begin{bmatrix} -8 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

In Exercises 25–34, use the matrix capabilities of a graphing utility to find the inverse of the matrix (if it exists).

25. $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ -5 & -7 & -15 \end{bmatrix}$

26. $\begin{bmatrix} 10 & 5 & -7 \\ -5 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

28. $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 2 \\ -4 & 4 & 3 \end{bmatrix}$

29. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

30. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}$

32. $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

33. $\begin{bmatrix} 1 & -2 & -1 & -2 \\ 3 & -5 & -2 & -3 \\ 2 & -5 & -2 & -5 \\ -1 & 4 & 4 & 11 \end{bmatrix}$

34. $\begin{bmatrix} 4 & 8 & -7 & 14 \\ 2 & 5 & -4 & 6 \\ 0 & 2 & 1 & -7 \\ 3 & 6 & -5 & 10 \end{bmatrix}$

35. If A is a 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. If $ad - bc \neq 0$, verify that the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

36. Use the result of Exercise 35 to find the inverse of each matrix.

(a) $\begin{bmatrix} 5 & -2 \\ 2 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 7 & 12 \\ -8 & -5 \end{bmatrix}$

In Exercises 37–40, use an inverse matrix to solve the system of linear equations. (Use the inverse matrix found in Exercise 11.)

37. $\begin{aligned} x - 2y &= 5 \\ 2x - 3y &= 10 \end{aligned}$

38. $\begin{aligned} x - 2y &= 0 \\ 2x - 3y &= 3 \end{aligned}$

39. $\begin{aligned} x - 2y &= 4 \\ 2x - 3y &= 2 \end{aligned}$

40. $\begin{aligned} x - 2y &= 1 \\ 2x - 3y &= -2 \end{aligned}$

In Exercises 41 and 42, use an inverse matrix to solve the system of linear equations. (Use the inverse matrix found in Exercise 19.)

41. $\begin{aligned} x + y + z &= 0 \\ 3x + 5y + 4z &= 5 \\ 3x + 6y + 5z &= 2 \end{aligned}$

42. $\begin{aligned} x + y + z &= -1 \\ 3x + 5y + 4z &= 2 \\ 3x + 6y + 5z &= 0 \end{aligned}$

In Exercises 43 and 44, use an inverse matrix and the matrix capabilities of a graphing utility to solve the system of linear equations. (Use the inverse matrix found in Exercise 33.)

43. $\begin{aligned} x_1 - 2x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 &= 1 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 &= -1 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 &= 2 \end{aligned}$

44. $\begin{aligned} x_1 - 2x_2 - x_3 - 2x_4 &= 1 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 &= -2 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 &= 0 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 &= -3 \end{aligned}$

In Exercises 45–52, use an inverse matrix to solve (if possible) the system of linear equations.

45. $\begin{aligned} 3x + 4y &= -2 \\ 5x + 3y &= 4 \end{aligned}$

46. $\begin{aligned} 18x + 12y &= 13 \\ 30x + 24y &= 23 \end{aligned}$

47. $\begin{aligned} -0.4x + 0.8y &= 1.6 \\ 2x - 4y &= 5 \end{aligned}$

48. $\begin{aligned} 13x - 6y &= 17 \\ 26x - 12y &= 8 \end{aligned}$

49. $\begin{aligned} 3x + 6y &= 6 \\ 6x + 14y &= 11 \end{aligned}$

50. $\begin{aligned} 3x + 2y &= 1 \\ 2x + 10y &= 6 \end{aligned}$

51. $\begin{aligned} 4x - y + z &= -5 \\ 2x + 2y + 3z &= 10 \\ 5x - 2y + 6z &= 1 \end{aligned}$

52. $\begin{aligned} 4x - 2y + 3z &= -2 \\ 2x + 2y + 5z &= 16 \\ 8x - 5y - 2z &= 4 \end{aligned}$

In Exercises 53–56, use the matrix capabilities of a graphing utility to solve (if possible) the system of linear equations.

53. $$\begin{aligned} 5x - 3y + 2z &= 2\\ 2x + 2y - 3z &= 3\\ -x + 7y - 8z &= 4 \end{aligned}$$

54. $$\begin{aligned} 2x + 3y + 5z &= 4\\ 3x + 5y + 9z &= 7\\ 5x + 9y + 17z &= 13 \end{aligned}$$

55. $$\begin{array}{rcrcrcrcr} 7x & - & 3y & & & + & 2w & = & 41\\ -2x & + & y & & & - & w & = & -13\\ 4x & & & + & z & - & 2w & = & 12\\ -x & + & y & & & - & w & = & -8 \end{array}$$

56. $$\begin{array}{rcrcrcrcr} 2x & + & 5y & & & + & w & = & 11\\ x & + & 4y & + & 2z & - & 2w & = & -7\\ 2x & - & 2y & + & 5z & + & w & = & 3\\ x & & & & & - & 3w & = & -1 \end{array}$$

Bond Investments **In Exercises 57–60, consider a person who invests in AAA-rated bonds, A-rated bonds, and B-rated bonds. The average yields are 6.5% on AAA-bonds, 7% on A-bonds, and 9% on B-bonds. The person invests twice as much in B-bonds as in A-bonds. Let x, y, and z represent the amounts invested in AAA-, A-, and B-bonds, respectively.**

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & \text{(total investment)}\\ 0.065x & + & 0.07y & + & 0.09z & = & \text{(annual return)}\\ & & 2y & - & z & = & 0 \end{array}$$

Use the inverse of the coefficient matrix of this system to find the amount invested in each type of bond.

57. Total investment = \$25,000
Annual return = \$1900

58. Total investment = \$45,000
Annual return = \$3750

59. Total investment = \$12,000
Annual return = \$835

60. Total investment = \$500,000
Annual return = \$38,000

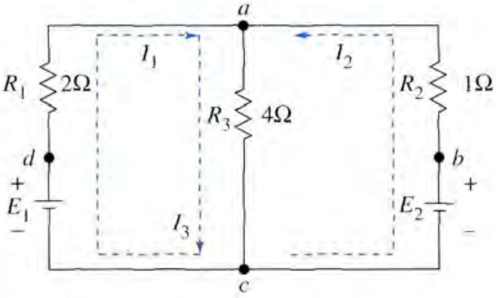
61. *Essay* Write a brief paragraph explaining the advantage of using an inverse matrix to solve the systems of linear equations in Exercises 37–44.

62. *True or False?* Multiplication of an invertible matrix and its inverse is commutative. Give an example to demonstrate your answer.

Circuit Analysis **In Exercises 63 and 64, consider the circuit in the figure. The currents I_1, I_2, and I_3, in amperes, are given by the solution of the system of linear equations**

$$\begin{array}{rcrcrcl} 2I_1 & & & + & 4I_3 & = & E_1\\ & & I_2 & + & 4I_3 & = & E_2\\ I_1 & + & I_2 & - & I_3 & = & 0 \end{array}$$

where E_1 and E_2 are voltages. Use the inverse of the coefficient matrix of this system to find the unknown currents for the given voltages.



63. $E_1 = 14\text{ V},\ E_2 = 28\text{ V}$

64. $E_1 = 10\text{ V},\ E_2 = 10\text{ V}$

65. *Exploration* Consider the matrices of the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & 0\\ 0 & a_{22} & 0 & 0 & \cdots & 0\\ 0 & 0 & a_{33} & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

(a) Write a 2×2 matrix and a 3×3 matrix in the form of A. Find the inverse of each.

(b) Use the result of part (a) to make a conjecture about the inverse of a matrix of the form of A.

8.4 The Determinant of a Square Matrix

The Determinant of a 2 × 2 Matrix / Minors and Cofactors / The Determinant of a Square Matrix / Triangular Matrices

The Determinant of a 2 × 2 Matrix

Every *square* matrix can be associated with a real number called its **determinant.** Determinants have many uses, and several will be discussed in this and the next section. Historically, the use of determinants arose from special number patterns that occur when systems of linear equations are solved. For instance, the system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

has a solution given by

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that $a_1b_2 - a_2b_1 \neq 0$. Note that the denominator of each fraction is the same. This denominator is called the **determinant** of the coefficient matrix of the system.

Coefficient Matrix	*Determinant*
$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\det(A) = a_1b_2 - a_2b_1$

The determinant of the matrix A can also be denoted by vertical bars on both sides of the matrix, as indicated in the following definition.

Definition of the Determinant of a 2 × 2 Matrix

The **determinant** of the matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

is given by

$$\det(A) = |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

Note In this text, $\det(A)$ and $|A|$ are used interchangeably to represent the determinant of A. Although vertical bars are also used to denote the absolute value of a real number, the context will show which use is intended.

A convenient method for remembering the formula for the determinant of a 2×2 matrix is shown in the following diagram.

$$\det(A) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Note that the determinant is given by the difference of the products of the two diagonals of the matrix.

EXAMPLE 1 The Determinant of a 2×2 Matrix

Find the determinant of each matrix.

a. $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$ **b.** $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ **c.** $C = \begin{bmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{bmatrix}$

Solution

a. $\det(A) = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$

b. $\det(B) = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0$

c. $\det(C) = \begin{vmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{vmatrix} = 0(4) - 2\left(\frac{3}{2}\right) = 0 - 3 = -3$

Note Notice in Example 1 that the determinant of a matrix can be positive, zero, or negative.

The determinant of a matrix of order 1×1 is defined simply as the entry of the matrix. For instance, if $A = [-2]$, $\det(A) = -2$.

Most graphing utilities can evaluate the determinant of a matrix. For instance, on a *TI-82* or *TI-83,* you can evaluate the determinant of

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

by entering the matrix as [A] and then choosing the "det" feature in the matrix math menu.

det [A] ENTER

The result should be 7, as in Example 1(a). Try evaluating determinants of other matrices. What happens when you try to evaluate the determinant of a nonsquare matrix?

Minors and Cofactors

To define the determinant of a square matrix of order 3 × 3 or higher, it is convenient to introduce the concepts of **minors** and **cofactors.**

Minors and Cofactors of a Square Matrix

If A is a square matrix, the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} of the entry a_{ij} is given by

$$C_{ij} = (-1)^{i+j}M_{ij}.$$

Sign Pattern for Cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3 × 3 matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4 × 4 matrix

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

n × n matrix

Note In the sign pattern for cofactors above, notice that *odd* positions (where $i + j$ is odd) have negative signs and *even* positions (where $i + j$ is even) have positive signs.

EXAMPLE 2 Finding the Minors and Cofactors of a Matrix

Find all the minors and cofactors of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

Solution

To find the minor M_{11}, delete the first row and first column of A and evaluate the determinant of the resulting matrix.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 0(2) = -1$$

Similarly, to find M_{12}, delete the first row and second column.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = -5$$

Continuing this pattern, you obtain the following minors.

$$\begin{array}{lll} M_{11} = -1 & M_{12} = -5 & M_{13} = 4 \\ M_{21} = 2 & M_{22} = -4 & M_{23} = -8 \\ M_{31} = 5 & M_{32} = -3 & M_{33} = -6 \end{array}$$

Now, to find the cofactors, combine these minors with the checkerboard pattern of signs shown at the left (for a 3 × 3 matrix) to obtain the following.

$$\begin{array}{lll} C_{11} = -1 & C_{12} = 5 & C_{13} = 4 \\ C_{21} = -2 & C_{22} = -4 & C_{23} = 8 \\ C_{31} = 5 & C_{32} = 3 & C_{33} = -6 \end{array}$$

The Determinant of a Square Matrix

The following definition is called **inductive** because it uses determinants of matrices of order $n - 1$ to define the determinant of a matrix of order n.

Determinant of a Square Matrix

If A is a square matrix (of order 2×2 or greater), the determinant of A is the sum of the entries in any row (or column) of A multiplied by their respective cofactors. For instance, expanding along the first row yields

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdot \cdot \cdot + a_{1n}C_{1n}.$$

Applying this definition to find a determinant is called **expanding by cofactors.**

Note Try checking that for a 2×2 matrix this definition yields

$$|A| = a_{11}a_{22} - a_{21}a_{12}$$

as previously defined.

EXAMPLE 3 The Determinant of a Matrix of Order 3 × 3

Find the determinant of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

Solution

Note that this is the same matrix that was given in Example 2. There you found the cofactors of the entries in the first row to be

$$C_{11} = -1, \quad C_{12} = 5, \quad \text{and} \quad C_{13} = 4.$$

Therefore, by the definition of the determinant of a square matrix, you have

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \text{First-row expansion} \\ &= 0(-1) + 2(5) + 1(4) \\ &= 14. \end{aligned}$$

In Example 3, the determinant was found by expanding by the cofactors in the first row. You could have used any row or column. For instance, you could have expanded along the second row to obtain

$$\begin{aligned} |A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} && \text{Second-row expansion} \\ &= 3(-2) + (-1)(-4) + 2(8) \\ &= 14. \end{aligned}$$

When expanding by cofactors, you do not need to find cofactors of zero entries, because zero times its cofactor is zero.

$$a_{ij}C_{ij} = (0)C_{ij} = 0$$

Thus, the row (or column) containing the most zeros is usually the best choice for expansion by cofactors. This is demonstrated in the next example.

EXAMPLE 4 The Determinant of a Matrix of Order 4 × 4

Find the determinant of

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & 2 \end{bmatrix}.$$

Solution

After inspecting this matrix, you can see that three of the entries in the third column are zeros. Thus, you can eliminate some of the work in the expansion by using the third column.

$$|A| = 3(C_{13}) + 0(C_{23}) + 0(C_{33}) + 0(C_{43})$$

Because C_{23}, C_{33}, and C_{43} have zero coefficients, you need only find the cofactor C_{13}. To do this, delete the first row and third column of A and evaluate the determinant of the resulting matrix.

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & 2 \end{vmatrix} \qquad \text{Delete 1st row and 3rd column.}$$

$$= \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & 2 \end{vmatrix} \qquad \text{Simplify.}$$

Study Tip

Although most graphing utilities can calculate the determinant of a square matrix, it is also important to know how to calculate them *by hand*.

Expanding by cofactors in the second row yields the following.

$$\begin{aligned} C_{13} &= 0(-1)^3\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} + 2(-1)^4\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} + 3(-1)^5\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} \\ &= 0 + 2(1)(-8) + 3(-1)(-7) \\ &= 5 \end{aligned}$$

Thus, you obtain

$$|A| = 3C_{13} = 3(5) = 15.$$

Note Try using a graphing utility to confirm the result of Example 4.

Triangular Matrices

Evaluating determinants of matrices of order 4 or higher can be tedious. There is, however, an important exception: the determinant of a **triangular** matrix. A square matrix is **upper triangular** if it has all zero entries below its main diagonal and **lower triangular** if it has all zero entries above its main diagonal. A matrix that is both upper and lower triangular is called **diagonal.** That is, a diagonal matrix is one in which all entries above and below the main diagonal are zero.

Upper Triangular Matrix *Lower Triangular Matrix*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

To find the determinant of a triangular matrix of any order, simply form the product of the entries on the main diagonal.

EXPLORATION

The formula for the determinant of a triangular matrix (discussed at the right) is only one of many properties of matrices. You can use a computer or calculator to discover other properties. For instance, how is $|cA|$ related to $|A|$? How are $|A|$ and $|B|$ related to $|AB|$?

EXAMPLE 5 The Determinant of a Triangular Matrix

a. $$\begin{vmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{vmatrix} = (2)(-2)(1)(3) = -12$$

b. $$\begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{vmatrix} = (-1)(3)(2)(4)(-2) = 48$$

Group Activity

The Determinant of a Triangular Matrix

Write an argument that explains why the determinant of a 3 × 3 triangular matrix is the product of its main-diagonal entries.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33}$$

8.4 /// EXERCISES

In Exercises 1–16, find the determinant of the matrix.

1. $[5]$

2. $[-8]$

3. $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 1 \\ 5 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 5 & 2 \\ -6 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

7. $\begin{bmatrix} -7 & 6 \\ \frac{1}{2} & 3 \end{bmatrix}$

8. $\begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 2 & -3 \\ -6 & 9 \end{bmatrix}$

11. $\begin{bmatrix} 2 & -1 & 0 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

12. $\begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 6 & 3 & -7 \\ 0 & 0 & 0 \\ 4 & -6 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

15. $\begin{bmatrix} -1 & 2 & -5 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 0 & 0 \\ -4 & -1 & 0 \\ 5 & 1 & 5 \end{bmatrix}$

In Exercises 17–20, use the matrix capabilities of a graphing utility to find the determinant of the matrix.

17. $\begin{bmatrix} 0.3 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 \\ -0.4 & 0.4 & 0.3 \end{bmatrix}$

18. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 6 & -6 \\ -2 & 1 & 4 \end{bmatrix}$

20. $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & -2 \end{bmatrix}$

In Exercises 21–24, find all (a) minors and (b) cofactors of the matrix.

21. $\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$

22. $\begin{bmatrix} 11 & 0 \\ -3 & 2 \end{bmatrix}$

23. $\begin{bmatrix} 3 & -2 & 8 \\ 3 & 2 & -6 \\ -1 & 3 & 6 \end{bmatrix}$

24. $\begin{bmatrix} -2 & 9 & 4 \\ 7 & -6 & 0 \\ 6 & 7 & -6 \end{bmatrix}$

In Exercises 25–30, find the determinant of the matrix by the method of expansion by cofactors. Expand using the indicated row or column.

25. $\begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{bmatrix}$ (a) Row 1 (b) Column 2

26. $\begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$ (a) Row 2 (b) Column 3

27. $\begin{bmatrix} 5 & 0 & -3 \\ 0 & 12 & 4 \\ 1 & 6 & 3 \end{bmatrix}$ (a) Row 2 (b) Column 2

28. $\begin{bmatrix} 10 & -5 & 5 \\ 30 & 0 & 10 \\ 0 & 10 & 1 \end{bmatrix}$ (a) Row 3 (b) Column 1

29. $\begin{bmatrix} 6 & 0 & -3 & 5 \\ 4 & 13 & 6 & -8 \\ -1 & 0 & 7 & 4 \\ 8 & 6 & 0 & 2 \end{bmatrix}$ (a) Row 2 (b) Column 2

30. $\begin{bmatrix} 10 & 8 & 3 & -7 \\ 4 & 0 & 5 & -6 \\ 0 & 3 & 2 & 7 \\ 1 & 0 & -3 & 2 \end{bmatrix}$ (a) Row 3 (b) Column 1

In Exercises 31–40, find the determinant of the matrix. Expand by cofactors on the row or column that appears to make the computations easiest.

31. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$

32. $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 4 \\ 1 & 0 & 2 \end{bmatrix}$

33. $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix}$

34. $\begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

35. $\begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 7 & 3 & 6 \\ 1 & 5 & 0 & 1 \\ 3 & 7 & 0 & 7 \end{bmatrix}$ **36.** $\begin{bmatrix} 3 & 6 & -5 & 4 \\ -2 & 0 & 6 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 3 & -1 & -1 \end{bmatrix}$

37. $\begin{bmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{bmatrix}$ **38.** $\begin{bmatrix} 1 & 4 & 3 & 2 \\ -5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \end{bmatrix}$

39. $\begin{bmatrix} 3 & 2 & 4 & -1 & 5 \\ -2 & 0 & 1 & 3 & 2 \\ 1 & 0 & 0 & 4 & 0 \\ 6 & 0 & 2 & -1 & 0 \\ 3 & 0 & 5 & 1 & 0 \end{bmatrix}$

40. $\begin{bmatrix} 5 & 2 & 0 & 0 & -2 \\ 0 & 1 & 4 & 3 & 2 \\ 0 & 0 & 2 & 6 & 3 \\ 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

In Exercises 41–48, use the matrix capabilities of a graphing utility to evaluate the determinant.

41. $\begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 8 & 1 & 6 \end{vmatrix}$ **42.** $\begin{vmatrix} 5 & -8 & 0 \\ 9 & 7 & 4 \\ -8 & 7 & 1 \end{vmatrix}$

43. $\begin{vmatrix} 7 & 0 & -14 \\ -2 & 5 & 4 \\ -6 & 2 & 12 \end{vmatrix}$ **44.** $\begin{vmatrix} 3 & 0 & 0 \\ -2 & 5 & 0 \\ 12 & 5 & 7 \end{vmatrix}$

45. $\begin{vmatrix} 1 & -1 & 8 & 4 \\ 2 & 6 & 0 & -4 \\ 2 & 0 & 2 & 6 \\ 0 & 2 & 8 & 0 \end{vmatrix}$ **46.** $\begin{vmatrix} 0 & -3 & 8 & 2 \\ 8 & 1 & -1 & 6 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{vmatrix}$

47. $\begin{vmatrix} 3 & -2 & 4 & 3 & 1 \\ -1 & 0 & 2 & 1 & 0 \\ 5 & -1 & 0 & 3 & 2 \\ 4 & 7 & -8 & 0 & 0 \\ 1 & 2 & 3 & 0 & 2 \end{vmatrix}$

48. $\begin{vmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{vmatrix}$

In Exercises 49–52, evaluate the determinants to verify the equation.

49. $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = -\begin{vmatrix} y & z \\ w & x \end{vmatrix}$

50. $\begin{vmatrix} w & cx \\ y & cz \end{vmatrix} = c\begin{vmatrix} w & x \\ y & z \end{vmatrix}$

51. $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = \begin{vmatrix} w & x + cw \\ y & z + cy \end{vmatrix}$

52. $\begin{vmatrix} w & x \\ cw & cx \end{vmatrix} = 0$

In Exercises 53 and 54, evaluate the determinant to verify the equation.

53. $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y - x)(z - x)(z - y)$

54. $\begin{vmatrix} a + b & a & a \\ a & a + b & a \\ a & a & a + b \end{vmatrix} = b^2(3a + b)$

In Exercises 55 and 56, solve for x.

55. $\begin{vmatrix} x - 1 & 2 \\ 3 & x - 2 \end{vmatrix} = 0$

56. $\begin{vmatrix} x - 2 & -1 \\ -3 & x \end{vmatrix} = 0$

In Exercises 57–62, evaluate the determinant, where the entries are functions. Determinants of this type occur in calculus.

57. $\begin{vmatrix} 4u & -1 \\ -1 & 2v \end{vmatrix}$ **58.** $\begin{vmatrix} 3x^2 & -3y^2 \\ 1 & 1 \end{vmatrix}$

59. $\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$ **60.** $\begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1 - x)e^{-x} \end{vmatrix}$

61. $\begin{vmatrix} x & \ln x \\ 1 & 1/x \end{vmatrix}$ **62.** $\begin{vmatrix} x & x \ln x \\ 1 & 1 + \ln x \end{vmatrix}$

In Exercises 63–66, find (a) $|A|$, (b) $|B|$, (c) AB, and (d) $|AB|$.

63. $A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

64. $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

65. $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

66. $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$

67. *Exploration* Find square matrices A and B to demonstrate that

$$|A + B| \neq |A| + |B|.$$

68. *Exploration* Consider square matrices in which the entries are consecutive integers. An example of such a matrix is

$$\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}.$$

(a) Use a graphing utility to evaluate four determinants of this type. Make a conjecture based on the results.

(b) Verify your conjecture.

69. *Essay* Write a brief paragraph explaining the difference between a square matrix and its determinant.

70. *Think About It* If A is a matrix of order 3×3 such that $|A| = 5$, is it possible to find $|2A|$? Explain.

In Exercises 71–73, a property of determinants is given. State how the property has been applied to the given determinants and use a graphing utility to verify the results.

71. If A and B are square matrices and B is obtained from A by interchanging two rows of A or interchanging two columns of A, then $|B| = -|A|$.

(a) $\begin{vmatrix} 1 & 3 & 4 \\ -7 & 2 & -5 \\ 6 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 3 \\ -7 & -5 & 2 \\ 6 & 2 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 3 & 4 \\ -2 & 2 & 0 \\ 1 & 6 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 6 & 2 \\ -2 & 2 & 0 \\ 1 & 3 & 4 \end{vmatrix}$

72. If A and B are square matrices and B is obtained from A by adding a multiple of a row of A to another row of A or by adding a multiple of a column of A to another column of A, then $|B| = |A|$.

(a) $\begin{vmatrix} 1 & -3 \\ 5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 0 & 17 \end{vmatrix}$

(b) $\begin{vmatrix} 5 & 4 & 2 \\ 2 & -3 & 4 \\ 7 & 6 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 10 & -6 \\ 2 & -3 & 4 \\ 7 & 6 & 3 \end{vmatrix}$

73. If A and B are square matrices and B is obtained from A by multiplying a row of A by a nonzero constant c or multiplying a column of A by a nonzero constant c, then $|B| = c|A|$.

(a) $\begin{vmatrix} 5 & 10 & 15 \\ 2 & -3 & 4 \\ 2 & -7 & 1 \end{vmatrix} = 5\begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & 4 \\ 2 & -7 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 8 & -3 \\ 3 & -12 & 6 \\ 7 & 4 & 9 \end{vmatrix} = 12\begin{vmatrix} 1 & 2 & -1 \\ 3 & -3 & 2 \\ 7 & 1 & 3 \end{vmatrix}$

8.5 Applications of Matrices and Determinants

Area of a Triangle / Lines in the Plane / Cramer's Rule / Cryptography

Area of a Triangle

In this section, you will study some additional applications of matrices and determinants. The first involves a formula for finding the area of a triangle whose vertices are given by three points on a rectangular coordinate system.

Area of a Triangle

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where the symbol (±) indicates that the appropriate sign should be chosen to yield a positive area.

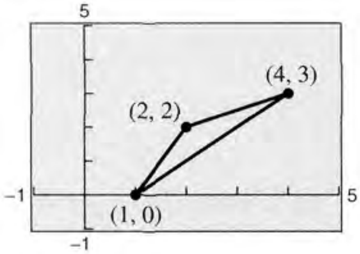
EXAMPLE 1 Finding the Area of a Triangle

Find the area of a triangle whose vertices are (1, 0), (2, 2), and (4, 3), as shown in Figure 8.1.

Solution

Let $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (2, 2)$, and $(x_3, y_3) = (4, 3)$. Then, to find the area of a triangle, evaluate the determinant

Figure 8.1



$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= 1(-1)^2\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 0(-1)^3\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix}$$

$$= 1(-1) + 0 + 1(-2)$$

$$= -3.$$

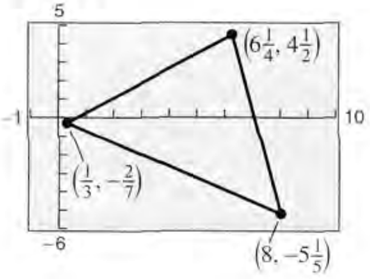
Using this value, you can conclude that the area of the triangle is

$$\text{Area} = -\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = -\frac{1}{2}(-3) = \frac{3}{2}.$$

EXAMPLE 2 Finding the Area of a Triangle

Find the area of the triangle whose vertices are $\left(\frac{1}{3}, -\frac{2}{7}\right)$, $\left(6\frac{1}{4}, 4\frac{1}{2}\right)$, and $\left(8, -5\frac{1}{5}\right)$, as shown in Figure 8.2.

Figure 8.2



Solution

Let $(x_1, y_1) = \left(\frac{1}{3}, -\frac{2}{7}\right)$, $(x_2, y_2) = \left(6\frac{1}{4}, 4\frac{1}{2}\right)$, and $(x_3, y_3) = \left(8, -5\frac{1}{5}\right)$. Then, to find the area of the triangle, evaluate the determinant

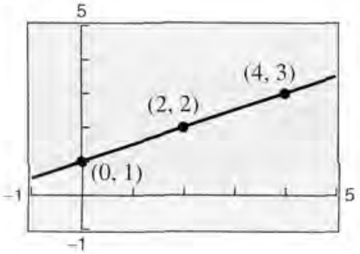
$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{2}{7} & 1 \\ 6\frac{1}{4} & 4\frac{1}{2} & 1 \\ 8 & -5\frac{1}{5} & 1 \end{vmatrix}.$$

Using the matrix capabilities of a graphing utility, you find the value of the determinant to be $-65.7\overline{6}$.

Now you can use this value to conclude that the area of the triangle is

$$\text{Area} = -\tfrac{1}{2}(-65.7\overline{6}) = 32.88\overline{3}.$$

Figure 8.3



Lines in the Plane

Suppose the three points in Example 1 had been on the same line. What would have happened had the area formula been applied to three such points? The answer is that the determinant would have been zero. Consider, for instance, the three collinear points (0, 1), (2, 2), and (4, 3), as shown in Figure 8.3. The area of the "triangle" that has these three points as vertices is

$$\frac{1}{2}\begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = \frac{1}{2}\left[0(-1)^2\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 1(-1)^3\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix}\right]$$

$$= \frac{1}{2}[0 + (-1)(-2) + 1(-2)] = 0.$$

This result is generalized as follows.

Test for Collinear Points

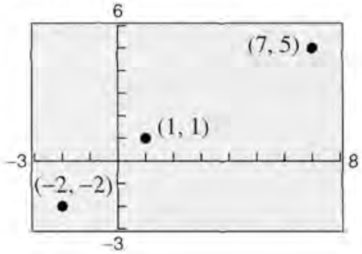
Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear (lie on the same line) if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

EXAMPLE 3 Testing for Collinear Points

Determine whether the points $(-2, -2)$, $(1, 1)$, and $(7, 5)$ lie on the same line. (See Figure 8.4.)

Figure 8.4



Solution

Letting $(x_1, y_1) = (-2, -2)$, $(x_2, y_2) = (1, 1)$, and $(x_3, y_3) = (7, 5)$, you have

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} -2 & -2 & 1 \\ 1 & 1 & 1 \\ 7 & 5 & 1 \end{vmatrix}$$

$$= -2(-1)^2\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} + (-2)(-1)^3\begin{vmatrix} 1 & 1 \\ 7 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix}$$

$$= -2(-4) + 2(-6) + 1(-2)$$

$$= -6.$$

Because the value of this determinant *is not* zero, you can conclude that the three points do not lie on the same line.

You can use the following steps on a *TI-82* or *TI-83* graphing calculator to check whether three points are collinear.

1. Plot the points by entering Pt-On(x, y) for each point. [Pt-On(can be found in the DRAW POINTS menu.]
2. Draw a line from the two farthest points by entering Line (x_1, y_1, x_2, y_2).

Use the steps above to check the results of Example 3. Explain how the graph shows that the points are not collinear. Why is Step 2 important in determining if points are collinear?

The test for collinear points can be adapted to another use. That is, if you are given two points on a rectangular coordinate system, you can find an equation of the line passing through the two points, as follows.

Two-Point Form of the Equation of a Line

An equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

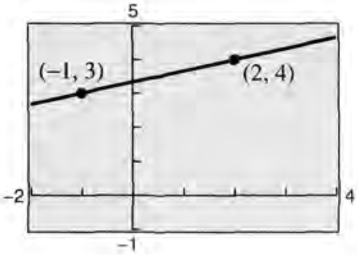
Note that this method of finding the equation of a line works for all lines, including horizontal and vertical lines. For instance, the equation of the vertical line through $(2, 0)$ and $(2, 2)$ is

$$\begin{vmatrix} x & y & 1 \\ 2 & 0 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0$$

$$-2x + 4 = 0$$

$$x = 2.$$

Figure 8.5



EXAMPLE 4 Finding an Equation of a Line

Find an equation of the line passing through the two points (2, 4) and (−1, 3), as shown in Figure 8.5.

Solution

Applying the determinant formula for the equation of a line produces

$$\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0.$$

To evaluate this determinant, you can expand by cofactors along the first row to obtain the following.

$$\begin{aligned} x(-1)^2\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} + y(-1)^3\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} &= x - 3y + 10 \\ &= 0 \end{aligned}$$

Therefore, an equation of the line is

$$x - 3y + 10 = 0.$$

Note There are a variety of ways to check that the equation of the line in Example 4 is correct. You can check it algebraically using the techniques you learned in Section P.3, or you can check it graphically by plotting the points and graphing the line in the same viewing rectangle.

Cramer's Rule

So far, you have studied three methods for solving a system of linear equations: substitution, elimination (with equations), and elimination (with matrices). You will now study one more method, **Cramer's Rule,** named after Gabriel Cramer (1704–1752). This rule uses determinants to write the solution of a system of linear equations. To see how Cramer's Rule works, take another look at the solution described at the beginning of Section 8.4. There, it was pointed out that the system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

has a solution given by

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that $a_1b_2 - a_2b_1 \neq 0$. Each numerator and denominator in this solution can be expressed as a determinant, as follows.

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Relative to the original system, the denominator for x and y is simply the determinant of the *coefficient* matrix of the system. This determinant is denoted by D. The numerators for x and y are denoted by D_x and D_y, respectively. They are formed by using the column of constants as replacements for the coefficients of x and y, as follows.

Coefficient Matrix	D	D_x	D_y
$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$	$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$	$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

EXAMPLE 5 Using Cramer's Rule for a 2 × 2 System

Use Cramer's Rule to solve the following system of linear equations.

$$4x - 2y = 10$$
$$3x - 5y = 11$$

Solution

To begin, find the determinant of the coefficient matrix.

$$D = \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} = -20 - (-6) = -14$$

Because this determinant is not zero, you can apply Cramer's Rule to find the solution, as follows.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix}}{-14} = \frac{(-50) - (-22)}{-14} = \frac{-28}{-14} = 2$$

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix}}{-14} = \frac{44 - 30}{-14} = \frac{14}{-14} = -1$$

Therefore, the solution is $x = 2$ and $y = -1$. Check this in the original system.

Cramer's Rule generalizes easily to systems of n equations in n variables. The value of each variable is given as the quotient of two determinants. The denominator is the determinant of the coefficient matrix, and the numerator is the determinant of the matrix formed by replacing the column corresponding to the variable (being solved for) with the column representing the constants. For instance, the solution for x_3 in the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

is given by

$$x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}.$$

Cramer's Rule

If a system of n linear equations in n variables has a coefficient matrix A with a *nonzero* determinant $|A|$, the solution is given by

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad . \; . \; . \; , \quad x_n = \frac{|A_n|}{|A|}$$

where the ith column of A_i is the column of constants in the system of equations. If the coefficient matrix is zero, the system has either no solution or infinitely many solutions.

EXAMPLE 6 Using Cramer's Rule for a 3 × 3 System

Use Cramer's Rule, if possible, to solve the following system of linear equations.

$$\begin{aligned} -x \qquad + z &= 4 \\ 2x - y + z &= -3 \\ y - 3z &= 1 \end{aligned}$$

Solution

Using the matrix capabilities of a graphing utility to evaluate the determinant of the coefficient matrix A, you find that Cramer's Rule cannot be applied because $|A| = 0$.

Cryptography

A **cryptogram** is a message written according to a secret code. (The Greek word *kryptos* means "hidden.") Matrix multiplication can be used to **encode** and **decode** messages. To begin, you need to assign a number to each letter in the alphabet (with 0 assigned to a blank space), as follows.

0 = _
1 = A
2 = B
3 = C
4 = D
5 = E
6 = F
7 = G
8 = H
9 = I
10 = J
11 = K
12 = L
13 = M
14 = N
15 = O
16 = P
17 = Q
18 = R
19 = S
20 = T
21 = U
22 = V
23 = W
24 = X
25 = Y
26 = Z

Then the message is converted to numbers and partitioned into **uncoded row matrices,** each having n entries, as demonstrated in Example 7.

EXAMPLE 7 Forming Uncoded Row Matrices

Write the uncoded row matrices of order 1×3 for the message

MEET ME MONDAY.

Solution

Partitioning the message (including blank spaces, but ignoring punctuation) into groups of three produces the following uncoded row matrices.

$$\begin{array}{ccccccccccccccc} [13 & 5 & 5] & [20 & 0 & 13] & [5 & 0 & 13] & [15 & 14 & 4] & [1 & 25 & 0] \\ M & E & E & T & & M & E & & M & O & N & D & A & Y & \end{array}$$

Note that a blank space is used to fill out the last uncoded row matrix.

To **encode** a message, choose an $n \times n$ invertible matrix A and multiply the uncoded row matrices by A (on the right) to obtain **coded row matrices.** Here is an example.

Uncoded Matrix *Encoding Matrix A* *Coded Matrix*

$$[13 \quad 5 \quad 5] \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [13 \quad -26 \quad 21]$$

This technique is further illustrated in Example 8.

EXAMPLE 8 Encoding a Message

Use the following matrix to encode the message MEET ME MONDAY.

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

Solution

The coded row matrices are obtained by multiplying each of the uncoded row matrices found in Example 7 by the matrix A, as follows.

Uncoded Matrix   Encoding Matrix A   Coded Matrix

$$[13 \quad 5 \quad 5]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [13 \quad -26 \quad 21]$$

$$[20 \quad 0 \quad 13]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [33 \quad -53 \quad -12]$$

$$[5 \quad 0 \quad 13]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [18 \quad -23 \quad -42]$$

$$[15 \quad 14 \quad 4]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [5 \quad -20 \quad 56]$$

$$[1 \quad 25 \quad 0]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [-24 \quad 23 \quad 77]$$

Thus, the sequence of coded row matrices is

$$[13 \ -26 \ \ 21][33 \ -53 -12][18 \ -23 -42][5 \ -20 \ \ 56][-24 \ \ 23 \ \ 77].$$

Finally, removing the matrix notation produces the following cryptogram.

$$13 \ -26 \ \ 21 \ \ 33 \ -53 -12 \ \ 18 \ -23 -42 \ \ 5 \ -20 \ \ 56 \quad -24 \ \ 23 \ \ 77$$

An efficient method for encoding the message at the right with your graphing utility is to enter A as a 3×3 matrix. Let B be the 5×3 matrix whose rows are the uncoded row matrices,

$$B = \begin{bmatrix} 13 & 5 & 5 \\ 20 & 0 & 13 \\ 5 & 0 & 13 \\ 15 & 14 & 4 \\ 1 & 25 & 0 \end{bmatrix}.$$

The product BA gives the coded row matrices.

For those who do not know the matrix A, decoding the cryptogram found in Example 8 is difficult. But for an authorized receiver who knows the matrix A, decoding is simple. The receiver need only multiply the coded row matrices by A^{-1} (on the right) to retrieve the uncoded row matrices. Here is an example.

$$\underbrace{[13 \quad -26 \quad 21]}_{\text{Coded}}A^{-1} = \underbrace{[13 \quad 5 \quad 5]}_{\text{Uncoded}}$$

EXAMPLE 9 Decoding a Message

Use the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$ to decode the cryptogram

$$13\ -26\ \ 21\ \ 33\ -53\ -12\ \ 18\ -23\ -42\ \ 5\ -20\ \ 56\ -24\ \ 23\ \ 77.$$

Solution

Partition the message into groups of three to form the coded row matrices. Then, multiply each coded row matrix by A^{-1} (on the right).

Coded Matrix *Decoding Matrix* A^{-1} *Decoded Matrix*

$$\begin{bmatrix} 13 & -26 & 21 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 13 & 5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 33 & -53 & -12 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 18 & -23 & -42 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -20 & 56 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 15 & 14 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -24 & 23 & 77 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 25 & 0 \end{bmatrix}$$

Thus, the message is as follows.

$$\begin{array}{ccccccccccccccc} [13 & 5 & 5] & [20 & 0 & 13] & [5 & 0 & 13] & [15 & 14 & 4] & [1 & 25 & 0] \\ \text{M} & \text{E} & \text{E} & \text{T} & & \text{M} & \text{E} & & \text{M} & \text{O} & \text{N} & \text{D} & \text{A} & \text{Y} & \end{array}$$

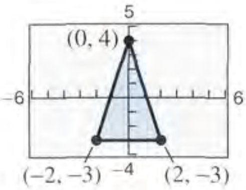
Group Activity *Cryptography*

Create your own numeric code for the alphabet (such as on page 643), and use it to convert a message of your own into numbers. Create an invertible $n \times n$ matrix A to encode your message. Exchange your numeric code, encoded message, and matrix A with another group. Find the necessary decoding matrix and decode the message you received.

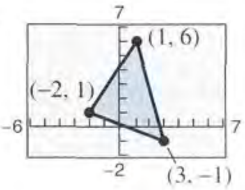
8.5 /// EXERCISES

In Exercises 1–10, use a determinant to find the area of the triangle with the given vertices.

1.



2.



3. $(0, 0), (1, 5), (3, 1)$

4. $(0, 0), (4, 5), (5, -2)$

5. $\left(0, \frac{1}{2}\right), \left(\frac{5}{2}, 0\right), (4, 3)$

6. $(0, 4), (2, 3), (5, 0)$

7. $(4, 5), (6, 1), (7, 9)$

8. $(0, -2), (-1, 4), (3, 5)$

9. $(-3, 5), (2, 6), (3, -5)$

10. $(-2, 4), (1, 5), (3, -2)$

In Exercises 11 and 12, find a value of x such that the triangle has an area of 4.

11. $(-5, 1), (0, 2), (-2, x)$

12. $(-4, 2), (-3, 5), (-1, x)$

In Exercises 13–16, use Cramer's Rule to solve (if possible) the system of equations.

13. $$\begin{aligned} 3x + 4y &= -2 \\ 5x + 3y &= 4 \end{aligned}$$

14. $$\begin{aligned} -0.4x + 0.8y &= 1.6 \\ 0.2x + 0.3y &= 2.2 \end{aligned}$$

15. $$\begin{aligned} 4x - y + z &= -5 \\ 2x + 2y + 3z &= 10 \\ 5x - 2y + 6z &= 1 \end{aligned}$$

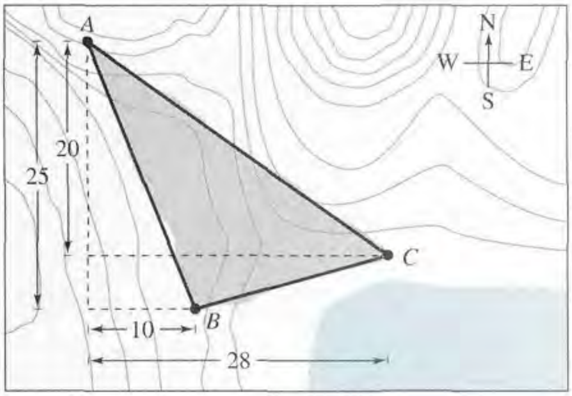
16. $$\begin{aligned} 4x - 2y + 3z &= -2 \\ 2x + 2y + 5z &= 16 \\ 8x - 5y - 2z &= 4 \end{aligned}$$

In Exercises 17 and 18, use a graphing utility and Cramer's Rule to solve (if possible) the system of equations.

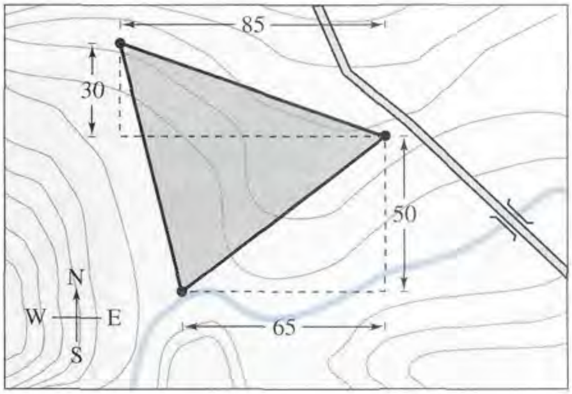
17. $$\begin{aligned} 3x + 3y + 5z &= 1 \\ 3x + 5y + 9z &= 2 \\ 5x + 9y + 17z &= 4 \end{aligned}$$

18. $$\begin{aligned} 2x + 3y + 5z &= 4 \\ 3x + 5y + 9z &= 7 \\ 5x + 9y + 17z &= 13 \end{aligned}$$

19. *Area of a Region* A large region of forest has been infected with gypsy moths. The region is roughly triangular, as shown in the figure. From the northernmost vertex A of the region, the distances to the other vertices are 25 miles south and 10 miles east (for vertex B), and 20 miles south and 28 miles east (for vertex C). Use a graphing utility to approximate the number of square miles in this region.



20. *Area of a Region* You own a triangular tract of land, as shown in the figure. To estimate the number of square feet in the tract, you start at one vertex, walk 65 feet east and 50 feet north to the second vertex, and then walk 85 feet west and 30 feet north to the third vertex. Use a graphing utility to determine how many square feet there are in the tract of land.



In Exercises 21–26, use the determinant feature of a graphing utility to decide if the points are collinear.

21. $(3, -1), (0, -3), (12, 5)$

22. $(-3, -5), (6, 1), (10, 2)$

23. $\left(2, -\frac{1}{2}\right), (-4, 4), (6, -3)$

24. $(0, 1), (4, -2), (-8, 7)$

25. $(0, 2), (1, 2.4), (-1, 1.6)$

26. $(2, 3), (3, 3.5), (-1, 2)$

In Exercises 27–32, use a determinant to find an equation of the line through the points.

27. $(0, 0), (5, 3)$

28. $(0, 0), (-2, 2)$

29. $(-4, 3), (2, 1)$

30. $(10, 7), (-2, -7)$

31. $\left(-\frac{1}{2}, 3\right), \left(\frac{5}{2}, 1\right)$

32. $\left(\frac{2}{3}, 4\right), (6, 12)$

In Exercises 33 and 34, find x such that the points are collinear.

33. $(2, -5), (4, x), (5, -2)$

34. $(-6, 2), (-5, x), (-3, 5)$

In Exercises 35 and 36, find the uncoded 1×3 row matrices for the message. Then encode the message using the matrix.

Message	*Matrix*
35. TROUBLE IN RIVER CITY	$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$
36. PLEASE SEND MONEY	$\begin{bmatrix} 4 & 2 & 1 \\ -3 & -3 & -1 \\ 3 & 2 & 1 \end{bmatrix}$

In Exercises 37–40, write a cryptogram for the message using the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}.$$

37. LANDING SUCCESSFUL

38. BEAM ME UP SCOTTY

39. HAPPY BIRTHDAY

40. OPERATION OVERLORD

In Exercises 41 and 42, use A^{-1} to decode the cryptogram.

41. $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

11, 21, 64, 112, 25, 50, 29, 53, 23, 46, 40, 75, 55, 92

42. $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$

9, −1, −9, 38, −19, −19, 28, −9, −19, −80, 25, 41, −64, 21, 31, 9, −5, −4

In Exercises 43 and 44, decode the cryptogram by using the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}.$$

43. 20, 17, −15, −12, −56, −104, 1, −25, −65, 62, 143, 181

44. 13, −9, −59, 61, 112, 106, −17, −73, −131, 11, 24, 29, 65, 144, 172

45. The following cryptogram was encoded with a 2×2 matrix.

8, 21, −15, −10, −13, −13, 5, 10, 5, 25,
5, 19, −1, 6, 20, 40, −18, −18, 1, 16

The last word of the message is _RON. What is the message?

46. The following cryptogram was encoded with a 2×2 matrix.

5, 2, 25, 11, −2, −7, −15, −15, 32,
14, −8, −13, 38, 19, −19, −19, 37, 16

The last word of the message is _SUE. What is the message?

Focus on Concepts

In this chapter, you studied several concepts that are required in the study of matrices and determinants and their applications. You can use the following questions to check your understanding of several of these basic concepts. The answers to these questions are given in the back of the book.

1. Describe the three elementary row operations that can be performed on an augmented matrix.

2. What is the relationship between the three elementary row operations on an augmented matrix and the operations that lead to equivalent systems of equations?

3. In your own words, describe the difference between a matrix in row-echelon form and a matrix in reduced row-echelon form.

In Exercises 4–7, the row-echelon form of an augmented matrix that corresponds to a system of linear equations is given. Use the matrix to determine whether the system is consistent or inconsistent, and if it is consistent, determine the number of solutions.

4. $\left[\begin{array}{ccccc} 1 & 2 & 3 & \vdots & 9 \\ 0 & 1 & -2 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$

5. $\left[\begin{array}{ccccc} 1 & 2 & 3 & \vdots & 9 \\ 0 & 1 & -2 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & 8 \end{array}\right]$

6. $\left[\begin{array}{ccccc} 1 & 2 & 3 & \vdots & 9 \\ 0 & 1 & -2 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & -3 \end{array}\right]$

7. $\left[\begin{array}{ccccccc} 1 & 2 & 3 & 10 & 6 & \vdots & 0 \\ 0 & 1 & -5 & -2 & 0 & \vdots & 5 \\ 0 & 0 & 1 & 12 & 0 & \vdots & -2 \\ 0 & 0 & 0 & 1 & 1 & \vdots & 0 \end{array}\right]$

In Exercises 8–10, determine if the matrix operations (a) $A + 3B$ and (b) AB can be performed. If not, state why.

8. $A = \begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 10 \\ 12 & 8 \end{bmatrix}$

9. $A = \begin{bmatrix} 5 & 4 \\ -7 & 2 \\ 11 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 12 \\ 20 & 40 \\ 15 & 30 \end{bmatrix}$

10. $A = \begin{bmatrix} 5 & 4 \\ -7 & 2 \\ 11 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 12 \\ 20 & 40 \end{bmatrix}$

11. Under what conditions does a matrix have an inverse?

12. Explain the difference between a square matrix and its determinant.

13. Is it possible to find the determinant of a 4 × 5 matrix? Explain.

14. What is meant by the cofactor of an entry of a matrix? How is it used to find the determinant of the matrix?

15. Three people were asked to solve a system of equations using an augmented matrix. Each person reduced the matrix to row-echelon form. The reduced matrices were

$$\left[\begin{array}{cccc} 1 & 2 & \vdots & 3 \\ 0 & 1 & \vdots & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc} 1 & 0 & \vdots & 1 \\ 0 & 1 & \vdots & 1 \end{array}\right]$$

and

$$\left[\begin{array}{cccc} 1 & 2 & \vdots & 3 \\ 0 & 0 & \vdots & 0 \end{array}\right].$$

Could all three be right? Explain.

8 /// REVIEW EXERCISES

In Exercises 1 and 2, form the augmented matrix for the system of linear equations.

1. $$\begin{aligned} 3x - 10y &= 15 \\ 5x + 4y &= 22 \end{aligned}$$

2. $$\begin{aligned} 8x - 7y + 4z &= 12 \\ 3x - 5y + 2z &= 20 \\ 5x + 3y - 3z &= 26 \end{aligned}$$

In Exercises 3 and 4, write the system of linear equations represented by the augmented matrix. (Use variables x, y, z, and w.)

3. $$\left[\begin{array}{rrr|r} 5 & 1 & 7 & -9 \\ 4 & 2 & 0 & 10 \\ 9 & 4 & 2 & 3 \end{array}\right]$$

4. $$\left[\begin{array}{rrrr|r} 13 & 16 & 7 & 3 & 2 \\ 1 & 21 & 8 & 5 & 12 \\ 4 & 10 & -4 & 3 & -1 \end{array}\right]$$

In Exercises 5 and 6, write the matrix in *reduced* row-echelon form.

5. $$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

6. $$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

In Exercises 7–10, use the matrix capabilities of a graphing utility to write the matrix in *reduced* row-echelon form.

7. $$\begin{bmatrix} 3 & -2 & 1 & 0 \\ 4 & -3 & 0 & 1 \end{bmatrix}$$

8. $$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 2 & 8 & 0 & 0 & 1 \end{bmatrix}$$

9. $$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 2 & 4 & 6 \end{bmatrix}$$

10. $$\begin{bmatrix} 4 & 8 & 16 \\ 3 & -1 & 2 \\ -2 & 10 & 12 \end{bmatrix}$$

In Exercises 11–22, use matrices and elementary row operations to solve (if possible) the system of equations.

11. $$\begin{aligned} 5x + 4y &= 2 \\ -x + y &= -22 \end{aligned}$$

12. $$\begin{aligned} 2x - 5y &= 2 \\ 3x - 7y &= 1 \end{aligned}$$

13. $$\begin{aligned} 2x + y &= 0.3 \\ 3x - y &= -1.3 \end{aligned}$$

14. $$\begin{aligned} 0.2x - 0.1y &= 0.07 \\ 0.4x - 0.5y &= -0.01 \end{aligned}$$

15. $$\begin{aligned} 2x + y + 2z &= 4 \\ 2x + 2y \phantom{{}+6z} &= 5 \\ 2x - y + 6z &= 2 \end{aligned}$$

16. $$\begin{aligned} 2x + 3y + z &= 10 \\ 2x - 3y - 3z &= 22 \\ 4x - 2y + 3z &= -2 \end{aligned}$$

17. $$\begin{aligned} 4x + 4y + 4z &= 5 \\ 4x - 2y - 8z &= 1 \\ 5x + 3y + 8z &= 6 \end{aligned}$$

18. $$\begin{aligned} 2x + 3y + 3z &= 3 \\ 6x + 6y + 12z &= 13 \\ 12x + 9y - z &= 2 \end{aligned}$$

19. $$\begin{aligned} -x + y + 2z &= 1 \\ 2x + 3y + z &= -2 \\ 5x + 4y + 2z &= 4 \end{aligned}$$

20. $$\begin{aligned} 3x + 21y - 29z &= -1 \\ 2x + 15y - 21z &= 0 \end{aligned}$$

21. $$\begin{aligned} x + 2y + 6z &= 1 \\ 2x + 5y + 15z &= 4 \\ 3x + y + 3z &= -6 \end{aligned}$$

22. $$\begin{aligned} x + 2y + w &= 3 \\ -3y + 3z &= 0 \\ 4x + 4y + z + 2w &= 0 \\ 2x + z &= 3 \end{aligned}$$

23. *Think About It* Describe the row-echelon form of an augmented matrix that corresponds to a system of linear equations that has a unique solution.

24. *Partial Fractions* Write the partial fraction decomposition for the rational expression

$$\frac{x+9}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$

In Exercises 25–28, use the matrix capabilities of a graphing utility to reduce the augmented matrix and solve the system of equations.

25. $\begin{aligned} x - 3y &= -2 \\ x + y &= 2 \end{aligned}$

26. $\begin{aligned} -x + 3y &= 5 \\ 4x - y &= 2 \end{aligned}$

27. $\begin{aligned} x + 2y - z &= 7 \\ -y - z &= 4 \\ 4x \quad - z &= 16 \end{aligned}$

28. $\begin{aligned} 3x \quad + 6z &= 0 \\ -2x + y \quad &= 5 \\ y + 2z &= 3 \end{aligned}$

In Exercises 29–36, perform the matrix operations. If it is not possible, explain why.

29. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & -4 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 & -6 \\ 0 & -2 & 5 \end{bmatrix}$

30. $-2\begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix} + 8\begin{bmatrix} 7 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 6 & -2 & 8 \\ 4 & 0 & 0 \end{bmatrix}$

32. $\begin{bmatrix} 1 & 5 & 6 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} 6 & -2 & 8 \\ 4 & 0 & 0 \end{bmatrix}$

33. $\begin{bmatrix} 1 & 5 & 6 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} 6 & 4 \\ -2 & 0 \\ 8 & 0 \end{bmatrix}$

34. $\begin{bmatrix} 4 \\ 6 \end{bmatrix}\begin{bmatrix} 6 & -2 \end{bmatrix}$

35. $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 4 & -3 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

36. $\begin{bmatrix} 2 & 1 \\ 6 & 0 \end{bmatrix}\left(\begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 0 & 4 \end{bmatrix}\right)$

In Exercises 37–40, use a graphing utility to perform the matrix operations.

37. $3\begin{bmatrix} 8 & -2 & 5 \\ 1 & 3 & -1 \end{bmatrix} + 6\begin{bmatrix} 4 & -2 & -3 \\ 2 & 7 & 6 \end{bmatrix}$

38. $-5\begin{bmatrix} 2 & 0 \\ 7 & -2 \\ 8 & 2 \end{bmatrix} + 4\begin{bmatrix} 4 & -2 \\ 6 & 11 \\ -1 & 3 \end{bmatrix}$

39. $\begin{bmatrix} 4 & 1 \\ 11 & -7 \\ 12 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 & 6 \\ 2 & -2 & -2 \end{bmatrix}$

40. $\begin{bmatrix} -2 & 3 & 10 \\ 4 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -5 & 2 \\ 3 & 2 \end{bmatrix}$

In Exercises 41–44, solve for X given

$$A = \begin{bmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{bmatrix}.$$

41. $X = 3A - 2B$

42. $6X = 4A + 3B$

43. $3X + 2A = B$

44. $2A - 5B = 3X$

45. Write the system of linear equations represented by the matrix equation

$$\begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -22 \end{bmatrix}.$$

46. Write the matrix equation $AX = B$ for the following system of linear equations.

$$\begin{aligned} 2x + 3y + z &= 10 \\ 2x - 3y - 3z &= 22 \\ 4x - 2y + 3z &= -2 \end{aligned}$$

In Exercises 47–50, use a graphing utility to find the inverse of the matrix (if it exists).

47. $\begin{bmatrix} 2 & 6 \\ 3 & -6 \end{bmatrix}$

48. $\begin{bmatrix} 3 & -10 \\ 4 & 2 \end{bmatrix}$

49. $\begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$

50. $\begin{bmatrix} 1 & 4 & 6 \\ 2 & -3 & 1 \\ -1 & 18 & 16 \end{bmatrix}$

In Exercises 51–58, evaluate the determinant. Use a graphing utility to verify your result.

51. $\begin{vmatrix} 50 & -30 \\ 10 & 5 \end{vmatrix}$

52. $\begin{vmatrix} 8 & 5 \\ 2 & -4 \end{vmatrix}$

53. $\begin{vmatrix} 10 & 8 \\ -6 & -4 \end{vmatrix}$

54. $\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$

55. $\begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{vmatrix}$

56. $\begin{vmatrix} 0 & 3 & 1 \\ 5 & -2 & 1 \\ 1 & 6 & 1 \end{vmatrix}$

57. $\begin{vmatrix} 3 & 0 & -4 & 0 \\ 0 & 8 & 1 & 2 \\ 6 & 1 & 8 & 2 \\ 0 & 3 & -4 & 1 \end{vmatrix}$

58. $\begin{vmatrix} -5 & 6 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ -3 & 4 & -5 & 1 \\ 1 & 6 & 0 & 3 \end{vmatrix}$

In Exercises 59–66, use a graphing utility to solve (if possible) the system of linear equations using the inverse of the coefficient matrix.

59. $\begin{aligned} x + 2y &= -1 \\ 3x + 4y &= -5 \end{aligned}$

60. $\begin{aligned} x + 3y &= 23 \\ -6x + 2y &= -18 \end{aligned}$

61. $\begin{aligned} -3x - 3y - 4z &= 2 \\ y + z &= -1 \\ 4x + 3y + 4z &= -1 \end{aligned}$

62. $\begin{aligned} x - 3y - 2z &= 8 \\ -2x + 7y + 3z &= -19 \\ x - y - 3z &= 3 \end{aligned}$

63. $\begin{aligned} x + 3y + 2z &= 2 \\ -2x - 5y - z &= 10 \\ 2x + 4y &= -12 \end{aligned}$

64. $\begin{aligned} 2x + 4y &= -12 \\ 3x + 4y - 2z &= -14 \\ -x + y + 2z &= -6 \end{aligned}$

65. $\begin{aligned} -x + y + z &= 6 \\ 4x - 3y + z &= 20 \\ 2x - y + 3z &= 8 \end{aligned}$

66. $\begin{aligned} 2x + 3y - 4z &= 1 \\ x - y + 2z &= -4 \\ 3x + 7y - 10z &= 0 \end{aligned}$

In Exercises 67–70, use Cramer's Rule to solve (if possible) the system of equations.

67. $\begin{aligned} x + 2y &= 5 \\ -x + y &= 1 \end{aligned}$

68. $\begin{aligned} 2x - y &= -10 \\ 3x + 2y &= -1 \end{aligned}$

69. $\begin{aligned} 20x + 8y &= 11 \\ 12x - 24y &= 21 \end{aligned}$

70. $\begin{aligned} 13x - 6y &= 17 \\ 26x - 12y &= 8 \end{aligned}$

In Exercises 71–74, use a graphing utility and Cramer's Rule to solve (if possible) the system of equations.

71. $\begin{aligned} 3x + 6y &= 5 \\ 6x + 14y &= 11 \end{aligned}$

72. $\begin{aligned} -0.4x + 0.8y &= 1.6 \\ 0.2x + 0.3y &= 2.2 \end{aligned}$

73. $\begin{aligned} 5x - 3y + 2z &= 2 \\ 2x + 2y - 3z &= 3 \\ x - 7y + 8z &= -4 \end{aligned}$

74. $\begin{aligned} 14x - 21y - 7z &= 10 \\ -4x + 2y - 2z &= 4 \\ 56x - 21y + 7z &= 5 \end{aligned}$

75. *Mixture Problem* A florist wants to arrange a dozen flowers consisting of two varieties: carnations and roses. Carnations cost \$0.75 each and roses cost \$1.50 each. How many of each should the florist use so that the arrangement will cost \$12.00?

76. *Mixture Problem* One hundred liters of a 60% acid solution is obtained by mixing a 75% solution with a 50% solution. How many liters of each must be used to obtain the desired mixture?

77. *Fitting a Parabola to Three Points* Find an equation of the parabola $y = ax^2 + bx + c$ that passes through the points $(-1, 2)$, $(0, 3)$, and $(1, 6)$.

78. *Break-Even Point* A small business invests \$25,000 in equipment to produce a product. Each unit of the product costs \$3.75 to produce and is sold for \$5.25. How many items must be sold before the business breaks even?

79. *Data Analysis* The median prices y (in thousands of dollars) of one-family houses sold in the United States in the years 1981 through 1993 are shown in the figure. The least squares regression line $y = a + bt$ for this data is found by solving the system

$$\begin{aligned} 13a + 91b &= 1107 \\ 91a + 819b &= 8404.7 \end{aligned}$$

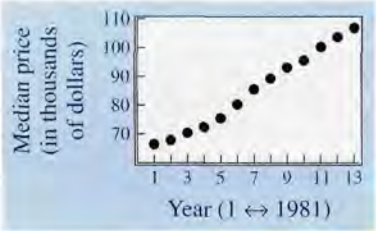
where $t = 1$ represents 1981. (Source: National Association of Realtors)

(a) Use a graphing utility to solve this system.

(b) Use a graphing utility to graph the regression line.

(c) Interpret the meaning of the slope of the regression line in the context of the problem.

(d) Use the regression line to estimate the median price of homes in 1995.



80. Solve the equation $\begin{vmatrix} 2-\lambda & 5 \\ 3 & -8-\lambda \end{vmatrix} = 0.$

In Exercises 81–84, use a determinant to find the area of the triangle with the given vertices.

81. (1, 0), (5, 0), (5, 8)

82. (−4, 0), (4, 0), (0, 6)

83. (1, 2), (4, −5), (3, 2)

84. $\left(\frac{3}{2}, 1\right), \left(4, -\frac{1}{2}\right), (4, 2)$

In Exercises 85–88, use a determinant to find an equation of the line through the given points.

85. (−4, 0), (4, 4)

86. (2, 5), (6, −1)

87. $\left(-\frac{5}{2}, 3\right), \left(\frac{7}{2}, 1\right)$

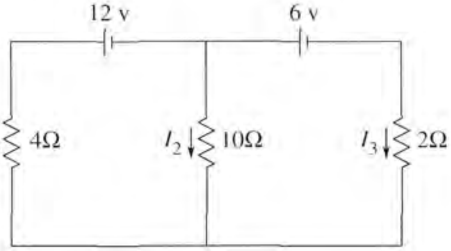
88. (−0.8, 0.2), (0.7, 3.2)

89. Verify that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+c_1 & a_{32}+c_2 & a_{33}+c_3 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

90. *Circuit Analysis* Consider the circuit in the figure. The currents I_1, I_2, and I_3 in amperes are given by the solution of the system of linear equations. Use the inverse of the coefficient matrix of this system to find the unknown currents.

$$\begin{aligned} I_1 + I_2 + I_3 &= 0 \\ 4I_1 - 10I_2 &= 12 \\ 10I_2 - 2I_3 &= -6 \end{aligned}$$



91. *Think About It* If A is a 3×3 matrix and $|A| = 2$, what is the value of $|4A|$? Give the reason for your answer.

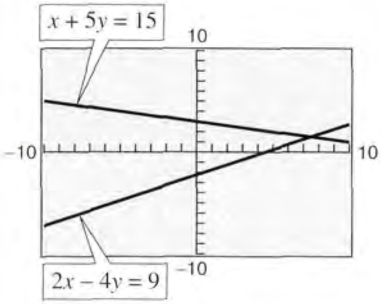
CHAPTER PROJECT *Row Operations and Graphing*

In this project, you will investigate the graphical interpretation of elementary row operations.

(a) Solve the following systems by hand using Gauss-Jordan elimination.

$$\begin{aligned} 2x - 4y &= 9 \\ x + 5y &= 15 \end{aligned} \qquad \begin{aligned} 6x + 2y &= -19 \\ 3x - y &= -5 \end{aligned}$$

(b) Enter the row operations program listed in the appendix into a graphing calculator. This program demonstrates how elementary row operations used in Gauss-Jordan elimination may be depicted graphically. For each system in part (a), run the program using a 2 × 3 matrix that corresponds to the system of equations. Compare the results of the program with those you obtained in part (a).



Graph of the system

$$\begin{aligned} 2x - 4y &= 9 \\ x + 5y &= 15 \end{aligned}$$

(c) During the running of the program, a row of the matrix is multiplied by a constant. What effect does this operation have on the graph of the corresponding linear equation?

(d) During the running of the program, a multiple of the first row of the matrix is added to the second row to obtain a 0 below the leading 1. What effect does this operation have on the graph of the corresponding linear equation?

(e) Each time the 2 × 3 matrix is transformed, the graph of the corresponding linear equations is displayed. What do you notice about the point of intersection each time?

Questions for Further Exploration

1. Is finding a point of intersection using the program more or less accurate than finding the point of intersection using the zoom and trace features? Explain your reasoning and give an example.

2. Run the program to find the solution to the following linear system.

$$\begin{aligned} 2x \qquad &= -15 \\ 3x + 5y &= 3 \end{aligned}$$

Why is only one line drawn in all but the last screen? Verify the program's solution by hand.

3. Run the program using the following linear system.

$$\begin{aligned} 2y &= -3 \\ -2x + y &= 3 \end{aligned}$$

Describe what happens and why.

4. A system of equations with three variables has a corresponding 3 × 4 augmented matrix. Write a program that will transform a 3 × 4 matrix into reduced row-echelon form. At the end of the program display the final matrix.

8 /// CHAPTER TEST

Take this test as you would take a test in class. After you are done, check your work against the answers given in the back of the book.

The *Interactive* CD-ROM provides answers to the Chapter Tests and Cumulative Tests. It also offers Chapter Pre-Tests (that test key skills and concepts covered in previous chapters) and Chapter Post-Tests, both of which have randomly generated exercises with diagnostic capabilities.

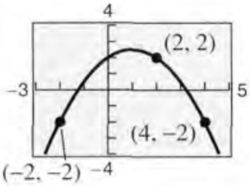
In Exercises 1 and 2, write the matrix in reduced row-echelon form. Use a graphing utility to verify your result.

1. $\begin{bmatrix} 1 & -1 & 5 \\ 6 & 2 & 3 \\ 5 & 3 & -3 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ 3 & 2 & -3 & 4 \end{bmatrix}$

3. Use the matrix capabilities of a graphing utility to reduce the augmented matrix and solve the system of equations.

$$\begin{aligned} 4x + 3y - 2z &= 14 \\ -x - y + 2z &= -5 \\ 3x + y - 4z &= 8 \end{aligned}$$

Figure for 4



4. Find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points in the figure. Use a graphing utility to verify your result.

5. Find (a) $A - B$, (b) $3A$, and (c) $3A - 2B$.

$$A = \begin{bmatrix} 5 & 4 & 4 \\ -4 & -4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 & 6 \\ -4 & 0 & -3 \end{bmatrix}$$

6. Find AB, if possible.

$$A = \begin{bmatrix} 2 & -2 & 6 \\ 3 & -1 & 7 \\ 2 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 4 \\ 3 & 2 \\ 1 & -2 \end{bmatrix}$$

7. Find A^{-1} for $A = \begin{bmatrix} -6 & 4 \\ 10 & -5 \end{bmatrix}$.

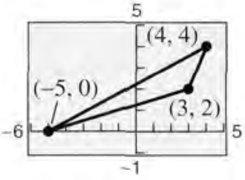
8. Use the result of Exercise 7 to solve the system.

$$\begin{aligned} -6x + 4y &= 10 \\ 10x - 5y &= 20 \end{aligned}$$

9. Evaluate the determinant of the matrix

$$\begin{bmatrix} 4 & 0 & 3 \\ 1 & -8 & 2 \\ 3 & 2 & 2 \end{bmatrix}.$$

Figure for 10



10. Use a determinant to find the area of the triangle in the figure.