TK-YUPANA

A SIMPLE INCAN ABACUS EMULATOR

By Kunturweb

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Kunturweb

Kunturweb is a project born from my passion for everything is concerned with history and culture of pre-Columbian civilizations.

Kuntur is the quechua name for the Andean Condor (*Vultur Gryphus*); **Web** is a synonymous of *World Wide Web* or *WWW*.

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Introductory Note

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1 - The Yupana

With the term **Yupana** we mean an abacus, used at the time of the Incas, by the accountants and treasurers of the empire, called *Khipu Camayuq*[1].

The name comes from the Quechua word Yupay (count) while on paternity has arisen over the years a lot of confusion not only among scholars, but also among researchers and professors: alcuni[§MOV] attribute it to the engineer *William Burns Glynn*, other[§RDP], at the time of the Incas, as the word is mentioned in an ancient Quechua language vocabulary compiled by *Diego Gonzalez Holguín*[§HOL] in 1608; but, Holguín, translates the term "***Yupana***" with "*Letra los numeros de guarismos*" which we can translate "*Numbers*"[2], while "***Yupana qqellca o qquipu***", as "*Las cuentas con nudos o por escrito*" meaning "Calculations with knots or in writing". Those who translates "*Table to count*" with "***Yupana o quippo***" is *Domingo de Santo Tomás* in his "*Lexicon, o Vocabulario de la lengua general del Peru*"[§SAT], dated 1560. So the word has Quechua origin and was used to indicate the instrument used for counting.

There are two classes of "objects" to which we refer when we speak of ***yupana***. The first class consists of a series of archaeological finds similar to boards of wood or stone whose boxes have different shapes and sizes; are thought (but is not certain[3]) that could be used as schedules. Henceforth we will refer to them with the name of "*yupane a casetta*".

The second class is actually made up of a single element: a design that appears in the manuscript "*El Primer Nueva Coronica y Buen Gobierno*" by *Felipe Guaman Poma de Ayala*, written in 1615, but found relatively recently in the library of Copenhagen[§POM]. This drawing depicts an accountant of the Inca empire and, at its foot, a board consisting of five rows and four columns, which boxes contain whites and blacks (or empty and full) circles.

The two classes of objects are, in my opinion, be treated separately because the artwork and archaeological finds are absolutely inconsistent. Furthermore, while the description of yupana of *Poma de Ayala* certifies its use as a table of calculation, there are not enough elements to establish with certainty the function of the "*yupana a casetta*".

1 Literally: Khipu (knot, meaning rope to which were made some knots to record accounting events) and Camayuq (teacher): teacher of the knots [A/N]

2 Some author translated it as "Digits of numbers, *or seeds corresponding to numbers to make knots on the Khipus*"[§LAU] and others with "*letter or numbers*"[§FLO] [N/A]

3 Some think it could be tables for gambling, others models of fortresses (cfr.[§RDP], chap. 3, page 11)

In this paper we will study the *yupana of Poma de Ayala* and the "*yupana a casetta*" will henceforth ignored.

1.1 - References to the Yupana by the Chroniclers of the Indies

In the next page of the drawing mentioned above, *Poma de Ayala* describes only approximately the method to make calculations used by the accountant of the empire. Poma de Ayala wrote:

"*Major accountant of the whole kingdom, Condor Chaua, son of gods: they called him Tawantin suyo rune quipoc Yncap, haziendan chasquicoc*[4] *major treasurer. It is said that he had a great ability. To test it, the Inca ordered him to number, count and adjust the natives of the kingdom. He associated the subjects with the wool of the Andean Deer*[5] *produced and with a food called quinoa and could determine how much quinoa and wool was produced. His skill was great, better that if he could write.*

Major accountant or hatun hucha quipoc[6] *and Accounting minor or Huchuy hucha quipoc*[7]*: they use tables, counting from one hundred thousand to ten thousand and one hundred and ten up to the unit. Everything that happens in this kingdom they shall annotate: parties, weeks, months and years. In every town there are these accountants and count starting from one, two and three: Suc [1], yscay [2], quinza [3], Taua [4], pichica [5], zocta [6], Canchis [7], puzac [8], yscon [9], chunga [10], yscay chunga [20], quinza chunga [30], Taua chonga [40], pisca chunga [50], zocta chunga [60], Canchis chunga [70], pozac chunga [80], yscon chunga [90], pachaca [100], uaranga [1000]*"[8]

Information collected:

- They used tables to count
- Probably had an upper limit of one hundred thousand and a lower limit to unity

4 The one who takes a census of the population of the Tawantinsuyo (Inca empire), the one who receives the taxes of the Inca, the one who keeps the books of the empire. [A/N]

5 It must be said that previous authors translate as "*He counted with strings made of wool of Andean deer*", a clear reference to the *khipu*, one of them is *Radicati di Primeglio* (op. cited above). But the type of wool used for Khipu, came from Llama or alpaca or cotton, so my translation seems more realistic, especially in light of the fact that "emparejar" means "to match"

6 The one that took into account the serious violations and shortages.[A/N]

7 He took account of the violations and shortages of minor importance [A/N]

8 "Contador mayor de todo este rreyno, *Condor Chaua*, hijo de *apo:* A éste le llamauan *Tawantin Suyo runa quipoc Yncap*, haziendan *chasquicoc*, tezorero mayor. Dize que este principal tenía grande auilidad; para sauer su auilidad el *Ynga* mandó contar y numirar, ajustar con los yndios deste rreyno. Con la lana del cierbo, taruga, enparexaua con la lana a los yndios y enparexaua con una comida llamado *quinua* [gramínea de altura], contaua la *quinua* y los yndios. Fue muy grande su auilidad, mejor fuera en papel y tinta. Contador mayor *hatun hucha quipoc*, contador menor *huchuy hucha quipoc*: Cuentan en tablas, numiran de cien mil y de dies mil y de ciento y de dies hasta llegar a una. De todo lo que pasan en este rreyno lo acienta y fiestas y domingos y meses y años. Y en cada ciudad y uilla y pueblos de yndios auía estos dichos contadores y tesoreros en este rreyno. Y contaua desta manera, comensando de uno, dos y tres: *Suc* [uno], *yscay* [dos], *quinza* [tres], *taua* [cuatro], *pichica* [5], *zocta* [6], *canchis* [7], *puzac* [8], *yscon* [9], *chunga* [10], *yscay chunga* [20], *quinza chunga* [30], *taua chonga* [40], *pisca chunga* [50], *zocta chunga* [60], *canchis chunga* [70], *pozac chunga* [80], *yscon chunga* [90], *pachaca* [100], *uaranga* [1000]". [§POM], page 361 (363) [translated by the Author]

- The numeral system used was decimal[9]

A second chronicler who wrote a few words about the counting system of the Incas, is the Jesuit *José de Acosta*, who in his work entitled "*Historia Natural y Moral de las Indias*" described, even approximately, how to use the abacus:

"These Indians took the grain and put one here, three there, eight in another location, move a grain from one mailbox to another exchange three other grains in order to obtain the perfect result, error-free"[10]

Information collected:

- They used seeds
- They moved and exchanged seeds from one box to another
- They were very precise

The third chronicler that I take into consideration is *Juan de Velasco*, who wrote:

"*The instrument used by these teachers was something like a series of trays, made of wood, stone or clay, with different separations, in which they put stones of different shapes and colours, even angular shapes*"[11]

Information collected:

- The instrument (singular) was composed of several parts (series of trays), each of them having different separations
- Used coloured stones with different shapes

Assuming that the three journalists were referring to the same object and the same method of calculation (after all why the accountants belonging to the same elite and educated in the same way, would have had to use different methods and tools to perform a specific calculation?) we can make further assumptions that are at the basis of some theories developed over the years on the Incan Abacus portrayed by *Poma de Ayala*.

- the black circles of the drawing by *Poma de Ayala* **may** be used as the equivalent of the grains in place of a certain amount;
- the number of holes (or seeds) present in the boxes **could** correspond to
    1. the value to be attached to the seeds present in a certain box
    2. the number of elements corresponding to a given value determined by the row/column of the box;

Still we do not have any certainty about the method used by accountants to perform arithmetic

9 The fact that the Incas were using a decimal system is also confirmed in a previous step of Nueva Coronica, when it comes to the division of the empire into classes. [§POM] - El capitulo de la visita general o censo, pages 195-236.

10 "*Y pondrán uno aquí, tres allá, ocho no sé donde; pasarán un grano de aquí, trocarán tres de allá y en efecto ellos salen con su cuenta hecha puntualisimamente sin errar un tilde*" [§ACO], book VI chap. XVIII – De los memoriales y cuentas que usaron los Indios del Perú, [translated by the author]

11 [§VEL] from pages 1841- 44, T.II cap. 7 [Translated by the author]

operations with the yupana of *Poma de Ayala* and interpretations continue to proliferate.

To understand the topics below, it is important that the reader focuses its attention on some essential concepts and that will hopefully deepen their own.

1.2 - Abacus and numeral systems

An **abacus** is an instrument invented and used by people of various cultures as an aid in mathematical calculations. The method of calculation used in the schedule is based on a specific *numeral system*.

By *numeral system* we refer to a way of representing the numbers using a series of symbols. The numeral systems are divided into two main categories: **sign-value** numeral systems and **positional** number systems.

In **sign-value** systems the value of the number represented is the sum of the values attributed to a limited series of basic symbols; the numbering of the Romans (at least in the initial version) is an example of sign-value notation systems, for it was defined a set of basic symbols: I, V , X, L, C, D, M, respectively corresponding to 1, 5, 10, 50, 100, 500, 1000. The value of the other numbers was obtainable from the sum of the values of the fundamental symbols:

1 = I
2 = I + I = II
3 = I + I + I = III
4 = I + I + I + I = IIII[12]
5 = V
6 = V + I = VI
7 = V + I + I = VII
8 = V + I + I + I= VIII
9 = V + I + I + I + I = VIIII[13]
10 = X
...

Note that in a purely additive system the representation of a number do not depend on the position of the basic symbols, i.e. the number 8 could also be written IVII or IIIV as the sum of the values of the symbols is the same as the conventional writing.

In **positional** systems the value of the symbols used to represent a number depends on the position occupied by the symbol. Among the positional numeral systems we distinguish different notations with respect to many types of base. The base is the number of digits that a unique positional numeral system used to represent all numbers.

An example of a positional numeral system is the *Arabic*, the most commonly used today. Each

12 Later IV, subtraction of number I from number V [N/A]
13 Later IX, subtraction of number I from number X [N/A]

number is represented by sequences of 10 digits (0, 1, 2, 3, ..., 9) and therefore the base is 10 and is said *decimal*. The right-most digit corresponds to the unit, the previous one to the tens, then hundreds, and so on. So the number can be written as a succession of previous figures, provided you comply with the Convention of positions: e.g. 5342 ($5 \times 10^3 + 3 \times 10^2 + 4 \times 10^1 + 2 \times 10^0$).

1.2.1 - Limitations of the sign-value notation

A sign-value numeral system is fine as long as you are dealing with small numbers. When you start to scale to tens of thousands, or millions, the number of symbols used to represent a certain value increases dramatically and we are forced to invent other symbols. For example, consider the number as above, and let's write it with Roman numerals:

5342 (4 symbols) = MMMMMCCCXXXXII (14 symbols)

Note that the sign-value numeral systems were used since ancient times even by primitive and prehistoric populations. The use of a sign-value numeral system, however, does not indicate a low degree of civilization, think for example to the Romans or the Greeks. However, a system of positional type, being more advantageous than sign-value one, is a winning choice over time and is intended to supplant it definitively.

2 - A century of theories

Here we will analyse the theories that have been proposed by various authors in nearly 100 years after the discovery of a copy of *Nueva Coronica of Poma de Ayala.*

Pay special attention to the following aspects, which will be highlighted and listed in a similar table at the beginning of each paragraph:

numeral System	Sign-value or positional?
Notation or **Base** (only of positional systems) **Powers of...** (only for sign-value system)	10, 12, 20, 40, 60, … Powers of 10
Table layout	Vertical: 5 rows x 4 columns Horizontal: 4 rows x 5 columns
Horizontal progression	Progression of numeral values assigned to each row
Vertical progression	Progression of numeral values assigned to each column

Consider also the following questions regarding the solution adopted; at the end of each chapter we will try to give an answer, that will ultimately serve to draw conclusions:

- Is it possible to represent all the numbers from one to a hundred thousand, with **only** a 5x4 table and the numeral system adopted?
- Are the operations to use the abacus simple?
- Is there any problem or inconsistency?

2.1 - Henry Wassen's Theory (1931)

Unfortunately I still have not found the original writings of Henry Wassen, therefore the arguments in this section are based on certain articles of other authors[§RDP] e [§MOV] who describe briefly the method by Wassen. In the future I hope to confirm what by now I just deduced and to make any changes to this document and Tk-yupana.

numeral System	Positional
Notation or **Base**	10
Table layout	Vertical: 5 rows x 4 columns
Horizontal progression	1, 5, 15, 30 (or 5, 15, 30, 30)
Vertical progression	1, 10, 100. 1000, 10000

Henry Wassen was the first to provide an interpretation of the drawing of the Yupana by *Poma de Ayala*. The hypothesis that is the basis of the theory of Wassen is that the white circles were holes where to deposit the seeds, while the black circles were such holes filled with a seed.

2.1.1 - Representing a number

The representation of a number in the yupana has a vertical progression of base 10 (decimal). This means that the number can be represented by placing the units in the first row (starting from the bottom), the tens digit in the second and so on. The value that each seed can assume, instead, depends on the column in which it is located, following the horizontal progression, based on the principle that in the first column it is possible to place a maximum number equal to 5 seeds having a value of 1 (for a total of 5 and equal the value of a single seed placed in column 2), in the second column a maximum number of three seeds having a value of 5 each (for a total of 15 and equal to the value of a single seed placed in column 3), in the third column a maximum of two seeds having the value of 15 each (for a total of 30 and equal to the value of a single seed placed in column 4). See Table 1.

ROW	POWER range values	COLUMN 1	COLUMN 2	COLUMN 3	COLUMN 4
		Value: 1	**Value: 5**	**Value: 15**	**Value: 30**
5	**10^4** **10,000-300,000**	10000-50000 ❍ ❍ ❍ ❍ ❍	50000-150000 ❍ ❍ ❍	150000-300000 ❍ ❍	300000 ❍
4	**10^3** **1,000-30,000**	1000-5000 ❍ ❍ ❍ ❍ ❍	5000-15000 ❍ ❍ ❍	15000-30000 ❍ ❍	30000 ❍
3	**10^2** **100-3,000**	100-500 ❍ ❍ ❍ ❍ ❍	500-1500 ❍ ❍ ❍	1500-3000 ❍ ❍	3000 ❍
2	**10^1** **10-300**	10-50 ❍ ❍ ❍ ❍ ❍	50-150 ❍ ❍ ❍	150-300 ❍ ❍	300 ❍
1	**10^0** **1-30**	1-5 ❍ ❍ ❍ ❍ ❍	5-15 ❍ ❍ ❍	15-30 ❍ ❍	30 ❍

Table 1: Scheme of the Yupana by H. Wassen – positional notation in base 10 with progression 1, 5, 15, 30

As an example for the representation of a number, choose 3,595. It fills the table from top to bottom and from left to right, bearing in mind that each digit corresponds to a row (positional system). We start with the thousands (4th row): three seeds (circles blacks) in the first section of the row, covering three holes (white circles); then continue with the hundreds (3rd row): five seeds in the first cell; then the tens (2nd row): since 9 is greater than 5, when five holes are filled (adding 5 seeds in the first box of the row), we can group the five seeds and replace them with one in the second box of the 2nd row; then proceed by adding the remaining four seeds in the first box of the 2nd row. Finally units (1st row): these also are five, then we can operate in the same way of the hundreds, or, in order to show that a number can be represented in different ways, one might group the five seeds and replace them with one positioned in the second box of the 1st row. See Table 2.

It should be noted that all numbers can be represented using only the first two columns (1 and 2). This obviously implies the uselessness of columns 3 and 4 and constitutes a limit of the theory of Wassen, if we consider true what Poma de Ayala wrote (cfr. Chapter 1.1 - upper limit of 100,000). By Wassens' theory we can represent all the numbers up to the upper limit of 888,880 (inclusion of all seeds or yupana completely filled).

The theory of Wassen, based on a positional notation system, was taken as a model by most subsequent authors, who, while rejecting his validity, never rejected the positional nature.

ROW	POWER range values	COLUMN 1	COLUMN 2	COLUMN 3	COLUMN 4
		Value: 1	Value: 5	Value: 15	Value: 30
5	10^4 Chunka Waranka (Tens of thousands) 10,000-300,000	10000-50000 ❍ ❍ ❍ ❍ ❍	50000-150000 ❍ ❍ ❍	150000-300000 ❍ ❍	300000 ❍
4	10^3 Waranqa (Thousands) 1,000-300,000	1000-5000 ● ❍ ● ❍ ●	5000-15000 ❍ ❍ ❍	15000-30000 ❍ ❍	30000 ❍
3	10^2 Pachak (Hundreds) 100-3,000	100-500 ● ● ● ● ●	500-1500 ❍ ❍ ❍	1500-3000 ❍ ❍	3000 ❍
2	10^1 Chunka (Tens) 10-300	10-50 ● ❍ ● ● ●	50-150 ● ❍ ❍	150-300 ❍ ❍	300 ❍
1	10^0 Huk (Units) 1-30	1-5 ❍ ❍ ❍ ❍ ❍	5-15 ❍ ❍ ●	15-30 ❍ ❍	30 ❍

Table 2: Representation of number 3,595 in the Yupana by H. Wassen

2.1.2 - Addition

To add two numbers the first addend should be represented in the yupana (see previous paragraph). Subsequently, starting from the column of units, add as many seeds as there are units of the second addend. If you complete the first column (five seeds), they are removed and replaced with a seed in the second column. When number ten is reached, all the seeds are removed, and a seed is added in the first column of the second row. Obviously the substitution principle also applies to the ciphers (rows) above.

COL. 1	COL. 2	COL. 1	COL. 2	COL. 1	COL. 2
(A)		(B)		(C)	
10000-50000 ❍ ❍ ❍ ❍ ❍	50000-150000 ❍ ❍ ❍	10000-50000 ❍ ❍ ❍ ❍ ❍	50000-150000 ❍ ❍ ❍	10000-50000 ❍ ❍ ❍ ❍ ❍	50000-150000 ❍ ❍ ❍
1000-5000 ● ❍ ● ❍ ●	5000-15000 ❍ ❍ ❍	1000-5000 ● ❍ ● ❍ ●	5000-15000 ❍ ❍ ❍	1000-5000 ● ❍ ● 🔵 ●	5000-15000 ❍ ❍ ❍
100-500 ● ● ● ● ●	500-1500 ❍ ❍ 🔴	100-500 ❍ ❍ ❍ ❍ ❍	500-1500 🔴 ❍ 🔴	100-500 ❍ ❍ ❍ ❍ 🔵	500-1500 ❍ ❍ ❍
10-50 ● 🔴 ● ● ●	50-150 ● ❍ ❍	10-50 ❍ ❍ ❍ ❍ ❍	50-150 ● ❍ 🔴	10-50 ❍ ❍ ❍ ❍ 🔵	50-150 ❍ ❍ ❍
1-5 🔴 🔴 🔴 🔴 🔴	5-15 ❍ ❍ ●	1-5 ❍ ❍ ❍ ❍ ❍	5-15 🔴 ❍ ●	1-5 ❍ ❍ ❍ ❍ ❍	5-15 ❍ ❍ ❍

Table 3: Operations for the addition: 3,595 + 515 = 4,110 in the Yupana by H. Wassen

As an example, we add the number 515 to the number represented in the previous paragraph. The first step of the procedure is to add the ciphers of units, tens and hundreds to the Yupana (see Table 3, Part A, red circles). The second step is to group the seeds that have filled the boxes of column 1 and replace them with a single seed on the second column of each row (see Table 3, Part B, red circles). Finally, grouping together the seeds that have filled the boxes in column 2 and will replace it with a single seed of box 1 of the next row (see Table 3, Part C, blue circles).

2.1.3 - Multiplication

TO DO

2.2 - Theory by Carlos Radicati di Primeglio (1979)

numeral System	Positional
Notation or **Base**	10
Table layout	Vertical: 5 rows x 4 columns
Horizontal progression	1, 1, 1, 1 (or 9, 9, 9, 9)
Vertical progression	1, 10, 100. 1000, 10000

2.2.1 - Representation of a number

Radicati di Primeglio settled his theory on a positional numeration system. Unlike Wassen, He did not think that the white and black circles, drawn by Poma de Ayala, were empty or full gaps, but seeds placed in the different boxes.

ROW	**POWER** range values	**COLUMN 1**	**COLUMN 2**	**COLUMN 3**	**COLUMN 4**
		Value: 1	**Value: 5**	**Value: 15**	**Value: 30**
5	**10^4** **10000-90000**	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍
4	**10^3** **1000-9000**	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍
3	**10^2** **100-900**	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍
2	**10^1** **10-90**	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍
1	**10^0** **1-9**	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍

Table 4: Scheme of the Yupana by C. Radicati – positional notation in base 10 with progression 1, 1, 1, 1

The representation of a number in the yupana has a vertical progression of base 10 (decimal). This means that the each number can be represented by placing the units in the a column, starting from the bottom (units), to the top (tens of thousands), or vice-versa. It was possible to put up to nine (9) seeds, all having value 1, in each box (see Table 4).

As an example for the representation of a number, choose 3,595. Let's begin filling the column 1 of the table from the top to the bottom, bearing in mind that each digit corresponds to a row (positional system). We start with the thousands (4th row): three seeds (black circles) in the first box; then continue with the hundreds (3rd row): five seeds in the first box of the row; then the tens (2nd row): as the seeds are nine, the box will be completely filled. Finally units (1st row): 5 seeds in the 1st box. See Table 5.

Note that you can represent all the numbers from 1 to 99999.

ROW	**POWER** range values	**COLUMN 1**	**COLUMN 2**	**COLUMN 3**	**COLUMN 4**
		Value: 1	**Value: 5**	**Value: 15**	**Value: 30**
5	**10^4** **10000-90000**	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍
4	**10^3** **1000-9000**	● ❍ ❍ ● ❍ ❍ ● ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍
3	**10^2** **100-900**	● ❍ ❍ ● ● ❍ ● ● ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍
2	**10^1** **10-90**	● ● ● ● ● ● ● ● ●	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍
1	**10^0** **1-9**	● ❍ ❍ ● ● ❍ ● ● ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍ ❍

Table 5: Representation of number 3,595 in the Yupana by C. Radicati di Primeglio

2.2.2 - Addition

To add two numbers, the two addends must be represented in the yupana (see previous paragraph), each of them in a corresponding column. Subsequently, starting from the column of units, move all the seeds of the row in the last column (column 4), keeping in mind that every time you reach the number 10, you can replace all the 10 seeds with a seed in the upper row. Obviously the substitution principle also applies to the ciphers (rows) above.

COL. 1	COL. 2	COL. 3	COL. 4		COL. 1	COL. 2	COL.3	COL.4
(A)					(B)			
❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍		❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍
❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍		❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍
❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍		❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍
● ❍ ❍	❍ ❍ ❍	🔵 ❍ ❍	● 🔵		● ❍ ❍	❍ ❍ ❍	🔵 ❍ ❍	● 🔵 ❍
● ❍ ❍	❍ ❍ ❍	🔵 ❍ ❍	● 🔵		● ❍ ❍	❍ ❍ ❍	🔵 ❍ ❍	● 🔵 ❍
● ❍ ❍	❍ ❍ ❍	🔵 ❍ ❍	● 🔵		● ❍ ❍	❍ ❍ ❍	🔵 ❍ ❍	● 🔵 🟠
● ❍ ❍	🔴 ❍ ❍	❍ ❍ ❍	● 🔴 🔴 🔴		● ❍ ❍	🔴 ❍ ❍	❍ ❍ ❍	🔵 🔵 ❍
● ● ❍	🔴 🔴 ❍	🔵 🔵 ❍	● ● 🔴 🔵 🔵		● ● ❍	🔴 🔴 ❍	🔵 🔵 ❍	🔵 🔵 ❍
● ● ❍	🔴 🔴 ❍	🔵 🔵 ❍	● ● 🔴 🔵 🔵		● ● ❍	🔴 🔴 ❍	🔵 🔵 ❍	🟠 ❍ ❍
● ● ●	❍ ❍ ❍	🔵 🔵 ❍	● ● ● 🔵 🔵 🔵		● ● ●	❍ ❍ ❍	🔵 🔵 ❍	🔵 🔵 ❍
● ● ●	❍ ❍ ❍	🔵 🔵 ❍	● ● ● 🔵 🔵 🔴		● ● ●	❍ ❍ ❍	🔵 🔵 ❍	🔵 🔵 🟠
● ● ●	🔴 ❍ ❍	🔵 🔵 🔵	● ● ● 🔵 🔵		● ● ●	🔴 ❍ ❍	🔵 🔵 🔵	🔵 🔵 🔵
● ❍ ❍	🔴 ❍ ❍	❍ ❍ ❍	● 🔴 🔴		● ❍ ❍	🔴 ❍ ❍	❍ ❍ ❍	❍ ❍ ❍
● ● ❍	🔴 🔴 ❍	❍ ❍ ❍	● ● 🔴		● ● ❍	🔴 🔴 ❍	❍ ❍ ❍	❍ ❍ ❍
● ● ❍	🔴 🔴 ❍	🔵 ❍ ❍	● ● 🔴 🔵		● ● ❍	🔴 🔴 ❍	🔵 ❍ ❍	🔵 ❍ ❍

Table 6: Operations for the addition: 3,595 + 515 + 3,471 = 7,581 in the Yupana by C. Radicati di Primeglio

As an example, we add to the number represented in the previous paragraph (seeds blacks, first column) the number 515 (red seeds, second column) and the number 3471 (blue seeds, third column). The first step of the procedure consists in representing said numbers in the first three columns (see Table 6, Part A, columns 1, 2 and 3). We start here from the units row, adding up all the blacks red and blue seeds, and moving them to the last column (Table 6, Part A, column 4). Since the total sum of the seeds is 11 (greater than 10), ten seeds (black + red seeds) can be replaced with a single seed in the top row (Table 6, Part B, line 2, column 4, seed orange) and in the box 1 × 4 will remain onlya blue seed. Similarly in row 2, the black seeds and the red seeds sum to give 10 tens, and can then be replaced by one seed (orange seed) in the hundreds row (Table 6, Part B, line 3, column 4, seed orange) while the blue seeds are moved in column 4. Even in row 3 the seeds red and blacks result in 10, and then may be replaced with a single seed of row 4 (Table 6, Part B, row 4, column 4, orange seed) and the blue seeds are moved to column 4. In row 5 you have only to move the blue and black seeds to column 4, because the number 10 is not reached. You can read the result vertically from top to bottom: 7581.

2.2.3 - Subtraction

To subtract two numbers, it is necessary to represent the minuend and the subtrahend in the first two columns of the yupana (see previous paragraph). Subsequently, starting from the column of units, the seeds of the first column must be moved in the third (or fourth) column, subtracting from time to time the seeds of column 2. If the result is negative, a seed belonging to the row (power) above must be transformed into 10 seeds of the current row (power) before subtracting the seeds of the 2nd column.

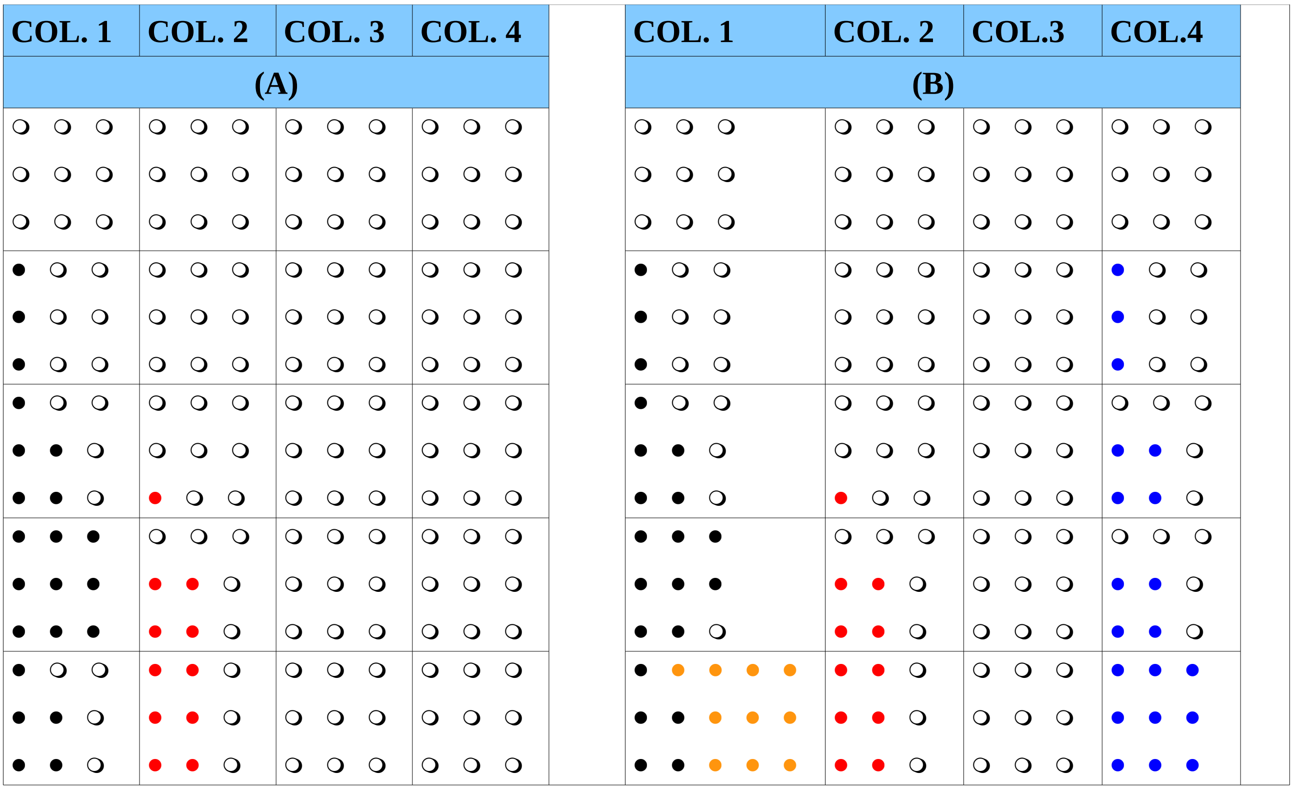


Table 7: Process for the subtraction of 3595 - 146 = 3449 in the yupana by C. Radicati di Primeglio

As an example, we subtract the number 146 (red seeds, second column) to the number shown in the previous section (black seeds, first column). The first step of the procedure, consists in representing said numbers in the first two columns (see Table 7, Part A, columns 1 and 2). We start here from the units row and and move the black seeds to the last column, while subtracting the red seeds; since the result would be negative, we transform a black seed of the next row (2nd row) into ten seeds of the 1st row (Table 7 part B, row 1, column 1, half orange). The result of the subtraction of row one are the blue seeds (Table 7, Part B, row 1, column 4, half blue). Proceed with the tens: 8 black seeds minus 4 red seeds, and report the results in column 4 (Table 7, Part B, line 2, column 4, semi blue). Similarly, for the hundreds we have to subtract a red seed to 5 black seeds (Table 7, Part B, line 4, column 4, blue seeds). Finally for the thousands is sufficient to report the three black seeds of column 1 to column 4, since there are no seeds to be subtracted in column 2. The result can be read vertically from top: 3449.

2.2.4 - Multiplication

TO DO

2.3 - William Glynn Burns' Theory (1981)

numeral System	Positional
Notation or **Base**	10
Table layout	Horizontal: 5 rows x 4 columns (vertical in the original version)
Horizontal progression	1, 1, 1, M (or 5, 3, 2, M)
Vertical progression	1, 10, 100, 1,000, 10,000

The theory developed by the textile engineer *William Glynn Burns* is based on a numeral system with positional notation in base 10, corresponding to the horizontal progression, growing from right to left[§MOV] & [§LES]. The vertical progression, instead, consists of the numbers 5, 3, 2 (from bottom to top), which, having sum 10, are sufficient to represent all the numbers from one to one hundred thousand. The last column, that of the one, is meant to be used as a "memory."

There are many variations of this numeral system proposed by various authors: some put the yupana horizontally (long side horizontal), other vertically, but the theory behind it is obviously the same.

2.3.1 - Representing a number

Since the horizontal arrangement of the Yupana has been more successful and has been used in many educational projects in various countries around the world, I decided to adopt it for the development of **Tk-yupana**.

Referring then to a table of 5 columns and 4 rows (longest side horizontal), the circles of the Yupana by *Poma de Ayala* are mnemonic indicating the maximum number of items (seeds) that can fit in a box (5 elements in the boxes in row 1 , 3 elements in the cells of row 2 and 2 elements in the cells of row 3, 1 element in the cells of row 1). The line 4 (the highest) has a different connotation than the other three below. According to the theory of Glynn, in fact, it was used as a ***memory*** in arithmetic operations, to reduce the possibility of error during substitutions (see for example the addition operation).

Each seed will have **unitary value**; this means that, in the interpretation of Glynn (as in that of Wassen) the circles of the yupana drawn by *Poma de Ayala* will have value 0 if empty and 1 when filled with a seed.

The ***horizontal progression*** is based on ten (decimal), and each column is associated with a power of ten. Starting from the right and growing to left, columns correspond to unit (10^0), tens (10^1), hundreds (10^2), thousands (10^3), tens of thousands (10^4). Each seed will therefore have different value (1 × 10^x) depending on the column in which it is located (see the diagram in Table 8).

5	4	3	2	1	COLUMN
10^4 (TENS THOUSANDS) hunu	10^3 (TOUSANDS) waranka	10^2 (HUNDREDS) pachaq	10^1 (TENS) chunka	10^0 (UNITS) huq	VALUE
❍	❍	❍	❍	❍	MEMORY
❍❍	❍❍	❍❍	❍❍	❍❍	Every circle (if filled) has value 1 multiplied by the value of the column
❍ ❍❍	❍ ❍❍	❍ ❍❍	❍ ❍❍	❍ ❍❍	
❍❍ ❍❍❍	❍❍ ❍❍❍	❍❍ ❍❍❍	❍❍ ❍❍❍	❍❍ ❍❍❍	

Table 8: Scheme of the numeral system by W. Glynn (positional notation in base 10)

The **vertical progression** is instead **1 1 1 M**, that is, every seed planted in any box of the first three rows of the table, starting from bottom, has the value 1, and has a value of M = memory = 10 × 10^{x-1} (with x equal to the number of the column starting from right to left) when located in the fourth row from bottom.

The representation of a number is very simple: we begin to fill the table from bottom to top and from right to left, bearing in mind that each digit corresponds to a column (positional system); in the first three rows (starting from low) fit the seeds (each one has value 1). See the example in Table 9.

10^4 (TENS THOUSANDS) hunu	10^3 (THOUSANDS) waranka	10^2 (HUNDREDS) pachaq	10^1 (TENS) chunka	10^0 (UNITS) huq
❍	❍	❍	❍	❍
❍❍	❍❍	❍●	❍❍	❍❍
❍ ❍❍	❍ ❍❍	● ●●	❍ ●●	❍ ❍❍
❍❍ ❍❍❍	❍❍ ❍❍●	●● ●●●	●● ●●●	❍❍ ❍❍●

Table 9: Representation of number 1,971

The number 0 is represented by not putting any seed into the yupana (empty table). The maximum number representable in the yupana by Glynn is 100,000 or 222,220 if also the memory holes are filled.

Note that with the theory of Glynn is possible to represent all numbers from 1 to 99,999; this representation, however, is not univoca[14]. The choice to use the number one as "memory", as well as providing an aid in the calculations, allows the numeral system to become purely decimal.

2.3.2 - Addition

To add two numbers, the first addend should be represented in the yupana (see previous paragraph).

Subsequently, starting from the column of units, add as many seeds as there are units of the second addend. If you complete the column (ten seeds), all the seeds placed are removed and replaced with a seed in the memory (that is 10^{x-1}, with x equal to the number of the column, starting from the right to the left); this seed also applies a seed in a box in the column immediately to the left. The three examples below are equivalent and are worth 100:

10 tens		**1 memory (value 100)**		**1 hundreds**	
10^2	**10^1**	**10^2**	**10^1**	**10^2**	**10^1**
❍	❍	❍	●	❍	❍
❍❍	●	❍❍	❍❍	❍❍	❍❍
❍ ❍❍	● ●●	❍ ❍❍	❍ ❍❍	❍ ❍❍	❍ ❍❍
❍❍ ❍❍❍	●● ●●●	❍❍ ❍❍❍	❍❍ ❍❍❍	❍❍ ❍❍●	❍❍ ❍❍❍

2.3.3 - Multiplication

The multiplication of two numbers according to Glynn use "magic numbers" 1, 2, 3 and 5 drawn by Poma de Ayala.

The first step is to find the four multiples of the multiplicand: M1, M2, M3, M5.

Secondly involves representing a multiplier in powers of 10, by the base 1, 2, 3, 5. Then x is the exponent of the power of ten corresponding to the digit in question (0 for the unit, one for the tens, 2 for the hundreds, and so on), all digits from 0-99,999 are representable, some in different

14 This is because the obliged "choice" of the numbers 2,3,5 does not meet *Zeckendorf's theorem* which states that every positive integer can be represented uniquely as the sum of one or more distinct non-consecutive Fibonacci numbers. As the numbers 2 and 3 are consecutive, the theorem is not satisfied and the representation is not unique. [N/A]

ways :

$0 = 0 \times 10^x$

$1 = 1 \times 10^x$

$2 = 2 \times 10^x$

$3 = 3 \times 10^x$

$4 = 2 \times 10^x + 2 \times 10^x$ (or $1 \times 10^x + 3 \times 10^x$)

$5 = 5 \times 10^x$

$6 = 5 \times 10^x + 1 \times 10^x$ (or $1 \times 10^x + 2 \times 10^x + 3 \times 10^x$)

$7 = 5 \times 10^x + 2 \times 10^x$ (or $1 \times 10^x + 3 \times 10^x + 3 \times 10^x$)

$8 = 5 \times 10^x + 3 \times 10^x$ (or $2 \times 10^x + 3 \times 10^x + 3 \times 10^x$)

$9 = 5 \times 10^z + 2 \times 10^x + 2 \times 10^x$ (or $1 \times 10^x + 3 \times 10^x + 5 \times 10^x$)

In general then a digit of the multiplier can be represented on the base 1,2,3,5, such as:

$$n_n = a_n \times 10^0 + b_n \times 10^1 + c_n \times 10^2 + d_n \times 10^3 + e_n \times 10^4 \; t.c.\; a, b, c, d, e \in \{0,1,2,3,5\} \wedge 0 \leq n \leq 4$$

Having decided on the decomposition of the multiplier, you can proceed to the actual calculation, multiplying each occurrence of non-zero digits of the multiplier by the corresponding multiplication factors (M1, M2, M3, M5) of the multiplicand and bringing in yupana the values, gradually doing the sums.

For example, suppose you want to perform the following multiplication $3 \times 12{,}359$.
We find the 4 multiples of the multiplicand: 3, 6, 9, 15.
Then decompose the multiplier 12359: we compose the following table, to which we have added as the last line, the multiples of the multiplicand that will help us later.

Power	1×	2×	3×	5×	**Result**
10^4	1				1×10^4
10^3		1			2×10^3
10^2			1		1×10^2
10^1				1	1×10^1
10^0		2		1	1×10^0
Multiplicative factors	**3**	**6**	**9**	**15**	

Table 10: Multiplication of $3 \times 12{,}359$

At this point we have all the elements to perform the calculation which will be reduced to simple sums. Starting from the unit, represent the multiplication factors (M1, M2, M3, M5) by the number of occurrences ($a_{0,} b_{0,} c_{0,} d_{0,} e_0$): $15 \times 1 + 6 \times 2$

❍	❍	❍	❍	❍
❍ ❍	❍ ❍	❍ ❍	❍ ❍	❍ ❍
❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍	❍ ● ●
❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ● ●	●● ●●●

Then the tens: 15×1

❍	❍	❍	❍	❍
❍ ❍	❍ ❍	❍ ❍	❍ ❍	❍ ❍
❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍	❍ ● ●	❍ ● ●
❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ●	●● ●●●	●● ●●●

Then hundreds: 9×1; note that adding 9 to the already present 1, a ten of hundreds is reached, so a thousand.

❍	❍	❍	❍	❍
❍ ❍	❍ ❍	❍ ❍	❍ ❍	❍ ❍
❍ ❍ ❍	❍ ❍ ❍	❍ ❍ ❍	❍ ● ●	❍ ● ●
❍ ❍ ❍ ❍ ❍	❍ ❍ ❍ ❍ ●	❍ ❍ ❍ ❍ ❍	●● ●●●	●● ●●●

Then thousands: 6×1

❍	❍	❍	❍	❍
❍ ❍	❍ ❍	❍ ❍	❍ ❍	❍ ❍
❍ ❍ ❍	❍ ● ●	❍ ❍ ❍	❍ ● ●	❍ ● ●
❍ ❍	●●	❍ ❍	●●	●●

❍ ❍ ❍	●●●	❍ ❍ ❍	●●●	●●●

Finally we add the tens of thousands: 3×1, thereby obtaining the result: 37,077

❍	❍	❍	❍	❍
❍ ❍	❍ ❍	❍ ❍	❍ ❍	❍ ❍
❍ ❍ ❍	❍ ● ●	❍ ❍ ❍	❍ ● ●	❍ ● ●
❍ ❍ ● ● ●	● ● ● ● ●	❍ ❍ ❍ ❍ ❍	● ● ● ● ●	● ● ● ● ●

2.4 - De Pasquale's Theory (2001)

numeral System	Positional
Notation or **Base**	40
Table layout	Vertical: 5 rows × 4 columns
Horizontal progression	1, 2, 3, 5 (Fibonacci's sequence)
Vertical progression	1, 40, 1,600, 64,000, 2,560,000

Nicolino de Pasquale proposes for the Abacus by *Poma de Ayala* a numeral system with **positional** notation in **base 40** and expects to have the yupana vertically (long side vertical)[§DEP].

2.4.1 - Representing a number

Each seed will have a value that depends on the row R and column C in which it is placed. Each row R of yupana corresponds to a power of 40, according to the formula:

$$f(R)=40^{R-1}, R\in[1,5] \quad 1.4.1$$

with R growing from bottom to top.

The horizontal progression of the values of the columns is **1, 2, 3, 5**, in correspondence to the number of circles in the drawing of *Poma de Ayala*, which is also equal to the maximum number of seeds that can be inserted in a given cell (R,C). In mathematical terms, the column value is expressed:

$$g(C)=C \text{ , where } C \in [1,4) \cup (4,5] \quad 1.4.2$$

with C growing from right to left.

The value of a seed placed in the box (R,C) is then given by:

$$V(R,C)=f(R)\cdot g(C)=C\cdot 40^{R-1} \text{ where } C \in [1,4) \cup (4,5] \text{ e } R\in [1,5] \quad 1.4.3$$

See also the scheme in Table 11.

ROW	POWER range values	COLUMN 5	COLUMN 3	COLUMN 2	COLUMN 1
		Value: 5	**Value: 3**	**Value: 2**	**Value: 1**
5	**40^4** **2,560,000-102,399,999**	12800000 ❍ ❍ ❍ ❍ ❍	7680000 ❍ ❍ ❍	5120000 ❍ ❍	2560000 ❍
4	**40^3** **64,000-2,559,999**	320000 ❍ ❍ ❍ ❍ ❍	192000 ❍ ❍ ❍	12800 ❍ ❍	64000 ❍
3	**40^2** **1,600-63,999**	8000 ❍ ❍ ❍ ❍ ❍	4800 ❍ ❍ ❍	3200 ❍ ❍	1600 ❍
2	**40^1** **40-1,599**	200 ❍ ❍ ❍ ❍ ❍	120 ❍ ❍ ❍	80 ❍ ❍	40 ❍
1	**40^0** **1-39**	5 ❍ ❍ ❍ ❍ ❍	3 ❍ ❍ ❍	2 ❍ ❍	1 ❍

Table 11: Scheme of the Yupana by De Pasquale – Positional notation in base 40

To represent a number you begin to fill the table from bottom to top and from right to left; the first row may contain a minimum value equal to zero (yupana empty) and a maximum equal to $1 \times 1 \times 40^0 + 2 \times 2 \times 40^0 + 3 \times 3 \times 40^0 + 5 \times 5 \times 40^0 = 1 + 4 + 9 + 25 = 39$. The number 40 corresponds to a yupana filled solely with a seed positioned in the box to the right of the second row. Even in this case there are numbers that can have multiple representations, as shown in Table 12 e Table 13 for number 100.

	Value: 5	Value: 3	Value: 2	Value: 1
40^1 **40-1,599**	200 ❍ ❍ ❍ ❍ ❍	120 ❍ ❍ ❍	80 ❍ ●	40 ❍
40^0 **0-39**	5 ❍ ● ❍ ● ❍	3 ● ● ●	2 ❍ ❍	1 ●

Table 12: First representation of number 100

$$2 \times 1 \times 40^1 + 5 \times 2 \times 40^0 + 3 \times 3 \times 40^0 + 1 \times 1 \times 40^0 = 80 + 10 + 9 + 1 = 100$$

Or:

	Value: 5	Value: 3	Value: 2	Value: 1
40^1 **40-1,599**	200 ❍ ❍ ❍ ❍ ❍	120 ❍ ❍ ❍	80 ❍ ●	40 ❍
40^0 **0-39**	5 ❍ ● ● ● ●	3 ❍ ❍ ❍	2 ❍ ❍	1 ❍

Table 13: Alternative representation of number 100

$$2 \times 1 \times 40^1 + 5 \times 4 \times 40^0 = 80 + 20 = 100$$

2.4.2 - Addition

TO DO

2.4.3 - Multiplication

TO DO

2.5 - Chirinos' Theory (2008)

Numeral System	Positional
Notation or **Base**	10
Table layout	Vertical: 5 rows × 4 columns
Horizontal progression	from 1 to 11
Vertical progression	0.1, 1, 10, 100, 1000

Andres Chirino Riverea proposed, for the Poma de Ayala's abacus, a numeral system with positional notation in base 10 and expects to have the yupana upright (vertical longest side)[§CHI].

2.5.1 - Rrepresentation of a number

Every seed will have a value which depends on the row **R**, on the column **C** and on the position in the box **C×R** it is occupying. Every row R of the yupana corresponds to a power of 10, according to the formula:

$$f(R)=40^{R-2}, R\in[1,5] \qquad 1.4.1$$

with R increasing from bottom to top.

The horizontal progression of the values is **1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11**, and it is based on the position of the circles present in the drawing by *Poma de Ayala*, so, for instance, in row 2, the seeds will have value 1 (column 1), values 2 and 3 (column 2), values 4, 5, 6 (column 3) e 7, 8, 9, 10, 11 (column 4). For details, please see Table 14.

Chirinos divides vertically every row of the yupana into six columns, called *huachos*, that correspond to the disposition of the cirlcles of the drawing by *Poma de Ayala*.

He then brings together the scores for quadrants, called *suyos*, and groups the suyos two by two, into two main sections, called *sayas*.

Finally, he points out that the sum of the seeds of the left-saya is 60, while the sum from the right-saya is 6 and concludes that the Inca, while adopting a positional numeral system in base 10, they also had the option of using the yupana as a sexagesimal calculation table.

ROW	POWER range of values	column 5		column 3		column 2	column 1
		Left Saya				Right Saya	
		Left Suyu		Right Suyu		Left Suyu	Right Suyu
5	10^3 1000-66000	❍ 11000 ❍ 10000 ❍ 9000	❍ 8000 ❍ 7000	❍ 6000 ❍ 4000	❍ 5000	❍ 3000 ❍ 2000	❍ 1000
4	10^2 100-6600	❍ 1100 ❍ 1000 ❍ 900	❍ 800 ❍ 700	❍ 600 ❍ 400	❍ 500	❍ 300 ❍ 200	❍ 100
3	10^1 10-660	❍ 110 ❍ 100 ❍ 90	❍ 80 ❍ 70	❍ 60 ❍ 40	❍ 50	❍ 30 ❍ 20	❍ 10
2	10^0 1-66	❍ 11 ❍ 10 ❍ 9	❍ 8 ❍ 7	❍ 6 ❍ 4	❍ 5	❍ 3 ❍ 2	❍ 1
1	10^0 0.1-0.66	❍ 0.11 ❍ 0.10 ❍ 0.9	❍ 0.8 ❍ 0.7	❍ 0.6 ❍ 0.4	❍ 0.5	❍ 0.3 ❍ 0.2	❍ 0.1
huachus		6	5	4	3	2	1

Table 14: Chirinos' Yupana scheme

Particular attention is paid to the so-called central circles, called unique or *chullas* , which are highlighted by blue color in Table 14. The remaining circles are called coupled-boxes, or *pitu* , and are indicated by the color black in Table 14.

In this theory there are number with multiple representations. Chirinos identifies three types:

1 . Coupled representation , when the number is represented only by pitu type boxes (see an example in Table 15) .
2 . Representation decoupled, when the number is represented both by pitu boxes that chullas boxes (see an example in Table 16).
3 . Rapprresentazione concrete , when it represents the number using up to five seeds for the numbers 1-9 , for each decimal place (see an example in Table 17).
4 . Concrete representation simplified form, where represents the number using the rules of step three , with the addition that the boxes of the sixth huachu all have equal value , equal to the average of the three seeds (see an example in Table 18).

1. **Coupled representation**, when the number is represented by only *pitu*-circles (see an example in Table 15).
2. **Uncoupled representation**, when the number is represented both by pitu-circles and chulla-circles (see an example in Table 16).
3. **Concrete representation**, when the number is represented by only five seeds for numbers from 1 to 9, for each decimal position (see an example in Table 17).
4. **Simplified concrete representation**, when the number is represented by the rules of step three, with the addition that the circles of the sixth-huachu all have equal value, that is the average value of the three seeds (see an example in Table 18).

10^1 **10-660**	○ 110 ○ 80 ○ 100 ○ 70 ○ 90	○ 60 ○ 50 ○ 40	○ 30 ● 20	○ 10
10^0 **1-66**	○ 11 ○ 8 ○ 10 ○ 7 ○ 9	○ 6 ○ 5 ○ 4	● 3 ○ 2	○ 1

Table 15: Coupled representation of number *23 (pitu)*

10^1 **10-660**	○ 110 ○ 80 ○ 100 ○ 70 ○ 90	○ 60 ○ 50 ○ 40	○ 30 ○ 20	○ 10
10^0 **1-66**	○ 11 ● 8 ○ 10 ○ 7 ○ 9	○ 6 ● 5 ● 4	● 3 ● 2	● 1

Table 16: Uncoupled representation of number *23 (chulla)*

10^1 **10-660**	○ 110 ○ 80 ○ 100 ○ 70 ○ 90	○ 60 ○ 50 ○ 40	○ 30 ○ 20	● 10
10^0 **1-66**	● 11 ○ 8 ○ 10 ○ 7 ○ 9	○ 6 ○ 5 ○ 4	○ 3 ● 2	○ 1

Table 17: Concrete representation of number *23. Note that unlike the previous, the number is represented with*

maximum five seeds in the circles from 1 to 9.

10^1 **10-660**	❍ 100 ❍ 100 ❍ 100	❍ 80 ❍ 70	❍ 60 ❍ 50 ❍ 40	❍ 30 ❍ 20	❍ 10
10^0 **1-66**	● 10 ❍ 10 ● 10	❍ 8 ❍ 7	❍ 6 ❍ 5 ❍ 4	● 3 ❍ 2	❍ 1

Table 18: Simplified concrete representation of number *23 Here the values of the circles belonging to the sixth huacho are all equal to the average value of the three seeds.*

2.5.2 - Addition

TO DO

2.5.3 - Multiplication

2.6 - Cinzia Florio's Theory (2008-2009)

In 2008-2009 *Cinzia Florio* proposed a completely different approach[§FLR] from the traditional positional point of view. Instead of formulating a theory of computation with a yupana five rows by four columns and the fixed set (1,2,3,5), Florio looked for a plausible solution to the exact arrangement of the circles in the design by *Poma de Ayala* and takes it as starting point for considering the non-random distribution.

The result of his treatment is surprising: when you consider the white circles as tens and the black circles as units, the first column of yupana provides sum 32, second 64, third 96 and the fourth 151 (Table 19). As long as you admit that *Poma de Ayala* made a "typing" error and has drawn a black dot instead of a white dot, the sum for the fourth column would become 160, or 96 + 64. This observation led Florio to the conclusion that the Fibonacci sequence shown in the figure was just a case related to the example and the yupana was used in that example as a multiplication-board. The multiplication represented would be 32 × 5, where 5 is broken down as 2 + 3 and then as a result of the application of the *distributive property of multiplication with respect to addition*, this would lead to the result:

32 × 5 = 32 x (2 + 3) = 32 × 2 + 32 × 3 = 64 + 96 = 160.

In addition to the clear strengths of being the only numeral system of calculation designed to have a hit in the same drawing by *Poma de Ayala*, the theory proposed by *Cinzia Florio* has, in my opinion, to its advantage the simplicity of use.

The points against him are two:

- You have to admit the error of Poma de Ayala to support the theory. However, as shown by Florio in her article cited in note 25, it is unlikely an error on the part of the chronicler. In any case the probability that Poma de Ayala has drawn circles "at random" and represented precisely the numbers 32, 64 and 96 is practically negligible.
- The yupana designed by Poma de Ayala (with five rows and four columns) would be contingent on the specific calculation drawn, and not all numbers can be represented using that table. The author solves this *impasse* whereas Guaman Poma would have drawn the rows and columns needed for that particular multiplication, but that yupanas had larger sizes. She also points out that for multiplicands where many digits between 6 and 9 appears, there would be needed a very large yupana or maybe more yupanas arranged side by side.

151 (160)	96	64	32
● ❍ ❍ ● ❍	❍ ● ❍	● ●	❍
❍ ❍ ❍ ❍ (●) → (❍)	❍ ❍ ●	❍ ❍	●
● ● ● ● ●	❍ ❍ ❍	● ❍	❍
❍ ❍ ● ❍ ❍	● ❍ ❍	● ❍	❍
● ❍ ● ❍ ❍	● ● ●	❍ ❍	●

Table 19: The calculation of Poma de Ayala according to the numeral system proposed by Cinzia Florio

2.6.1 - Representing a number

Although the example in *Nueva Coronica* refers to a particular multiplication, *Cinzia Florio* also proposes the use of yupana as abacus for addition and division (op. cited in note 25). It is important to keep in mind that the numeral system under consideration is additive by powers of 10. This means that during the representation of a number the *Khipu Camayuq* decomposed in units, tens, hundreds, etc.

Suppose we want to represent the number 3,204. It was broken down as: 3 × 1,000 + 2 × 100 + 0 × 10 + 4 × 1. Than the *Khipu Camayuq* needed 3 rows to represent the number. He chose the seeds coloured according to the power of ten (base 10) and distributed them in the first three boxes of column 1, grouping them by colour up to the complete number (see Table 20).

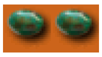
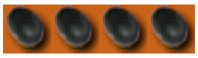
COLUMN 1	ROWS
 3 red seeds = 3,000	**1**
 2 green seeds = 200	**2**
 4 black seeds = 4	**3**
3,204	**TOTAL**

Table 20: Yupana by Florio: representing number 3,204

Note that the insertion order from the thousands to the units adopted in the example is not decisive for the representation of the number (in a sign-value notation system the representation of numbers is not dependent on the position of the digits).

2.6.2 - Addition

The addition according to the theory of *Cinzia Florio* occurred representing two or more numbers in as many columns and then summing them row by row. The result was shown in the last column (the first from left), that of the result.

Suppose you want to add the number 3,204 to number 2,847. The procedure was as follows:

1. Number 3,204 is represented in the yupana, as we have already shown in the previous section (see Table 20 or first column of Table 21).
2. Then, we add a second column in which we represent the number 2,847, as 2 × 1000 + 8 × 100 + 4 × 10 + 7. For this number we will need 4 rows (see the second column of Table 21).
3. Add the seeds of the same colour and returns the result in the next column (see the third column of Table 21)
4. Finally, where possible, regroup ten seeds of the same colour and replace it with one of the colour power of the ten higher (see the fourth column of Table 21).

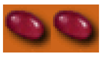
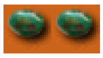
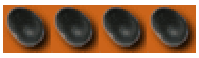
COLUMN 4 FINAL RESULT	COLUMN 3 PARTIAL RESULT	COLUMN 2 SECOND ADDEND	COLUMN 1 FIRST ADDEND
			
			
			
			
6051	**6051**	**2847**	**3204**

Table 21: Yupana by Florio: representation of numbers 3,204 (column 1) and 2,847 (column 2) and their sum (columns 3 and 4).

2.6.3 - Multiplication

Let us now see the operation principle of the yupana used as multiplication-table according to the theory by *Cinzia Florio*. In this case the abacus was used as an aid to perform multiplications and what appears to be drawn in *Nueva Coronica* by *Poma de Ayala* is to be considered on the calculation carried out (32 × 5).

Let us then consider the multiplication of two terms **M × m**, where we denote by **M** the multiplicand and by **m** the multiplier.

The process executed by the *Khipu Camayuq* is as follows:

1)
a) first thing is to break down the multiplicand M in base 10. This corresponds to write it as a sum of N terms:

1) BREAKDOWN AND INCLUSION OF MULTIPLICAND

a) first thing is to break down the multiplicand **M** in *base 10*. This corresponds to write it as a sum of N terms: $M=\sum x_j \cdot 10^j, t.c.\ j=0,1,2,\ldots,N$
b) you assign to each *power of ten* a coloured seed; **N** different types of seeds (or different colours) will be necessary, one for each power. Later we will call these seeds **seeds-j** or **powers-j**. Their type or colour will then account the relative power of ten.
c) start drawing a column of R rows. where R corresponds to the sum of the digits of

the multiplicand: $R=\sum x_j t.c.\ j=0,1,2,\ldots,N$

d) each rows of column 1 has value 1

e) let us insert[15] in *column 1* (first column from the right) x_j seeds, each for every type of ***power-j***;[16] the number of seeds of the same colour therefore corresponds to the coefficients of the summations.

2) BREAKDOWN AND INCLUSION OF THE MULTIPLIER

a) The numeral system by Florio is based on the *distributive property of multiplication with respect to addition*; therefore, we must factor the multiplier in the sum of two or more terms **not** necessarily grouped by the same power of ten:

$m=\sum a_k \cdot 10^s,\ t.c.\ k=0,1,2,3,\ldots,K\ e\ s=0,1,2,3,\ldots K$.

The number K and the value a_k of the terms is at the discretion of the *Khipu Camayuq* and obviously depends on his experience (in general it may be economical, for the purposes of the calculation speed, decompose the multiplier in a series of addends whose multiplication with the multiplicand is almost immediate, as for example 16 = 10 + 6, or 165 = 100 + 30 + 30 + 5).

b) the terms in which the multiplier has been decomposed fix the values a_k and the number K of the columns subsequent to the first. To the left of *column 1* are created K columns (one for each addend) and the respective **value** (corresponding to the number of seeds that can be inserted in each box) and a **power-s** (corresponding to the power of ten of the addend under consideration) are associated to each column.

c) This would result in a checker-board of N rows, the number increasing from top to bottom, multiplied by K+1 columns (1 column of the multiplicand + K columns of the multiplier), the number increasing from right to left. Since here on, each square of the board will be identified by the words **n × k** ($1\leq n\leq N\ e\ 1\leq k\leq K+1$) and will have a **value** a_k and **power-s** corresponding to the associated column (Table 22)

15 Note that entering a number in the method proposed by *Cinzia Florio* takes place from top to bottom, in accordance with José Acosta: "... they do not write by rows, but from the top to down" [§ACO], Book VI, Chapter IX [N/A]

16 You can do a reverse discourse, starting from the yupana represented by *Poma de Ayala*. Given that it has only five rows, you can enter the numbers 1, 2, 3, 4, 5, but not 6 because it would have required an extra row; while it is still possible to enter the numbers 10, 11, 12, 13 and 14 , but not on 15, and so on. Generalizing, the multiplicands that can be inserted in N rows are all M that $1\cdot 10^j\leq M\leq (N-j)\cdot 10^j,\ j\geq 0$ [N/A]

Multiplicator m (K columns)			Multiplicand M	Rows
K+1	...	2	1	
Value a_K Power s_K		Value a_1 Power s_1	1 × seed-1	**1**
			...	**2**
			1 × seed-2	**3**
			...	**...**
			1 × seed-N	**N**

Table 22: Yupana N rows × K+1 columns

3) **MULTIPLICATIONS (column 1 × column k)**
   a) for each row of the yupana multiply *column 1* by *column k* and add a_k weights in the box **n × k**, taking care to respect the following rule: if the column has ***power-s*** (s ≥ 0), you must enter in the box **n × k** a_k seeds equal to ***power-s* + *power-j***.
   b) The procedure is repeated for **K** columns inserted.

4) **ADDITION (column 2 + column 3 + … + column K+1)**
   a) Create a column to the left of the *column K+1*. In this column, which we will call *column K+2*, or *Product column*, will be written the sums of the previous columns (Table 23)
   b) For each row of the yupana sum the number of seeds of the same ***power-k*** of every column ***k*** with $2 \leq k \leq K+1$ and put an equal number of seeds in *column K+2*.

5) **PRODUCT**
   a) The sum of all the weights of *column K+2* is the result of the multiplication, or product.
   b) Start by grouping the seeds of **minor power**: whenever ten seeds of the same colour are reached, these are replaced by 1 seed of higher colour/power.
   c) After having done so up to the maximum weight, You will get R seeds s_i , $t.c.\, 0 \leq s_i \leq 9$
   d) Starting by the weights of greater power You can read the result as:
   $p = \sum s_i \cdot 10^i \, , t.c.\, i = 0,1,2,3,\ldots, P$

Product p	Multiplicator m (K columns)			Multiplicand M	Rows
K+2	K+1	...	2	1	
				1 × seed-1	1
				...	2
				1 × seed-2	3
				...	...
				1 × seed-N	N

Table 23: Yupana N rows × K+1 columns + Product column

Example 1: 32×5 (drawing by Poma de Ayala)

As a first example, we want to execute the multiplication drawn by *Poma de Ayala*: 32 × 5

1. Breakdown of 32
   a) 32 = 2 × 100 + 3 × 101
   b) let us assign the black colour to the units (power 0) and the white colour to tens (power 1)
   c) then draw a yupana of 5 rows, which are sufficient to contain the multiplicand, in fact 5 is given by the sum of the digits of the multiplicand 3+2
   d) then three white balls and two black are placed in the first column of the yupana (Table 24)[17]

Multiplicand A	Rows
1	
❍	1
❍	2
❍	3
●	4
●	5

Table 24: 32 x 5 – Entering the multiplicand

2. Decomposition and integration of the multiplier
   a) decompose the multiplier into the sum of two terms: 5 = 3 + 2[18]
   a) we will enter in the table other two columns of equal power 0 and distinct values 3

17 Note that since the numeral system is ***additive*** and not positional, the arrangement of the balls can be random and the result will always be the same. However, as a matter of order and as written by José de Acosta, we suggest inserting the multiplicand from the top of the yupana and proceeding downwards [N/A]

18 Theoretically I could also choose 1 + 4 or 2 + 2 + 1, and so on. [N/A]

and 2 (Table 26)

Multiplicator: 5		Multiplicand :32	Rows
3	2	1	
3×	2×	❍	1
		❍	2
		❍	3
		●	4
		●	5

Table 25: 32×5 – Decomposition of the multiplicator

3. Multiplications
   a) for each row of yupana multiply the values in *column 1* by the multiplication factor in *column 2* and then add 2 seeds of the same power in box **n × 2**. Then, for each row, multiply the values in *column 1* by the multiplication factor in *column 3* and add 3 seeds of the same power in box **n × 3** (Table 26).

Multiplicator: 5		Multiplicand :32	Rows
3 (3×)	2 (2×)	1	
❍❍❍	❍❍	❍	1
❍❍❍	❍❍	❍	2
❍❍❍	❍❍	❍	3
●●●	●●	●	4
●●●	●●	●	5

Table 26: 32×5 – Multiplications of column 1 by columns 2 and 3

4. Summations
   a) create *column 4* where will be written the sum of the previous columns.
   b) For each row of the yupana sum the ***seeds-k*** of the columns 2 and 3 and put as many seeds in *column 4* (Table 18).

Result	Multiplicator: 5		Multiplicand :32	Rows
4 (3 + 2)	3 (3×)	2 (2×)	1	
❍❍❍❍❍	❍❍❍	❍❍	❍	1
❍❍❍❍❍	❍❍❍	❍❍	❍	2
❍❍❍❍❍	❍❍❍	❍❍	❍	3
●●●●●	●●●	●●	●	4
●●●●●	●●●	●●	●	5

Table 27: 32×5 – Summation of columns 2 and 3 and result in column 4

5. Result
   a) You can already read the result in column 4: fifteen white balls + ten black balls, namely: (5 × 3) 101 + 5 × 2 = 150 + 10 = 160.
   b) However, wanting to get a more natural reading, starting from the seeds of lower weight, let us group ten units and convert the corresponding ten seeds in a seed of higher weight.
   c) We get two different weights (red for the hundreds and white for the tens) with their associated values (respectively 1 and 6).

d) start from the higher power and read the result 100 + 60 = 160 (Table 28)

Result	Multiplier: 5		Multiplicand :32	Rows
4 (3 + 2)	3 (3×)	2 (2×)	1	
❍	❍❍❍	❍❍	❍	1
❍❍❍❍❍❍	❍❍❍	❍❍	❍	2
	❍❍❍	❍❍	❍	3
	●●●	●●	●	4
	●●●	●●	●	5

Table 28: 32x5 – Grouping of the seeds of the same colour and reading of the result (product) in the 5th column

Example 2: 133x97

Let us decompose the multiplicand 133 = 3 + 3 × 101 + 1 × 102
Assign the various powers of ten to the following colours:

- Black = unit (10^0) = × 1
- White = tens (10^1) = × 10
- Red = hundred (10^2) = 100 ×
- Yellow = thousand (10^3) = 1000 ×
- Blue = tens of thousands (10^4) = × 10000

Then put a red ball, three white and three black balls in the first column of the yupana, for a total of 7 lines, corresponding to the sum of the digits of the multiplicand (Table 29).

Multiplicand 133	Rows
1	
❍ (red)	1
❍	2
❍	3
❍	4
●	5
●	6
●	7

Table 29: 133 x 97 – Entering the multiplicand

The second step is to decide the decomposition of the multiplier. The number 97 is decomposable in the sum 50 + 40 + 4 + 3 (Table 30), but theoretically I could also choose other decompositions. I need therefore 4 intermediate columns and one for the result for a total of 6 columns.

Result	Multiplicator: 97				Multiplicand 133	Rows
6	5	4	3	2	1	
	50X	40X	4X	3X	❍ (red)	1
					❍	2
					❍	3
					❍	4
					●	5
					●	6
					●	7

Table 30: 133x97 – Decomposition of the multiplier (97) in 50 + 40 + 4 + 3

For each row, multiply the value of the seed present in column 1 by the values of the columns 2 and 3 and distribute an adequate number of seeds in said columns. Put attention, because when multiplying a row of the multiplicand for the values of the columns 4 and 5, the result increases to higher power (from the units to the tens, from tens to hundreds and so on); You must then use a ball of the appropriate colour, according to the scale of conventions adopted above (Table 31).

Result	Multiplicator: 97				Multiplicand 133	Rows
6	5 (50x)	4 (40x)	3 (4x)	2 (3x)	1	
	○○ ○ ○○	○○ ○○	○○ ○○	○○○	○	1
	○○ ○ ○○	○○ ○○	○○ ○○	○○○	○	2
	○○ ○ ○○	○○ ○○	○○ ○○	○○○	○	3
	○○ ○ ○○	○○ ○○	○○ ○○	○○○	○	4
	○○ ○ ○○	○○ ○○	●● ●●	●●●	●	5
	○○ ○ ○○	○○ ○○	●● ●●	●●●	●	6
	○○ ○ ○○	○○ ○○	●● ●●	●●●	●	7

Table 31: 133x97 – Multiplication of column 1 by columns 2, 3, 4 e 5

Finally, collect the seeds of the same colour and put them in the last column, taking care to replace ten seeds of the same colour with a seed of the colour of power higher (Table 32). The procedure should be done step by step:

- We can now move the nine yellow balls in the box corresponding to thousands of row 1 in column 6
- then we add the seeds belonging to column 5 (which having as a coefficient a number that can be broken down into 5 x 10 will give a sum of 10 and then can be easily transformed into a ball of “higher” colour); see the red and yellow areas in Table 32.

Result	Multiplicator: 97				Multiplicand 133	Rows
6	5 (50x)	4 (40x)	3 (4x)	2 (3x)	1	
○○○○ ○○○○○			○○ ○○	○○○	○	1
	○○ ○ ○○	○○ ○○	○○ ○○	○○○	○	2
	○○ ○○ ○○ ○○ ○○	○○ ○○	○○ ○○	○○○	○	3
		○○ ○○	○○ ○○	○○○	○	4
	○○ ○ ○○	○○ ○○	●● ●●	●●●	●	5
	○○ ○○ ○○ ○○ ○○	○○ ○○	●● ●●	●●●	●	6
		○○ ○○	●● ●●	●●●	●	7

Table 32: 133x97 – Summations

We can see the result in *Table 33*, where also are shown the balls that will be added next, and promoted to the higher power (yellow area in which 10 red balls are added and then promoted to a yellow ball; red area, where ten white balls are added and promoted to a red)

Result	Multiplicator: 97				Multiplicand 133	Rows
6	5 (50x)	4 (40x)	3 (4x)	2 (3x)	1	
○○○○ ○○○○○			○○ ○○	○○○	○	1
	○○ ○ ○○	○○ ○○	○○ ○○	○○○	○	2
○		○○ ○○	○○ ○○	○○○	○	3
		○○ ○○	○○ ○○	○○○	○	4
	○○ ○ ○○	○○ ○○	●● ●●	●●●	●	5
	○	○○ ○○	●● ●●	●●●	●	6
		○○ ○○	●● ●●	●●●	●	7

Table 33: 133x97 – Further summations

Continuing the process, and gradually adding all the balls, we get the following result:

Result	Multiplicator: 97				Multiplicand 133	Rows
6	5 (50x)	4 (40x)	3 (4x)	2 (3x)	1	
○					○	1
○○					○	2
○○○○ ○○○○○					○	3
					○	4
●					●	5
					●	6
					●	7

Table 34: 133x97 – Result

The result is: a blue ball (10,000) + two yellow balls (2,000) + nine red balls (900) + no white ball (0) + a black ball (1): 12901

2.7 - Kak's Theory (2014)

Numeral System	Positional, non-uniform representation
Notation or **Base**	144 with powers' progression stepping by 2
Table layout	Vertical: 5 rows × 4 columns
Horizontal progression	1, 6, 24, 72 (for the first row)
Vertical progression	12^0, 12^2, 12^4, 12^6, 12^8

Subhash Kak proposes for the *Poma de Ayala's Yupana* a **non-uniform** representation of numbers, with **positional notation** in **base 144** with progression 12^0, 12^2, 12^4, 12^6, 12^8 and set the yupana in a vertical position (long side vertical)[§KAK].

In his work, Kak suppose that the abacus were used as a tool to perform astronomical calculus by the Incas. He gives his own explanation of the position of the seeds drawn by *Poma de Ayala*, and finds a connection with the sinodic sub-periods of some planets.

Kak explains also how the abacus may be used to represent, add and subtract numbers.

2.7.1 - Representation of a number

Kak based his theory on a *non-uniform representation system* of numbers.

The author attributes to blacks circles, designed by *Poma de Ayala*, the concept of full holes, while to the white ones the concept of empty holes. So in each box there will be a different number of gaps/seeds (5 in the first box, 3 in the second, 2 in the third, 1 in the fourth).

He starts to fill the yupana from left to right and from bottom to top. Then we number the boxes of the first row below as 1, 2 , 3 and 4 . In the 1st box you can enter 5 seeds with a value of 1 (unit = minimum value = least significant), for a total value TOT_1 = 5.

From TOT_1 he determines the value of the individual gaps in the next box, or TOT_1 + 1 = 6. In this box there are 3 holes, for a total value TOT_2 = (TOT_1 × 3) = (6 × 3) = 18.

From the sum of TOT_1 TOT_2 he can determine the value of the individual gaps in the next box, namely: TOT_1 + TOT_2 + 1 = 24 . In this box there are 2 gaps, for a total value TOT_3 = (TOT_3 × 2) = (24 × 2) = 48.

Finally, the value of a single gap in the next box is TOT_4 = TOT3 + TOT_2 + TOT_1 + 1 = 72.

Each hole/seed of column 1 has value 1.
Each hole/seed of column 2 has value 6, ossia (5 × 1) + 1
Each hole/seed of column 3 has value 24 = (5 × 1) + (6 × 3) + 1
Each hole/seed of column 4 has value 72 = (5 × 1) + (6 × 3) + (24× 2) + 1

In general, given a set of occurrences of holes: a, b, ..., z is possible to assign the least significant value to the first gap (a) and then determine the values, v_b, ..., v_z of the subsequent holes using the formula:

$$v_i = \cdot \begin{matrix} 1, i=a \\ 1+\sum_{j<i}^{z} j \times v_j , i \geq a \end{matrix}$$

This ensures the uniquness of the representation.

ROW	POWER range of vslues	COLUMN 4	COLUMN 3	COLUMN 2	COLUMN 1
5	12^8 429981696-61917364223	429981696 ❍ ❍ ❍ ❍ ❍	2579890176 ❍ ❍ ❍	10319560704 ❍ ❍	30958682112 ❍
4	12^6 2985984-429981695	2985984 ❍ ❍ ❍ ❍ ❍	17915904 ❍ ❍ ❍	71663616 ❍ ❍	214990848 ❍
3	12^4 20736-2985983	20736 ❍ ❍ ❍ ❍ ❍	124416 ❍ ❍ ❍	497664 ❍ ❍	1492992 ❍
2	12^2 144-20735	144 ❍ ❍ ❍ ❍ ❍	864 ❍ ❍ ❍	3456 ❍ ❍	10368 ❍
1	12^0 1-143	1 ❍ ❍ ❍ ❍ ❍	6 ❍ ❍ ❍	24 ❍ ❍	72 ❍

Table 35: Kak's Yupana

Also to determine the value of a single hole on the first box of the next line (line 2) we proceed in the same way, considering previous boxes all the boxes of the previous row (row 1), then:

$(1 \times 5) + (6 \times 3) + (24 \times 2) + 72 + 1 = 144 = 12^2$.

Filling the second row will get the starting value of the third, 12^4, and so on, up to the fifth line. We notice that the vertical progression of the power is not continuous, but occurs in steps of two, i.e. the system is positional, base 144, with progression X^n, such that n = 0 , 2, 4 , 6, 8. The results are shown in Table 35.

To represent a number you start to fill the table from the bottom upwards and from left to right , the first row may contain a minimum value of zero (yupana empty) and a maximum equal to $5 \times 1 \times 12^0 + 3 \times 6 \times 12^0 + 2 \times 24 \times 12^0 + 1 \times 72 \times 12^0 = 5 + 18 + 48 + 72 = 143$. The number 144 corresponds to a yupana filled solely with a seed placed in a hole of the box to the left of the second row. Each number from 144 to 287 (144 + 143) is represented with a single seed in the first box of the second row (value 144) and with a number of seeds present in the first row, up to a maximum of 143 seeds (first line full). See an example of representation in Table 36

	COLUMN 1	COLUMN 2	COLUMN 3	COLUMN 4
12^8	❍ ❍ ❍ ❍ ❍	❍ ❍ ❍	❍ ❍	❍
12^6	❍ ❍ ❍ ❍ ❍	❍ ❍ ❍	❍ ❍	❍
12^4	❍ ❍ ❍ ❍ ❍	❍ ❍ ❍	❍ ❍	●
12^2	❍ ❍ ❍ ● ●	❍ ❍ ❍	❍ ●	❍
12^0	❍ ❍ ❍ ❍ ❍	❍ ❍ ●	❍ ❍	❍

Table 36: Representation of 1,496,742

The representation of each number is univoque.

2.7.2 - The yupana as a computational tool for astronomy

In his article Kak also proposes that the yupana could be used for astronomical calculations.

Kak consider the arrangement of the circles of the drawing by *Poma de Ayala*. He supposes that each row corresponds to the representation of a number between 1 and 143 (so as five rows with power 12^0). See Table 37.

● ❍ ❍ ● ❍	❍ ● ❍	● ●	❍	**56**
❍ ❍ ❍ ❍ ●	❍ ❍ ●	❍ ❍	●	**79**
● ● ● ● ●	❍ ❍ ❍	● ❍	❍	**29**
❍ ❍ ● ❍ ❍	● ❍ ❍	● ❍	❍	**31**
● ❍ ● ❍ ❍	● ● ●	❍ ❍	●	**92**

Table 37: Representation of the five numbers of the drawing by Poma de Ayala, according to Kak's theory.

Kak proposes the existence of a strong relationship between the numbers shown in the table above and astronomy, also referring to some numbers found in Maya's astronomy. Refer to in this regard.

N	Pianeta	Spiegazione
56	Jupiter and Saturn	A first explanation is related to the Mayan calendar, since 56 × 117 = 819 × 8, and then 8 cycles of the Mayan calendar. A second explanation links the number to the synodic periods of the gaseous planets, since 56 = 7 × 8, which are ***almost*** perfect divisors of 399 (synodic period of Jupiter) and 378 (synodic period of Saturn)[19]
79	Venus	79 × 12 = 948. This value decreased by 365 (days in the calendar year) gives 583 which differs by 1 from the synodic period of Venus[20]
29	Mercury	29 × 4 = 116 which is the synodic period of Mercury
31	Month	31 is the number of days present in some months of the calendar year[21]
92	1/4 of a year	Number of days between the autumn equinox and the winter solstice
sum = 287	Mars	287 = 365 - 78, with 78 which should be 1/10 of the synodic period of Mars

Table 38: Explanation of the numbers represented by Poma de Ayala, according to Kak's theory.

Weaknesses

The weaknesses of the Kak's theory are more than one and highlighted by the author himself in his article (see notes 19, 20 and 21). However, other objections can be raised:

- not a number is directly associated with a synodic period of a planet;
- only one number is perfectly associable with a synodic period of a planet (the synodic period of Mercury is directly proportional to the number 29);
- if all the numbers had a uniform and direct correspondence with the synodic periods of the planets, this would provide a strong basis for his theory, but unfortunately multiplications,

19 8 is not a divisor of 399 nor of 378. Kak writes: "*This interpretation rests on special significance being given to 8 cycles of the 819-day period. Alternatively, the Inca may have held to the theory that 56 codes the synodic periods of Jupiter and Saturn because its factors 8 and 7 almost exactly divide 399 (Jupiter) and 378 (Saturn), respectively*" [n.d.A.]

20 The author writes that this discrepancy of 1 in the product, is a weakness of his theory, but that there might be astronomical reasons for that value.

21 The author points out that this may be just a coincidence and that it cannot be a significant number, but used by the Incas to get the 287 as a sum.

and corrections are necessary;

Frankly, this theory is weak and is widely open to criticism.

2.7.3 - Addition

TO DO

2.7.4 - Multiplication

TO DO

2.8 - Which theory should we choose?

The debate about the numeral system that best fit the Incan Abacus shown in “*Nueva Coronica*” is far from exhausted. The purpose of Tk-yupana is to provide an overview of all the possible solutions and not to make judgements on the solutions adopted, however, some aspects, that can support a certain theory and exclude others, can be considered:

- the purpose of the abacus is to assist the user, simplifying complex calculations and should be used in an intuitive and simple way. Are therefore to be preferred simple methods of calculation to complex ones (see for example the complexity of the calculations of the multiplication in Radicati's theory). For the same reason are to be preferred simple numeral systems to complex ones (see for example the complexity of numbers representation in the theories of Kak or De Pasquale).
- The importance of the need, on the part of the *Khipu Camayuq*, to keep under visual control the displacements of the seeds on the yupana is advocated both by *Cinzia Florio*[§FLR] and by supporters of the *Glynn* theory[§LES]. This visual control is achieved by breaking down the addends into "small" numbers in one case, and with the help of "*memory*" in the other.
- All Spanish chroniclers, from *Poma de Ayala* on, describe the numeral system of the Incas in base 10; this would exclude the numeral system of *De Pasquale*.[22] This view is strongly expressed in an interesting article published in the *Journal of Mathematics & Culture* by *M. Leonard* and *C. Shakiban*[§LES]. Here the numeral system of Glynn is reviewed in the light of linguistic and cultural considerations. This study, which expects to have the yupana in an upright position (as shown by *Poma de Ayala*) and that places in high regard the importance of the five fingers of the hand and then number five in the Andean culture, highlights two factors not negligible. First of all, thanks to the vertical arrangement of the yupana, there is a direct correspondence between it and the *khipu* (Another tool used by the Incas to record events and quantity)[§LES]. Secondly, based on the statements of the linguist *Pilares Casas*[§PIL] about the way of counting of the Ayamara-speaking populations, there is a correlation with the sequence of numbers 5, 3, 2, 1 of the yupana. These populations, in fact adopt a sign-value numeral system, or count in a manner similar to that of the Romans: the number 5 (qallqu) which would represent the first completion (fingers of the hand), are derived the word 6 (maqallqu) formed by 5 and 1 (ma), 7 (paqallqu) 5 and 2 (pa), the number 8 (kinsakallqu) 5 and 3 (kinsa) and 9 (llatunka) corresponds to 10 subtracted 1.[23] It should be noted that while supporting the thesis of Pilares Casas (and therefore a clearly additive numeral system), the two authors support also the theory (positional) by Glynn.
- Also according to Shakiban and Leonard, the infamous Fibonacci series (of which the inventors of the yupana were certainly *unaware*) turns out to be (see work cited in note 25) the real value of the abacus Inca, but its presence on the yupana should **not** be sought in something transcendental. In fact, thanks to the division of yupana in groups of units, it was

22 It is Guaman Poma who wrote that the Camayuq counted in thousands, hundreds, tens and units (compare the text cited in note 9). Below, in the same book, the author describes how the Inca empire was divided and once again numbering based on 10 returns. [N/A]

23 This number refers to a probable proto-Aymara from which (with some contamination) is derived the Aymara spoken today

easier for the accountant, from a **visual** point of view, to move the seeds on the table in order to perform the basic operations. The first group of seeds belonging to the first column (total value 5) is in fact clearly separable into three + two, namely the two groupings of successive columns (Table 39).

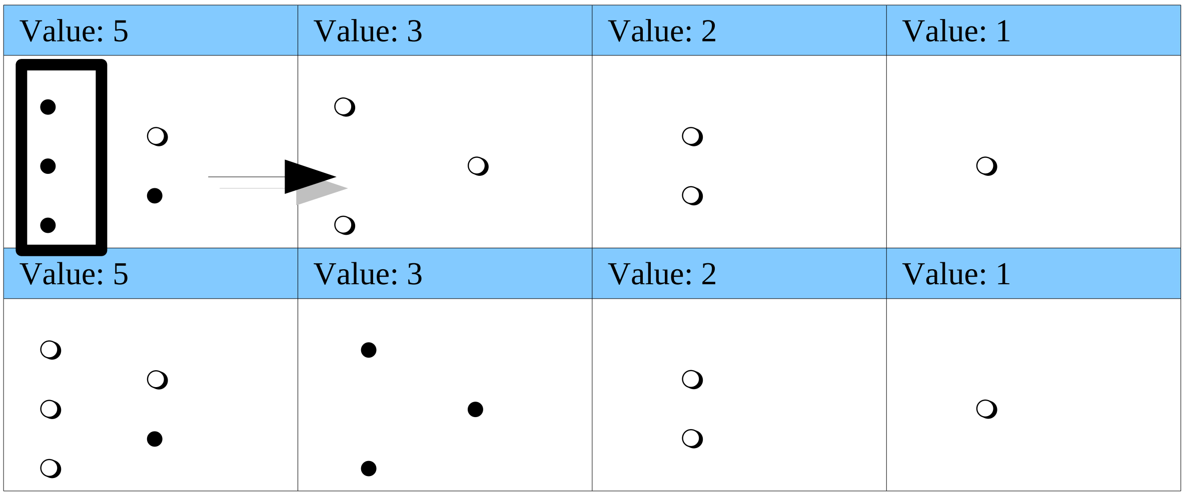


Table 39: Moving three seeds from one box to another (above) and result (below) - numeral system of Glynn

- Only the numeral system by *Florio*, so far, is compliant with the distribution of the balls on the table drawn by *Poma de Ayala*.
- According to *Florio*, the piece of *Fibonacci series* is a pure coincidence and even contingent on the particular calculation done.

We hope that **Tk-yupana** can allow users to experience and enjoy the use of different numeral systems applicable to the Incan Abacus and choose the one that best suits their needs.

3 - Tk-yupana

Tk-yupana is a simple program that shows various theories applicable to the yupana drawn by *Poma de Ayala* on the *El Primer Nueva Coronica y Buen Gobierno*. These theories are based on both positional numeral systems (yupana base 10 by *Henry Wassen* and *William Burns Glynn* and yupana in base 40 by *Nicolino De Pasquale*) and sign-value numeral systems (yupana by *Cinzia Florio*).

3.1 - Requirements

Tk-yupana is a tcl/tk script; so it is necessary to install tcl and tk programs for your OS before running the program.

- **LINUX**

  According to your distribution, you should install tkl-8.4 and tk-8.4 (or greater versions)

  For example with Debian (or Ubuntu), open a terminal and type:

  # sudo apt-get install tcl tk

- **MINDOMS**

  Install tcl/tk from this page: *http://www.activestate.com/activetcl/downloads*

3.2 - Download, install and running the program

PAY ATTENTION: Before running the program you must install tcl and tk ad described in the previous chapter.

- **LINUX (debian/ubuntu)**

  1. Download the file ***tkyupana_<rev>_all.deb*** from the web page: *http://kunturweb.altervista.org/pag/it/tk-yupana.html*
  2. Open a terminal, enter the directory where the file was downloaded and type:

     # sudo dpkg -i tkyupana_<rev>_all.deb
  3. In order to run the program type:

     # tkyupana
  4. For the man page type:

     # man tkyupana

- **LINUX (other distros)**
  1. Download the file ***tkyupana_<rev>.tar.gz*** from the web page :
     http://kunturweb.altervista.org/pag/it/tk-yupana.html
  2. Open a terminal, enter the directory where the file was downloaded and untar the file:

     # tar -xzf tkyupana_<rev>.tar.gz

     A directory ***tkyupana_<rev>*** will be created. It contains the tcl script and all the files.
  3. No enter the directory ***tkyupana_<rev>***

     # cd tkyupana_<rev>

     and type:

     # ./tkyupana.tcl
  4. For the manual, type:

     # man ./tkyupana.1

- **MINDOMS**
  1. Download the file ***tkyupana_<rev>.tar.zip*** from the web page:
     http://kunturweb.altervista.org/pag/it/tk-yupana.html

     and put it in a folder.
  2. Unzip the file ***tkyupana_<rev>.zip*** with your favourite unzip-program.

     A directory called ***tkyupana_<rev>***, will be created. It contains the tcl script and all the files.
  3. Enter the directory ***tkyupana_<rev>*** by double clicking on it.
  4. Run the program ***tkyupana.tcl*** by double clicking on it

3.3 - The main menu

Figure 1: Menu

When you start the program, an icon representing a stylized condor is displayed (Figure 1). Clicking with the left mouse button on the condor (or pressing the **F10** key), you receive a "drop-down menu" (Illustration 2), from which you can access the various functions of the program by topic, which are described in Table 40.

Illustration 2: Drop down menu of tkYupana

Note that while for the "Yupana by Florio" and "Yupana by Glynn" the algorithms for calculating the sum and the multiplication have been implemented, for the other theories the work is still in progress and the yupanas run **only** to represent numbers or for the addition.

3.4 - Conventions

Some conventions have been adopted both in the development of tk-yupana and writing this document.

1. With the term of “**mask**” (or “window”) we mean a graphical to user interface (GUI) intended for a specific function, which may be to allow the user to make a choice through a menu, or to perform an operation with the abacus, etc.
2. With the term “**table**” we mean a particular zone of the mask which reproduce schematically the Yupana drawn by *Poma de Ayala*
3. Each table is equipped with a number of **rows** (indicated with RX, where X is an integer: 1, 2, 3,...) and **columns** (indicated with CY, where Y is an integer: 1, 2, 3, …). The numbers are variable depending on the theory chosen.
4. A **box** (or house) is an element of the table obtained by crossing a row and a column. With the agreements of paragraph 3, a box may be written as **RX×CY**.
5. In each box there is a variable number of **gaps** that correspond to the empty spaces where it is possible to deposit the colored seeds. The number of these gaps (and their values) varies depending on the theory that you are considering.
6. Unless otherwise stated, to **put** a seed into a gap, simply click the gap with the left button of the mouse.
7. Unless otherwise stated, to **remove** a seed from a gap, you need to click the gap with the right button of the mouse while holding down the CTRL key.

Function	Description	Short Key
Sign-value notation Systems	Sub-menu about Sign-value notation systems	
Yupana by Florio (2008)	Sub-menu of yupana by *C. Florio* (powers of 10)	
Representation	Representation of numbers in Florio's system	**F1**
Addition	Addition of numbers in Florio's system	**F2**
Multiplication	Multiplication of numbers in Florio's system	**F3**
Positional notation Systems	Sub-menu about positional notation systems	
Yupana by Wassen (1931)	Sub-menu of yupana by *H. Wassen* (base 10, progression 1, 5, 15, 30)	
Representation	Representation of numbers in Wassen's system	**F1**
Addition	Addition of numbers in Wassen's system	**F2**
Yupana by Radicati (1979)	Sub-menu of the yupana by C. *Radicati di Primeglio* (base 10, progression 1, 1, 1, 1)	
Representaiton	Representaiton of numbers in Radicati's system	**F1**
Addition	Addition of numbers in Radicati's system	**F2**
Subtraction	Subtraction of numbers in Radicati's system	**F3**
Multiplication	Multiplication of numbers in Radicati's system	**F4**
Yupana by Glynn (1980)	Sub-menu of yupana by *Burns Glynn* (base 10, progression 1, 1, 1, M)	
Representation	Representation of numbers in Glynn's system	**F1**
Addition	Addition of numbers in Glynn's system	**F2**
Multiplication	Multiplication of numbers in Glynn's system	**F3**
Yupana by De Pasquale (2001)	Sub-menu of the Yupana by *De Pasquale* (base 40, progression 1,2,3,5)	

Function				**Description**	**Short Key**
			Representation	Representation of numbers in De Pasquale's system	**F1**
		Yupana by Chirinos (2008)		Sub-menu of the Yupana by *Chirinos* (base 10, progression 1-11)	
			Representation	Representation of numbers in De Pasquale's system	**F1**
	Author & License			Information about Kunturweb & GPL 3 License	**F12**
	Exit			Exit the program	**ESC**

Table 40: Descriptions of Tk-yupana Menu

3.5 - Yupana by Wassen

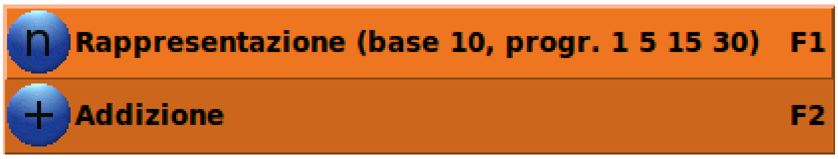


Illustration 3: Sub-menu of the "Yupana by Wassen"

The sub-menu **Yupana by Wassen** (see Illustration 3) has two choices (also selectable via the function keys F1 and F2) which correspond to the following functions:

1. **Representation**: is used to represent a number on the yupana
2. **Addition**: Allows you to add two numbers

3.5.1 - Representation

Illustration 4: Yupana by Wassen: empty table, number zero

The program presents an empty table in a vertical position, corresponding to the number zero (Illustration 4). The size of the table is different depending on the resolution of the screen and is able to operate even on a net-book with resolution 800x480[24]

24 Tk-yupana has been tested on an Asus Eee PC 701 4G with Linux Xandros operating system. [N/A]

This window provides for representation of numbers in the positional system developed by *H. Wassen*, with vertical progression of powers of 10 (base 10) and horizontal progression of the weights 1, 5, 15, 30.

Entering a number

The boxes in the table have different weights depending on the column in which they are found and different power of 10 depending on the row in which they are located.

The first column from the left has weight 1, and it is possible to put into it a maximum of 5 seeds, the second column has weight 5 and it is possible to put a maximum of 3 seeds, the third column has weight 15 and it is possible to put a maximum of 2 seeds and the last column has weight 30 and it is possible to put a maximum of one seed.

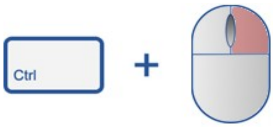
The first line from the bottom corresponds to units, the second to tens, the third to hundreds, the fourth and the fifth to the thousands and tens of thousands.

Clicking with the left mouse button on a box you put a seed in the Yupana

To enter a number you have to click with the left button of the mouse on one of the boxes. Note that the order of insertion is important (the system is positional) and You must proceed from bottom to top and left to right, starting with the units and rising gradually. When the box is selected, you will see a seed in it, the corresponding value is added to the total, and it is then displayed next to the table.

Illustration 5: Entering number 652

To enter the number 652, for example, we start from the units and insert two seeds in the box in the first row (from the bottom) and first column (from left), and then add the tens, by placing five seeds in the box in the second row and first column, and finally add the hundreds, by inserting a seed in the box of the third row and second column (value 5) and a seed in the box of the third row and first column (value 1; total: 5+1 = 6). See Illustration 5.



While holding down the CTRL key and clicking with the right mouse button on a cell a seed is removed from the Yupana

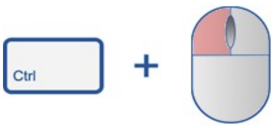
To clear a check box (decrease the number of the corresponding quantity) you must click on the corresponding seed/box by holding down the Control key (CTRL).

You can enter all the numbers from 1 to 99,999, (using the first two columns), or up to 888,880 (using all the boxes), and of course the zero (empty table).

The result is displayed in the "*Total*" field present on the table.

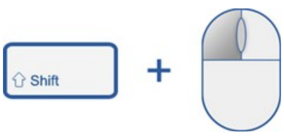
Operations of displacement of the seeds

Once you have filled a box R × C (row R, column C), you can "promote" all the seeds of that box, the next box (R × C+1), or turn all the seeds in a single seed of the box of next column; to do so, simply click on any of a seed present in the full box, holding down the CTRL key. The operation is only possible if there is at least one free space in the next box.



While holding down the CTRL key and clicking with the left mouse button on a box full of seeds, all the seeds are transformed into a single seed belonging to the box of the same row and next column.

Once you have filled a box R × C (row R, column C), you can "promote" all the seeds (or part them) of that box, to the box of next power (R+1 × 1), or turn all seeds in one or more seeds of the box into one of the next line, column 1; to do so, simply click on any of seed of the full box, holding down the SHIFT key. The operation is only possible if there is at least one free space in the first box of the next power.



Holding down the SHIFT key and clicking the left mouse button on a box full of seeds, all the seeds are transformed into one or more seeds, belonging to the cell of the first column of the next row

3.5.2 - Addition

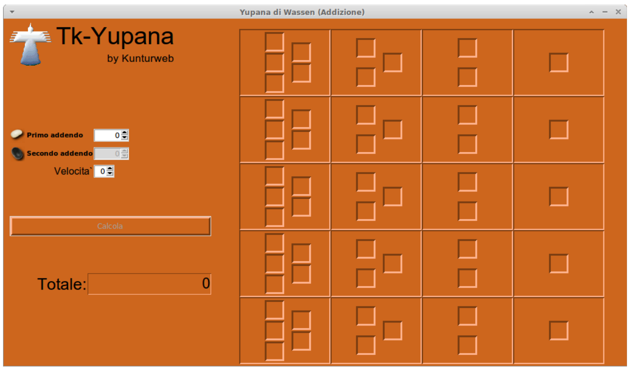


Illustration 6: Yupana by Wassen, Addition

The window allows you to add two numbers. The numbers are inserted through the panel on the left, which involves the insertion of the first and second addend, as well as the ability to select the speed with which the yupana will be filled (see Illustration 6).

Entering the first term

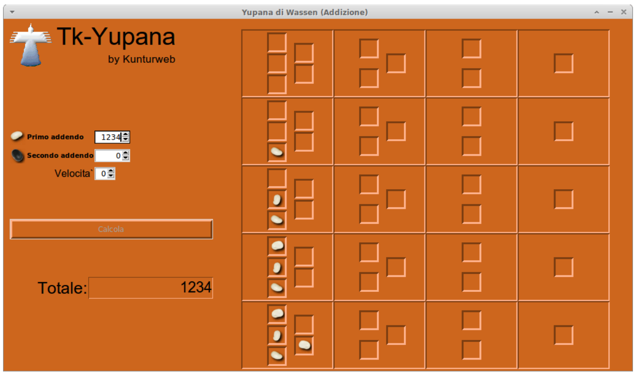


Illustration 7: Entering 1st addend (1,234)

You can enter the first term using the arrows in the field, or by entering the number directly using the keypad. Once you leave the field, the second term is enabled (otherwise disabled) and the first number is represented in yupana with white seeds according to the rules described in the previous paragraph.

Entering the 2nd term

Once you have entered the first term (and only then) you can enter the second term, using the arrows or by directly entering the field with the keyboard. When you leave the field the "Calculate" button is enabled.

Addition of terms

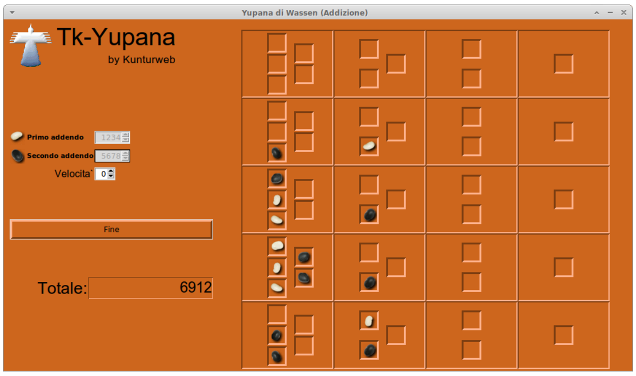


Illustration 8: Addition

By clicking on the "Calculate" button the operation of addition of the two numbers previously entered begins. If the speed is set to 0, the calculation is done instantly and the result displayed on the yupana. The seeds for the second addend are black to distinguish them from those of the first term (see previous paragraph). If the speed is set with a value greater than zero, the filling of the table is done step-by-step and with increasing speed according to the number entered in field.

3.5.3 - Multiplication

TO DO

3.5.4 - Menu functions

Clicking on the condor (upper left picture) a drop-down menu is displayed, from which you can select the following functions:

1. **Clean the Yupana**: remove all the seeds from the table and clears the counter of the total (**F1**).
2. **Help**: Displays a help file (**F12**)
3. **Close**: Closes the window (**F8**).

3.6 - Yupana by Radicati

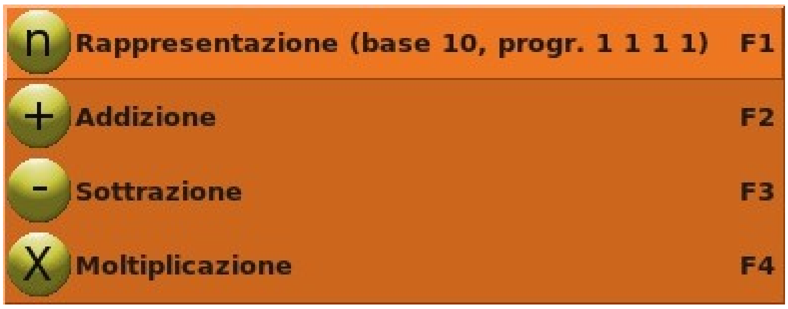


Illustration 9: Radicati di Primeglio sub-menu (possible choice)

The sub-menu **Yupana Radicati** (see Illustration 9) has four choices (also selectable via the function keys F1, F2, F3 and F4) which correspond to the following functions:

1. **Representation**: allows to represent a number on the yupana
2. **Addition**: Allows to perform the addition operation between two or three numbers
3. **Subtraction**: allows you to perform the subtraction operation between two numbers
4. **Multiplication** allows you to perform the multiplication of two numbers

3.6.1 - Representation

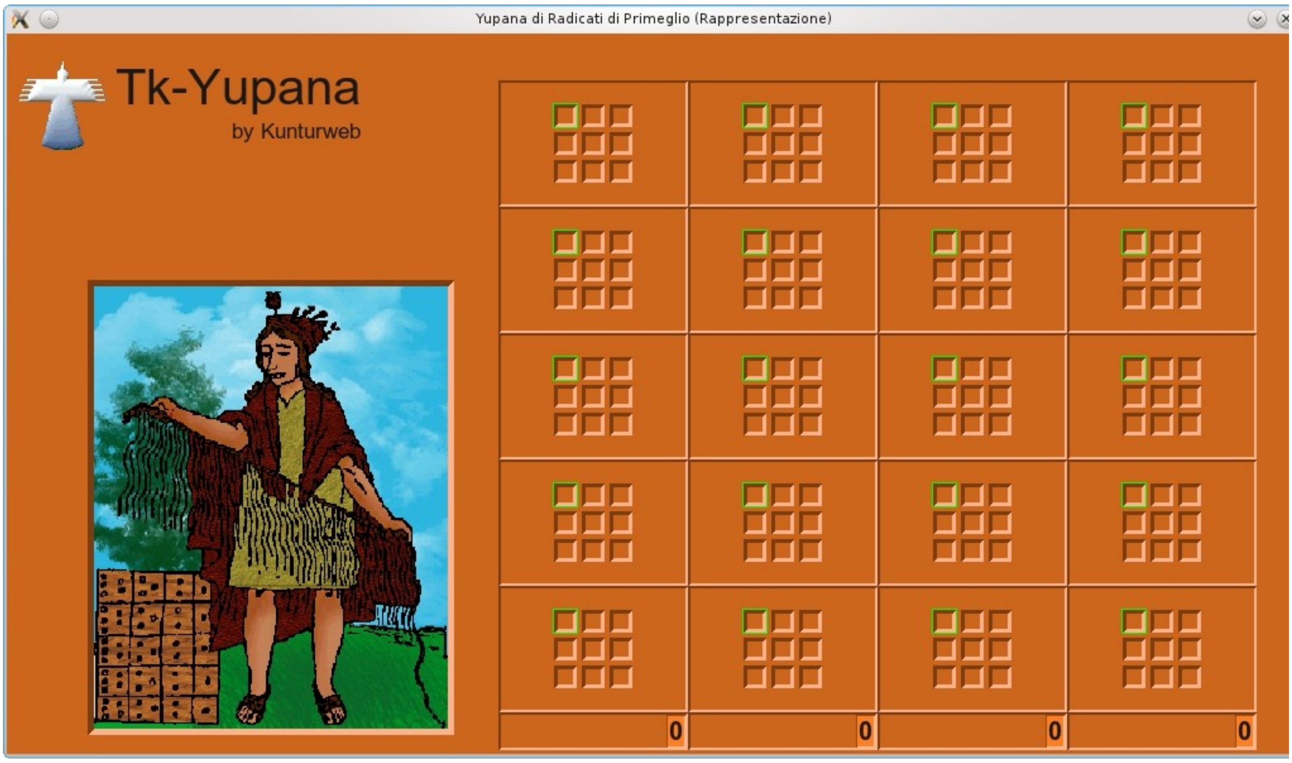


Illustration 10: Empty Yupana, corresponding to number 0 (zero)

The program shows a table in an upright position and empty, corresponding to the number zero (Illustration 10). The size of the table is different depending on the resolution of the screen and is able to work even on a netbook with resolution 800x480[25]

This window allows to enter and represent four numbers in base 10 positional system, one for each column.

Entering a number

In each of the four colums it is possible to represent a number. Each row, starting from the bottom, represent a power of tens. In each cell is it possible to insert up to 9 seeds, each of them with value one (1).

Clicking with the left mouse button on a box you put a seed in the Yupana

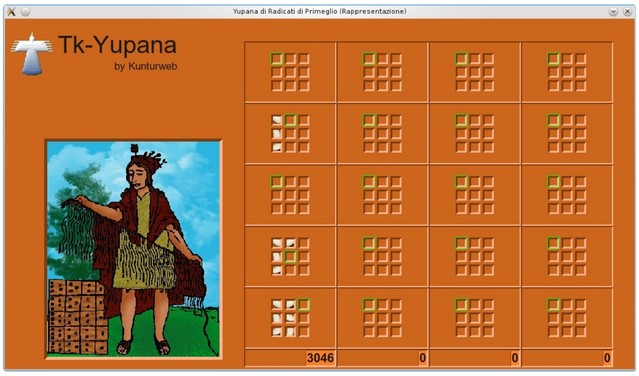
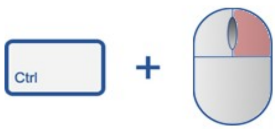


Illustration 11: Inserimento del numero 3046

To enter a number you must click with the left mouse button on a free box (a green frame shows which of them are enabled): a seed will appear right in the box and its value will be added to the total; the total is shown under the corresponding column. The color of the seeds is white for the odd coluns (starting from left) and black for the even (this color difference is only to distinguish the represented numbers).

To enter number 3046 in column 1, for example, you need to select some boxes in the square R2×C1 (it sums to 4 tens) and six boxes in the square R1×C1 (total of 6 units). The total is shown under column C1. Please, see 'Illustration 11.

It is possible to enter all numbers from 1 to 99999.



While holding down the CTRL key and clicking with the right mouse button on a seed is removed from the Yupana

To remove a seed from the table (and decrease the total of the corresponding quantity) you must click with the right button of the mouse on the seed, while holding the *control key* (CTRL).

25 Tk-yupana è stato testato su un Asus eee PC 701 4G con sistema operativo Linux Xandros. [n.d.A.]

The order of removal is opposite to the insertion direction, so you can not remove the seeds that are not adjacent.

3.6.2 - Addizione

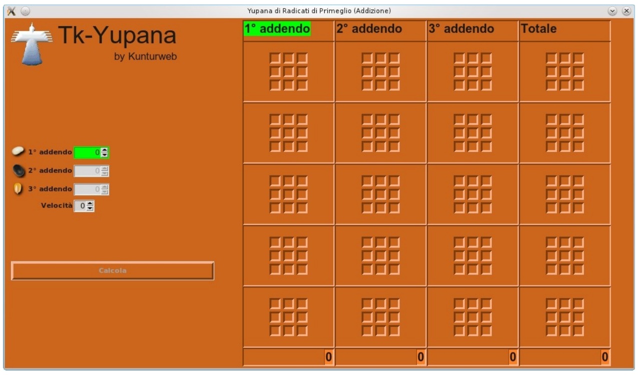


Illustration 12: Yupana di Radicati: addizione

The window allows you to add three numbers together. The numbers are inserted through the panel on the left, which involves the insertion of the first, second and third addend, besides the possibility to select the speed with which is filled the yupana (please, see Illustration 12).

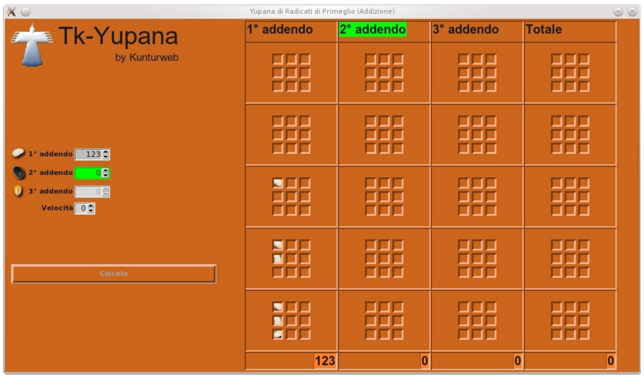


Illustration 13: Inserimento del primo addendo (123)

Entering the first addend

You can enter the first term using the arrows in the related field, or by entering the number directly using the keyboard. Once you leave the field, the second term is enabled (otherwise disabled) and the first number is represented in yupana with the white seeds according to the rules described in the previous paragraph.

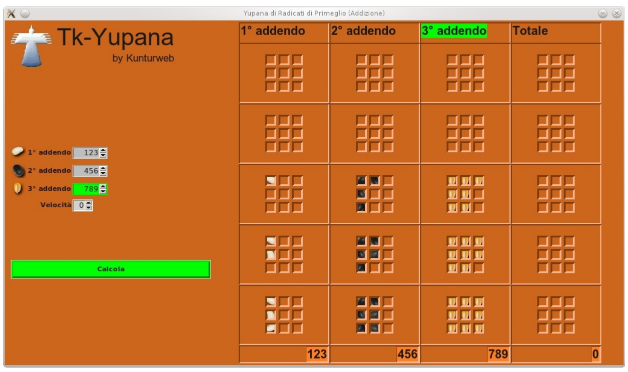


Illustration 14: Inserimento del secondo (456) e del terzo addendo (789) e somma

Entering the second and third addend

Once inserted the first term (and only then) you can enter the second term, using the arrows or by directly filling the field with the keyboard. When you leave the field, the number is displayed in the second column by black seeds and the next field is enabled for entering the third summand. Once you have entered the third summand, when you leave the respective field, the number is represented in the third column with yellow seeds, and the "Calculate" button is enabled, which allows you to compute the sum.

Sum of the addends

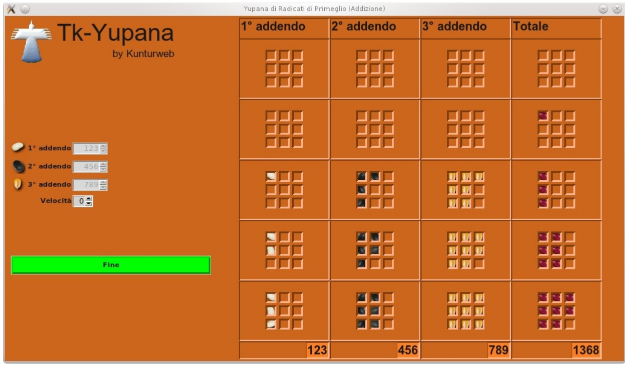


Illustration 15: Somma degli addendi (1268)

By clicking on the "Calculate" button, the sum of the three numbers you entered previously is made. If the speed is set to 0, the calculation is done instantly and the result appears in the fourth column of the yupana. If the speed is set to a value greater than zero, the filling of the table is done gradually and with increasing speed according to the number entered in field speed.

3.6.3 - Subtraction

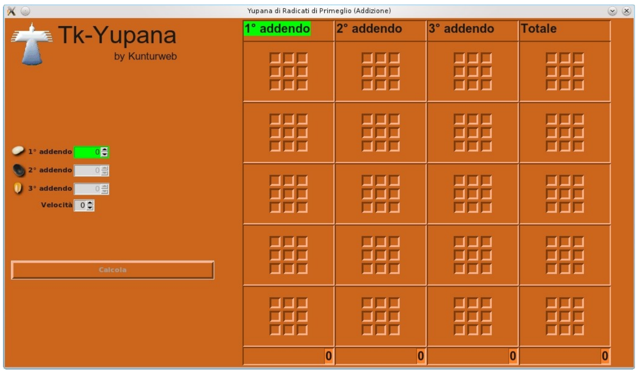


Illustration 16: Yupana di Radicati: sottrazione

The window allows you to subtract two numbers together. The numbers (minuend and subtracting) are inserted through the panel on the left, you can also select the speed with which it is filled with the yupana (please, see Illustration 24).

Entering the minuend

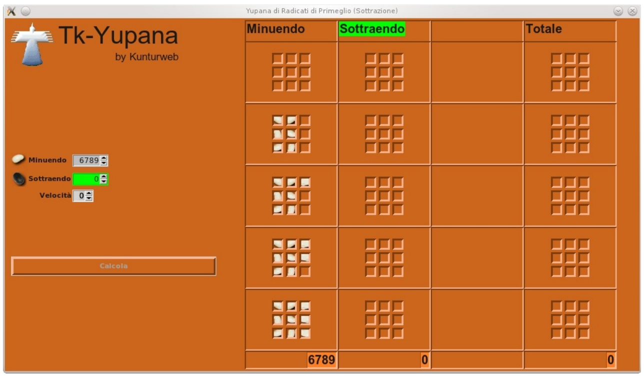


Illustration 17: Inserimento del primo addendo (6789)

You can enter the minuend using the arrows in the related field, or by entering the number directly using the keyboard. Once you leave the field, the number is represented in yupana with white seeds, according to the rules described in section2.2.3 - , and the subtrahend input field is enabled (otherwise disabled). Please, see Illustration 21.

Entering the subtrahend

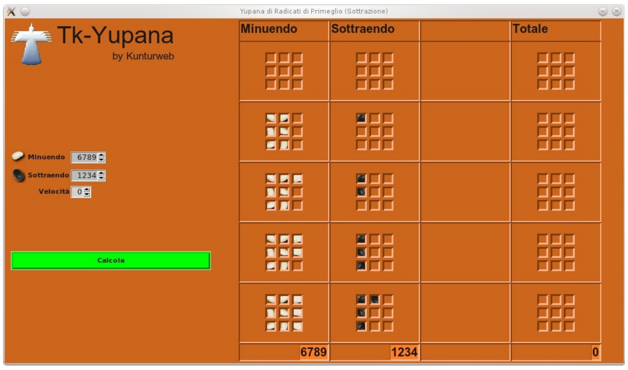


Illustration 18: Inserimento del sottraendo (1234)

Once inserted the minuend (and only then) you can enter the subtrahend, using the arrows or by directly entering the field with the keyboard. When you leave the field, the number is represented in the second column by black seeds, and the "Calculate" button is enabled, which allows you to compute the sum. Please, see Illustration 25.

Subtraction

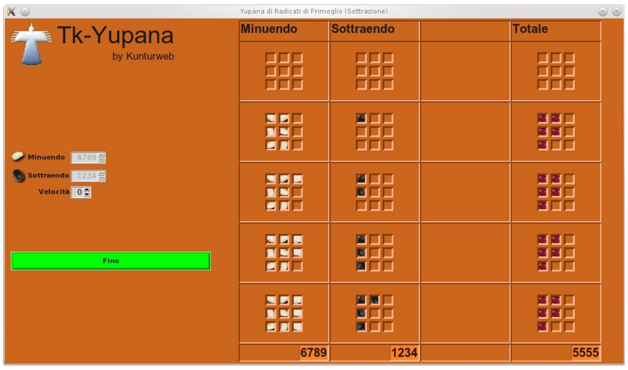


Illustration 19: Sottrazione (5555)

By clicking on the "Calculate" button the subtraction of the two numbers you entered previously is performed. If the speed is set to 0, the calculation is done instantly and the result (difference) appears in the fourth column of the yupana. If the speed is set to a value greater than zero, the filling of the table is done gradually and with increasing speed according to the number entered in field speed. Please, see Illustration 23.

3.6.4 - Multiplication

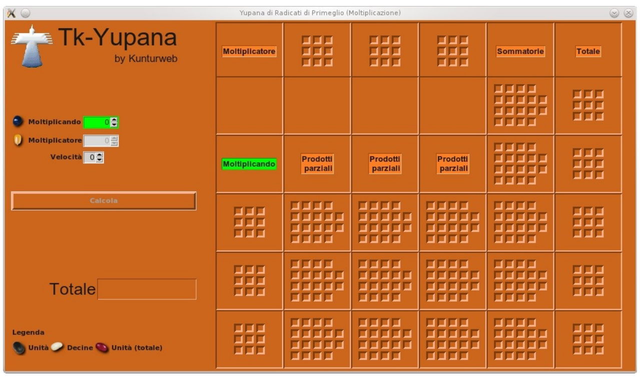


Illustration 20: Yupana di Radicati: moltiplicazione

This window allows to multiplicate two numbers together. The numbers are inserted through the panel on the left, which involves the insertion of the multiplicand, multiplier, as well as the ability to select the speed with which the yupana will be filled (please, see Illustration 20).

Entering the multiplicand

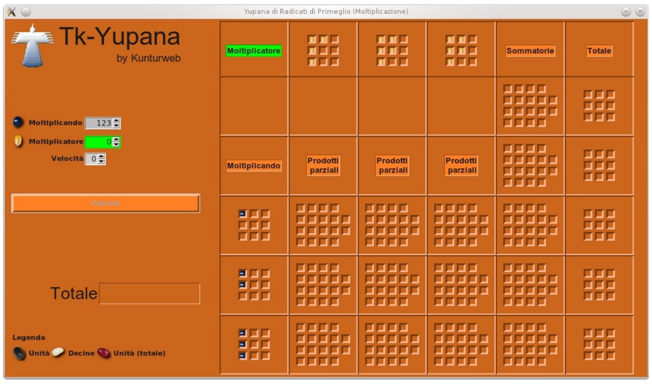


Illustration 21: Inserimento del moltiplicando (123)

You can enter by multiplying by acting on the arrows on the field, or by entering the number directly using the keyboard. Once you leave the field, the number is represented in the first column from the left with purple seeds and the multiplier field is enabled (otherwise disabled). Please, see Illustration 21. You can only represent numbers between 1 and 999 and such that the product does not exceed 99999.

Entering the multiplier

Once inserted the multiplicand (and only then) you can enter the multiplier, using the arrows or by directly entering the field with the keyboard. When you leave the field, the number is represented, with yellow seeds, in the first row of the yupana (columns 2, 3 and 4 from the left). Also in this case can only represent numbers between 1 and 999 and such that the product does not exceed 99999. The "Calculate" button is enabled. See Illustration 22.

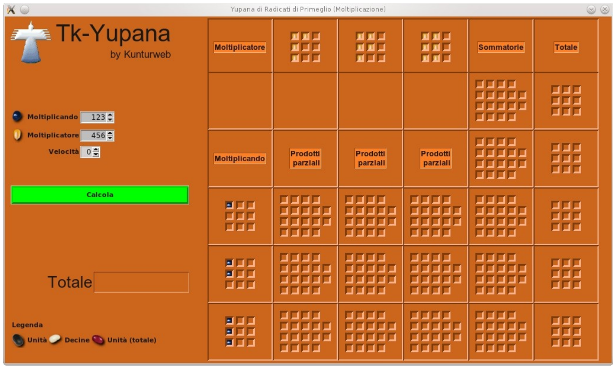


Illustration 22: Inserimento del moltiplicando (456)

1st step: partial products

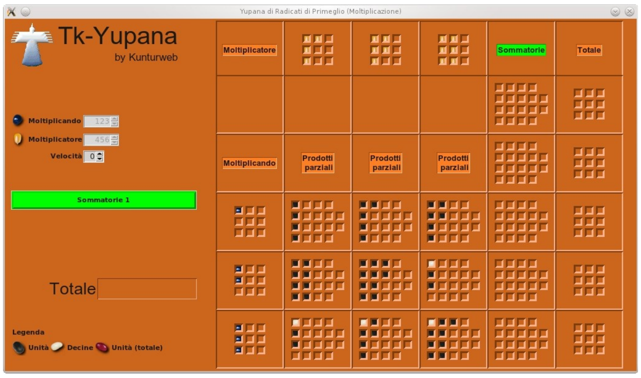


Illustration 23: Riempimento dei prodotti parziali

By clicking on the "Calculate" button, the multiplication of the two numbers you have entered previously will begin. The operation will be carried out step by step. On the first pressure of the button, the fields "partial products" are highlighted (but the operation is not executed), just to give the user the opportunity to understand the method. At a subsequent press of the button, the partial products are calculated (see Illustration 23), and the central cells (corresponding to rows 1, 2 and 3 from the bottom and columns 2, 3 and 4 from the left, are filled.

2nd step: Summations

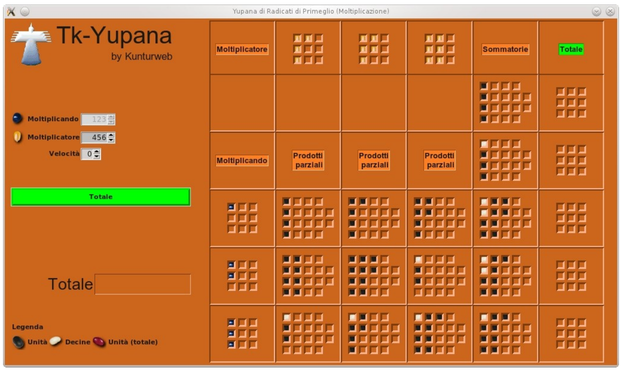


Illustration 24: Calcolo delle sommatorie

Next, at a further pressure of the button, diagonal summations of the partial products are carried out. The results are shown in column 5. Each press will change the message of the button itself, to indicate the step under consideration. Note that if the speed is set to 0, every step of the calculation is done instantly and the result suddenly displayed on the yupana, otherwise the seeds will be placed in the yupana with increasing speed, depending on the number entered in the speed-field.

3rd step: Product

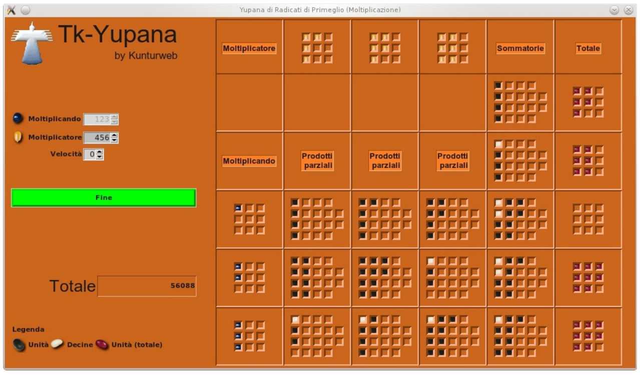


Illustration 25: Primo passo dell'operazione

Finally, on further clicking on the "Calculate" button, the operation ends. The calculation of the product of the two factors is carried out and the number is shown in column 6, with red seeds. The product is also numerically shown at the bottom left of the window. Please, see Illustration 25.

A further press of the "Calculate" button allows you to perform another operation.

3.6.5 - Menu functions

Clicking on the condor (upper left picture) a drop-down menu is displayed, from which you can select the following functions:

1. **Clean the Yupana**: remove all the seeds from the table and clears the counter of the total (**F1**).
2. **Help**: Displays a help file (**F12**)
3. **Close**: Closes the window (**F8**).

3.7 - Yupana by Glynn

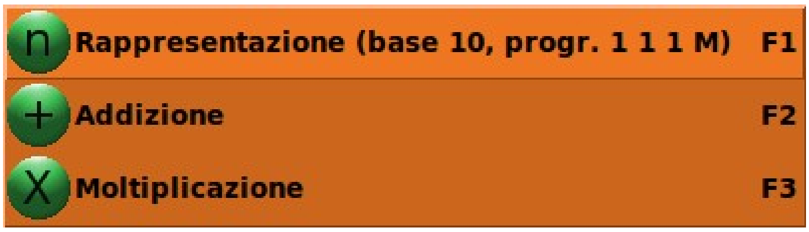


Illustration 26: Sub-menu of "Yupana by Glynn"

The sub-menu **Yupana by Glynn** (see Illustration 26) has two choices (also selectable via the function keys F1 and F2) which correspond to the following functions:

1. **Representation**: is used to represent a number on the yupana
2. **Addition**: Allows you to add two numbers
3. **Multiplication**: allows you to multiply two numbers

3.7.1 - Representation

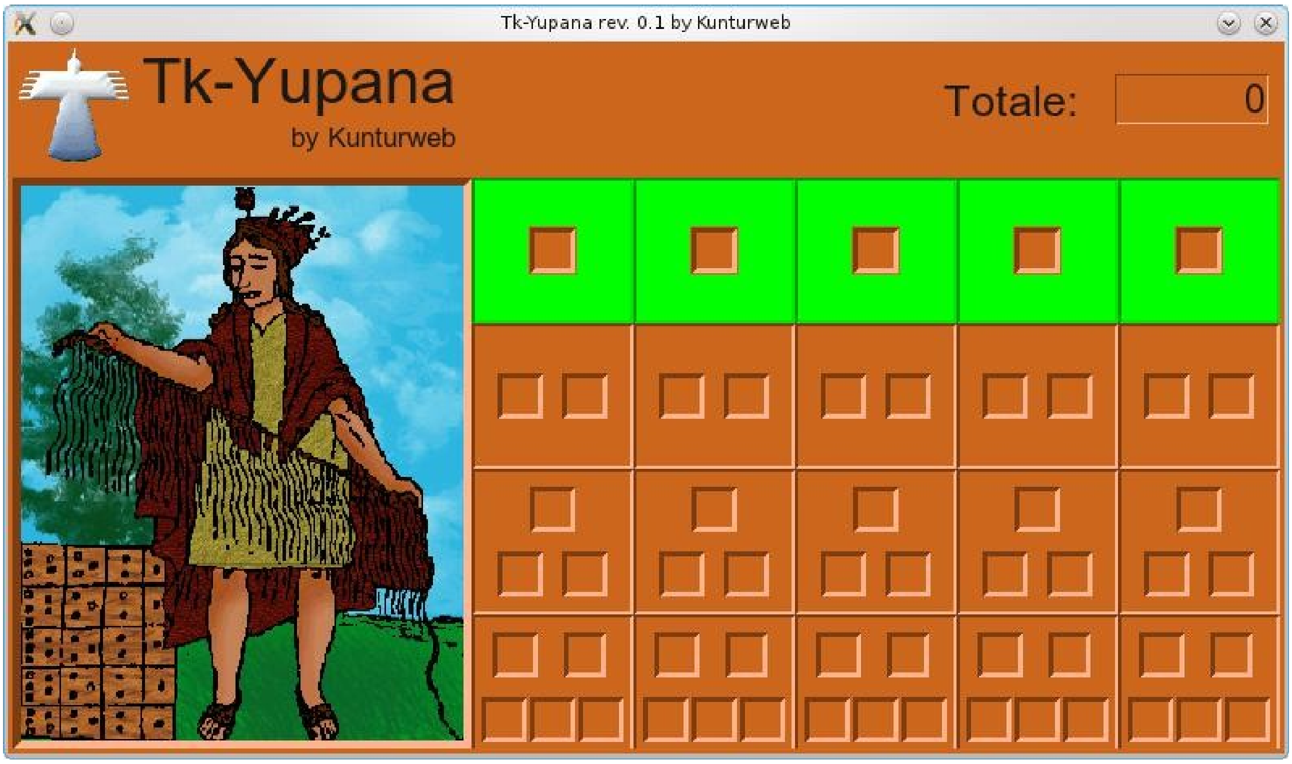


Illustration 27: Empty yupana (number zero)

The window shows an empty table in a horizontal position, corresponding to the number zero (Illustration 27). The size of the table is different depending on the resolution of the screen and is able to operate even on a net-book with resolution 800x480[26]

The program allows to represent numbers in base 10 on positional system.

Entering a number

Each box of the table has a unit weight; the first column counting from the right corresponds to the column of the unit, the second of tens, the third of hundreds, the fourth and the fifth of thousands and tens of thousands.

Clicking with the left mouse button on a box you put a seed in the Yupana

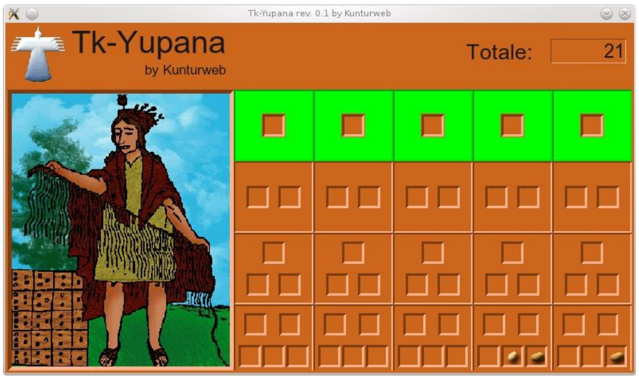
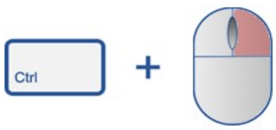


Illustration 28: Entering number 21

To enter a number you have to click with the left button of the mouse on one of the boxes. Note that the order of insertion is important[27] and must proceed from bottom to top and from right to left. When the check box is selected, you will see a seed in it, the corresponding value is added to the total, and it is then displayed above the table.

To enter the number 21, for example, will need to click on the box at the bottom right of the fourth column (Total: 10), then click on the box immediately to its left (Total: 20), then click on the box at the bottom right of the fifth column (total: 21). See Illustration 28.



While holding down the CTRL key and clicking with the right mouse button on a seed is removed from the Yupana

To clear a seed (decrease the number of the corresponding quantity) you must click on the corresponding seed/box by holding down the Control key (CTRL).

26 Tk-yupana has been tested on an Asus Eee PC 701 4G with Linux Xandros operating system. [N/A]
27 It is not possible in version 0.4 to transfer pairs or sets of seeds in the highest slots of the yupana. This feature will be possibly included in a later version. [N/A]

You can enter any number from 1 (one) to 111,110 (or up to 222,220 using also memories). The result is displayed in the "Total" field present above the table.

Use of memory

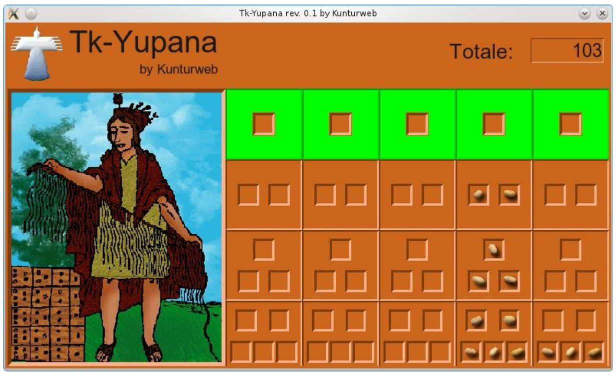


Illustration 29: Number entered: 103

Once a column is totally filled with seeds (10 seeds), it is possible to promote all the seeds in the memory box in order to be able to perform arithmetic operations without having to worry about keeping in mind the seeds moved.

Suppose for example that you have set in Yupana the number 103 and therefore you have completed the column 4 (corresponding to ten tens, or a hundred) and filled three boxes of column 5 (Illustration 29).

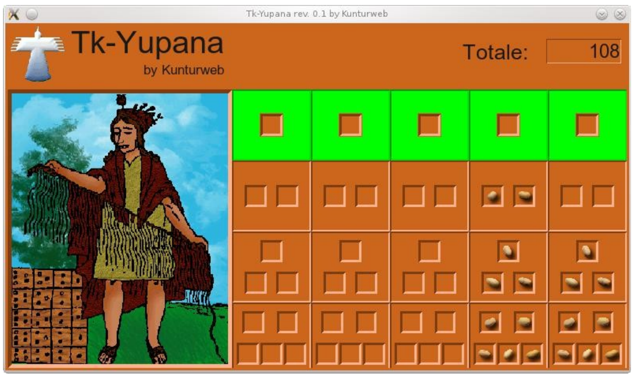


Illustration 30: Adding 5 units

If, at this point, you wanted to add 35 to the amount already entered, would be sufficient to select 5 boxes of column 5 and 3 cells in column 4.

The operation on column 5 does not present any problems and can be carried out immediately (Illustration 30) by clicking with the left mouse button on the five boxes above the three already filled.

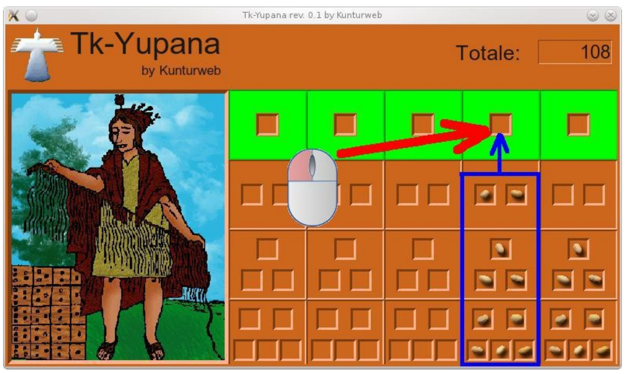


Illustration 31: Click with the left button on the memory box to clear the seeds of the column

Because the fourth column is filled (Illustration 30), before performing the operation on the tens (add three seeds), you must free the necessary space, replacing the ten seeds of column 4 with a seed memory (equal to one hundred).

We thus replace the ten seeds with a seed of memory in column 4 (Illustration 31). To do this, simply click on the *Memory* of column 4: all the seeds below will be replaced with a single seed in the green box. The result is shown in Illustration 32.

When a column has been filled with ten seeds, you can click with the left mouse button on a memory box in order to replace the ten seeds with one in the memory

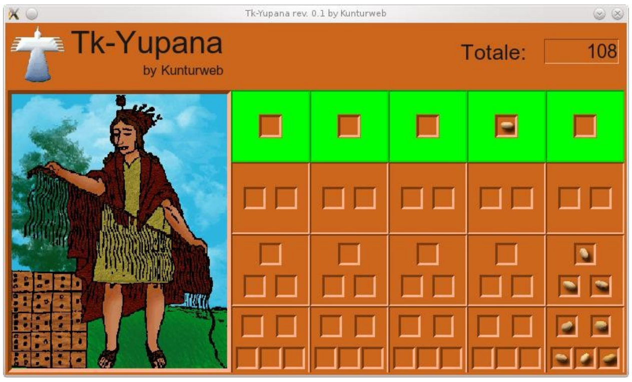


Illustration 32: Replace ten seeds of column 4 with a seed in the memory in the same column

Note that if you have not filled all ten boxes below, the operation is not possible.

Illustration 33: Adding tens (3 seeds) to column 4

Then, to complete the operation, You can add three seeds in the (now) free boxes of column 4 (Illustration 33), which corresponds, as we have said, to the three tens of the number 35

A seed in memory can be moved to the first empty space of the next column, as long as there is one free by clicking on the seed itself with the left mouse button

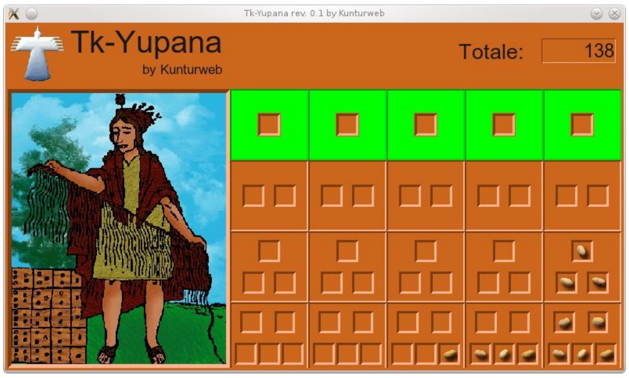


Illustration 34: Moving the seed stored in the first free cell to the next column

To conclude, You can move the seed from the memory of column 3 to the first free box (in this case the first box in the lower right) of column four. To accomplish this, simply click with the left mouse button on the seed in memory of column 4: the seed is removed and placed automatically in the first free box of column 3 (free box (in this case the first box in the lower right)Illustration 34).

If the operation is not possible (e.g. when the next column is completely filled) the seed will remain in the memory box. To move the seed, You must promote all the seeds of the next column in its memory.

Inverse operations

When a seed is in memory, you can also redistribute its value in the boxes of the column below, if it they are **all** free. To accomplish this, just click on the seed into memory with the *right mouse button*: the seed disappear from the memory box and ten boxes below will be filled.

A seed in the memory box can be redistribute in the boxes below, where they are all free, by clicking with the right button on the memory box.

Similarly, the last seed entered in column X, can be moved in the memory of the previous column, by clicking on it with the right mouse button. The seed disappears from the box to appear in the memory of the previous column. Since this is only possible on the highest suit (last entered), if you click on one of the other seeds, nothing happens

Any seed of a given column X, can be moved to the memory of the previous column as long as the corresponding box is free.

3.7.2 - Addition

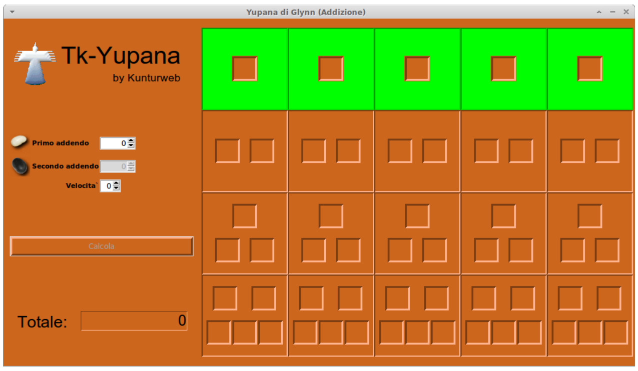


Illustration 35: Yupana by Glynn: addition

The window allows you to add two numbers. The numbers are inserted through the panel on the left, which involves the insertion of the first and second addend, as well as the ability to select the speed with which the yupana will be filled (see Illustration 35).

Entering the first term

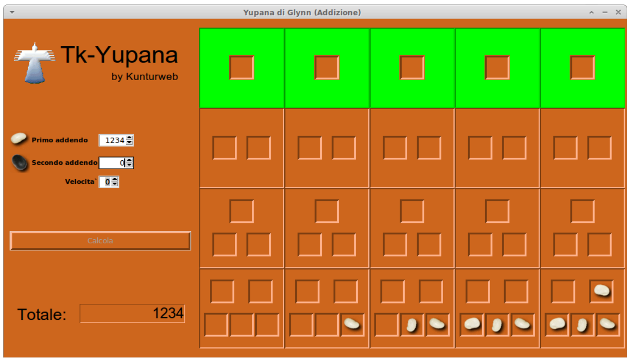


Illustration 36: Yupana by Glynn: entering number 1,234

You can enter the first term using the arrows in the field, or by entering the number directly using the keypad. Once you leave the field, the second term is enabled (otherwise disabled) and the first number is represented in yupana with white seeds according to the rules described in the previous paragraph.

Entering the second term

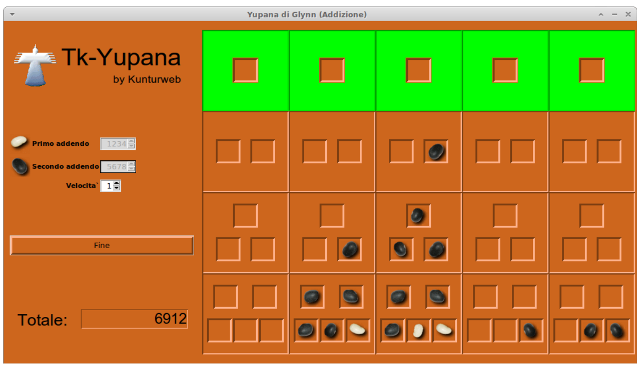


Illustration 37: Yupana by Glynn: entering 2nd addend (5,678) and addition

Once you have entered the first term (and only then) you can enter the second term, using the arrows or by directly entering the field with the keyboard. When you leave the field is enabled the "Calculate" button.

Addition of the terms

By clicking on the "Calculate" button the addition of the two numbers previously entered begins. If the speed is set to 0, the calculation is done instantly and the result displayed on the yupana. The seeds for the second addend are black to distinguish them from those of the first addend (see Illustration 37). If the speed is set with a value greater than zero, the filling of the table is performed step by step and with increasing speed according to the number entered in field speed.

3.7.3 - Multiplication

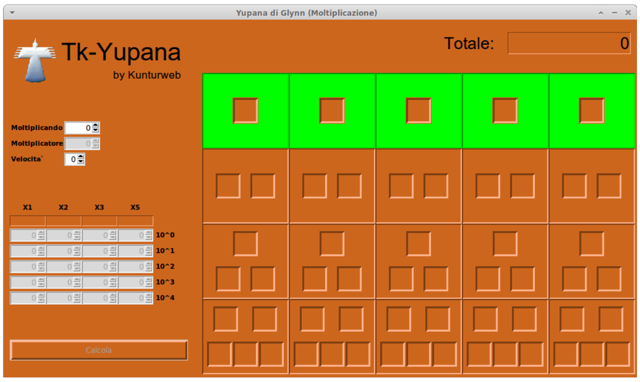


Illustration 38: Yupana by Glynn: multiplication

The window allows you to multiply two numbers. The numbers are inserted through the panel on the left, which involves the insertion of the multiplicand, multiplier, as well as the ability to select the speed with which the yupana is filled (see Illustration 38).

Entering the Multiplicand

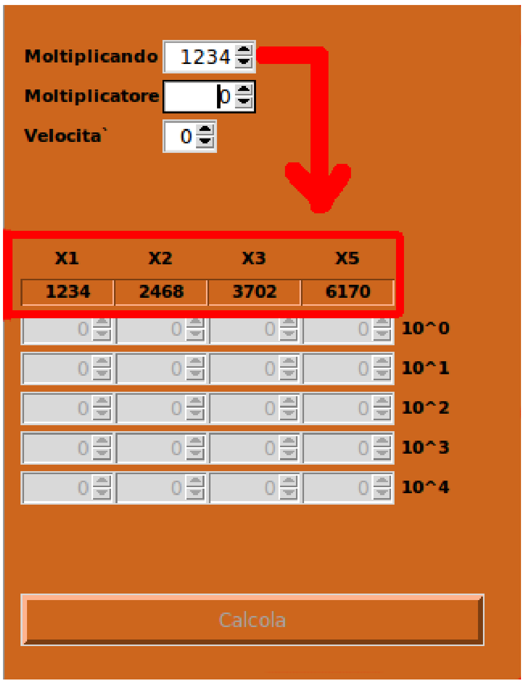


Illustration 39: Entering the multiplicand

You can enter the multiplicand using the arrows in the field, or by entering the number directly using the keypad. Once you leave the field, the multiplier is enabled (otherwise disabled) and the multiples 1x, 2x, 3x and 5x are calculated automatically and shown in the table below. (Illustration 39).

Entering the multiplier

Once inserted the multiplicand (and only then) you can enter the multiplier using the arrows or by directly entering the field with the keyboard. When you leave the field, the multiplier entered is broken down according to predefined rules and shown in the board below; the "Calculate" button is then enabled (Illustration 40).

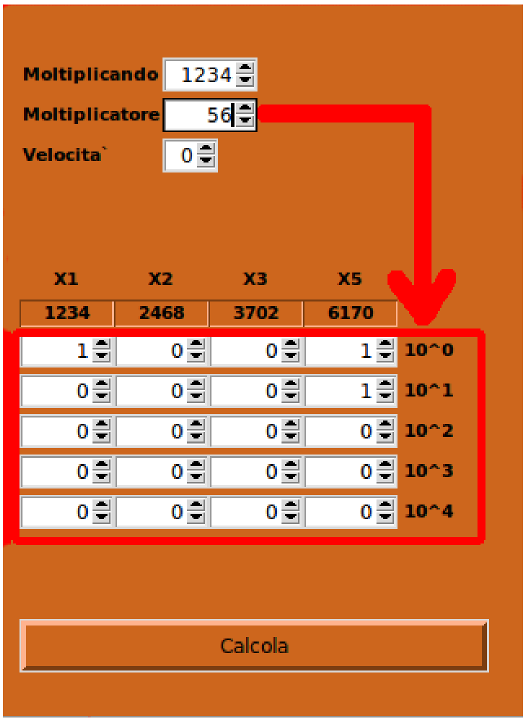


Illustration 40: Entering the multiplicator

Adding the terms

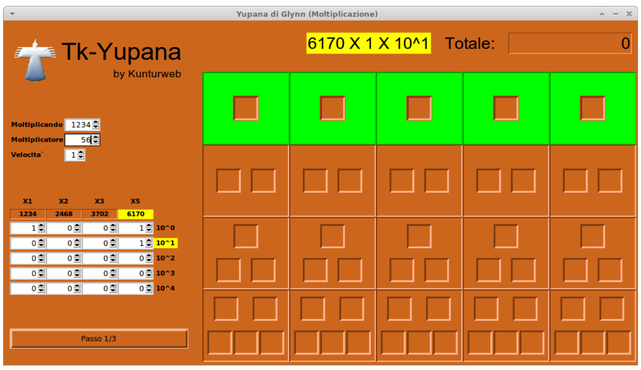


Illustration 41: First press of calculate button; next step is shown in yellow

By clicking on the "Calculate" button, the multiplication of two numbers entered previously begins; the operation will be performed step by step, in particular, at the first press of the “Calculate” button, is presented the first step that will be run: the row and column, and the operation are highlighted in yellow (Illustration 41). The words of the "Calculate" button is changed with the number of the step in question (e.g. "*Step 1/3*"). To continue with the operation you need to press the button again.

Note that if the speed is set to 0, every step of the calculation is done instantly and the result displayed on the yupana, otherwise the seeds will be placed in yupana with increasing speed depending on the number entered in the Speed field.

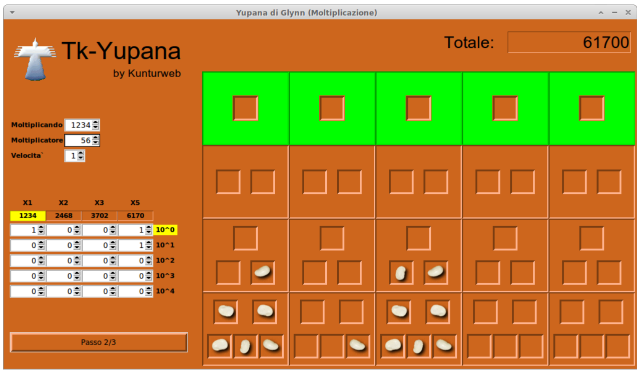


Illustration 42: 1st step (additions)

The seeds placed in each step of the calculation have different colours: this not because they have different weights, but for educational purposes only.

By clicking on the "*Step 1/3*" take the filling of yupana and the words change to "Step 2/3", and so on (Illustration 42 and following).

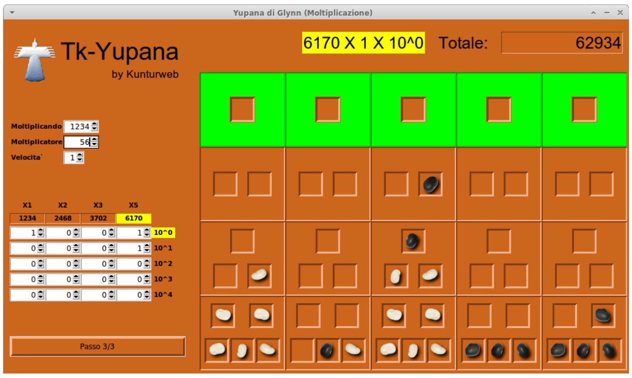


Illustration 43: 2nd step (additions)

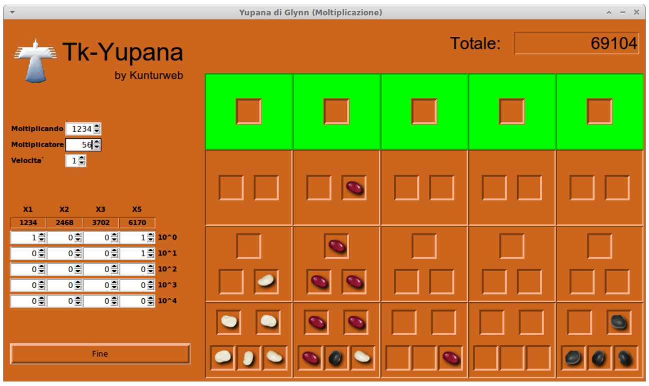


Illustration 44: 3rd step (result)

Once concluded the calculation, the final result (product) is shown in the upper right corner of the window (Illustration 44).

3.7.4 - Menu functions

Clicking on the condor (upper left picture) a drop-down menu is displayed, from which you can select the following functions:

1. **Clean the Yupana**: remove all the seeds from the table and clears the counter of the total (**F1**).
2. **Help**: Displays a help file (**F12**)
3. **Close**: Closes the window (**F8**).

3.8 - Yupana by De Pasquale

Illustration 45: Empty Yupana (number zero)

When you open the window **Yupana by De Pasquale**, the program presents an empty table in an upright position, corresponding to the number zero (Illustration 45). The size of the table is different depending on the resolution of the screen and is able to operate even on a net-book with resolution 800x480[28]

In this case have not yet been implemented algorithms, but it is **only** possible to represent numbers in the positional numeral system in base 40.

3.8.1 - Representation

The numbering of the columns increases from right to left and that of rows grows from the bottom upwards.

The boxes of the table have a different weight depending on the row and the column in which they are located, according to the scheme shown in section *2.4.1 -* .

28 Tk-yupana has been tested on an Asus Eee PC 701 4G with Linux Xandros operating system. [N/A]

Clicking with the left mouse button on a box you put a seed in the Yupana

To enter a number you have to click with the left button of the mouse on any of the boxes.

When the check box is selected, you will see a seed in it, the corresponding value is added to the total, and it is then displayed next to the table.

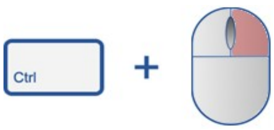
Since the numbering is in base 40, in the first row at the bottom you can enter any number from 0 (row and empty table) to number 39 (full row). To enter the number 40 must clear the entire first row and insert one seed in the first box to the right of the second column.

Every number does not have a unique representation, as can be seen in Illustration 46 and Illustration 47, where are shown two representations of number 9.

Illustration 46: A representation of the number 9 given by 3x2 + 2x1 + 1

Illustration 47: Another representation of the number 9 given by 3x3

To clear a seed (decrease the number of the corresponding amount) you have to click with the ***right button of the mouse*** on the corresponding seed/box by holding down the Control key (CTRL).



While holding down the CTRL key and clicking with the right mouse button on a seed, it is removed from the Yupana

You can enter any number from 1 (one) to 102,399,990.

The result is displayed in the "Total" field on the upper-left of the window.

3.8.2 - Menu functions

Clicking on the condor (upper left picture) a drop-down menu is displayed, from which you can select the following functions:

1. **Clean the Yupana**: remove all the seeds from the table and clears the counter of the total (**F1**).
2. **Help**: Displays a help file (**F12**)
3. **Close**: Closes the window (**F8**).

3.9 - Yupana by Chirinos (2008)

Illustration 48: Empty yupana, corresponding to number 0 (zero)

When you open the window **Yupana by Chirinos**, the program presents an empty table in an upright position, corresponding to the number zero (Illustration 48). The size of the table is different depending on the resolution of the screen and is able to operate even on a net-book with resolution 800x480[29]

In this case have not yet been implemented algorithms, but it is **only** possible to represent numbers in the positional numeral system in base 10, with progression 1-11.

3.9.1 - Representation

The numbering of the columns grows from right to left, and that of the rows from bottom to top.

The little boxes (seeds) in the table have *different weight* depending on the row and the column in which they are placed; please refer to paragraph *2.5.1 -* for details. Note that row 1, corresponding to the lower row, stands for decimals.

29 Tk-yupana has been tested on an Asus Eee PC 701 4G with Linux Xandros operating system. [N/A]

Clicking with the left mouse button on a box you put a seed in the Yupana

To enter a number you have to click with the left button of the mouse on any of the boxes.

When the check box is selected, you will see a seed in it, the corresponding value is added to the total, and it is then displayed next to the table.

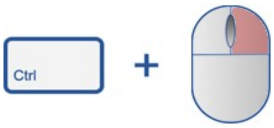
Since the numbering is in base 10, and it starts from decimals, in the first row at the bottom you can enter any number from 0 (row and empty table) to 6.6 (full row).

Each number does not have a unique representation, as can be seen in Illustration 49 e Illustration 50, where are shown two representations of number 9,8.

Illustration 49: A representation of number 9,8 given from 6 + 3 + 0.8

Illustration 50: Another representation of number 9 given from 1 + 2 + 5 (second row) + 1 + 0,8 (first row)

To clear a seed (decrease the number of the corresponding amount) you have to click with the ***right button of the mouse*** on the corresponding seed/box by holding down the Control key (CTRL).



While holding down the CTRL key and clicking with the right mouse button on a seed, it is removed from the Yupana

You can enter any number from 1 (one) to 73332,6.

The result is displayed in the "Total" field on the upper-left of the window.

3.9.2 - Menu functions

Clicking on the condor (upper left picture) a drop-down menu is displayed, from which you can select the following functions:

1. **Clean the Yupana**: remove all the seeds from the table and clears the counter of the total (**F10 e F1**).
2. **Help**: Displays a help file (**F10 e F12**)
3. **Close**: Closes the window (**F10 e F8**).

3.10 - Yupana di Kak (2014)

Image 51: Empty Yupana, corrispondente al numero 0 (zero)

When you open the window **Yupana by Kak**, the program presents an empty table in an upright position, corresponding to the number zero (Image 51). The size of the table is different depending on the resolution of the screen and is able to work even on a netbook with 800x480 resolution.[30]

In this case have not yet been implemented algorithms, but it is **only** possible to represent numbers in the *non-uniform positional numeral system*.

3.10.1 - Representation

The numbering of the columns grows from left to right and the rows grows from the bottom upwards.

The boxes of the table have a different weight depending on the row and column in which they are located, according to the scheme shown in section *2.7.1 -* .

30 TkYupana è stato testato su un Asus eee PC 701 4G con sistema operativo Linux Xandros. [n.d.A.]

Clicking with the left mouse button on a box you put a seed in the Yupana

To enter a number you have to click with the left button of the mouse on any of the boxes.

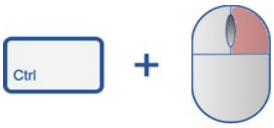
When the box is selected, you will see a seed inside it and the corresponding value is added to the total, and the total is then displayed next to the table.

The first row at the bottom corresponds to the power 12^0, and the values of the boxes varies according to the progression 1, 6, 24, 72; so in this row it is possible to enter all the numbers from 0 (bare table and row) to number 143 (full line).

Each number has a unique representation.

Image 52: Representation of the number 223 = 144 (2nd row) + 1 + 6 + 72 (1st row)

To clear a box (decrease the total of the corresponding quantity) you have to click with the right button of the mouse on the corresponding seed/box while holding down the Control key (Ctrl).



While holding down the CTRL key and clicking with the right mouse button on a seed, it is removed from the Yupana

You can enter all the numbers from 1 (one) to 61,917,364,223 (sixty-one billion, nine hundred seventeen million, three hundred and sixty-four thousand, two hundred and twenty-three).

The result is displayed in the "Total" field, on the left side of the table.

3.10.2 - Menu functions

Clicking on the condor (upper left picture) a drop-down menu is displayed, from which you can select the following functions:

1. **Clean the Yupana**: remove all the seeds from the table and clears the counter of the total (**F10 & F1**).
2. **Help**: Displays a help file (**F10 & F12**)
3. **Close**: Closes the window (**F10 & F8**).

3.11 - Yupana by Florio (2008)

Illustration 53: Sub-menu Florio

The sub-menu Yupana Florio (see Illustration 53) has three choices (also selectable via the function keys F1, F2 and F3) which correspond to the following functions:

1. **Representation**: is used to represent a number on the yupana
2. **Addition:** allows to add two numbers
3. **Multiplication**: allows to multiply two numbers

3.11.1 - Representing a number

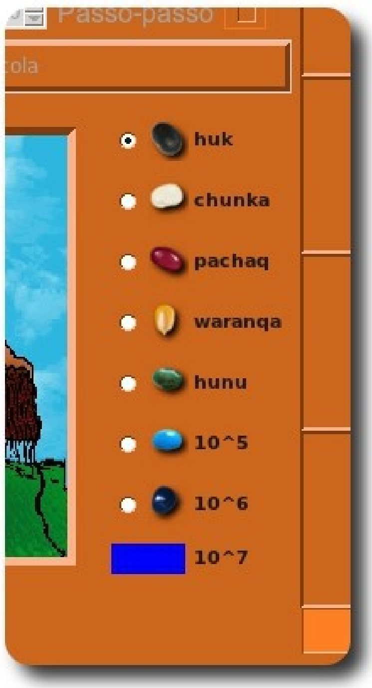


Illustration 54: List of seeds

To insert a number in the table You need to break it down into units, tens, hundreds, etc.. and represent it in the first column from the right as a sum of powers of ten.

To change the power of ten You must select the corresponding seed from the list of seeds next to the table (Illustration 54). Some names are written in Quechua and their meanings are shown in Table 41.

The power of 10^7 (blue, without seed) is an upper limit (not due to Florio's algorithm, but to reasons of programming).

Compared to the writings by *Cinzia Florio*, that are mainly about the algorithm of multiplication, which is limited and relevant to the particular multiplication portrayed by *Poma de Ayala*[31], in **Tk-yupana** you can enter <u>all the numbers from 1 to 10000</u> (limit of five rows) as the program allows you to enter the digits from 1 to 9 in each cell of the first column. See, in this regard, the warnings about the limitations in the representation of the multiplicand (always in paragraph 2.6.3 -).

31 See section 2.6.3 - for details

Quechua	**Translation**	**Representation by Powers of tens**	**Representation as number**
Huk	Units	10^0	1
Chunka	Tens	10^1	10
Pachaq	Hundreds	10^2	100
Waranqa	Thousands	10^3	1,000
Hunu[32]	Tens of Thousands	10^4	10,000
		10^5	100,000
		10^6	1,000,000
		10^7	10,000,000

Table 41: Correspondence between numbers and names in Quechua

Once you have selected the power of ten associated to the cipher to be entered, just click with the left mouse button in any *gap* of any *box* of the first column: a seed of the same power selected will appear in the *gap* and its value will be added to the total of what was introduced before and that will also be displayed at the bottom of the column itself.

Clicking with the left mouse button on a gap of one of the boxes in the first column, you can insert a seed in the Yupana. The value of the seed will depend on the selected colour to the left of the table.

NOTE:

- The parts of the number that is inserted do not depend on the position (in fact the numeral system is additive).
- The order of filling should be done from top to bottom so as reported by *José de Acosta* (see note 15)
- As long as you do not enter at least one seed, you can not change the value of the multiplier (the fields at the top left of the window are disabled).

For example, let us enter the number 1,291 = $1 \times 10^3 + 2 \times 10^2 + 9 \times 10^1 + 1 \times 10^0$. We select the black seed (× 1, unit) and click any gap in the extreme right column (last column); let's begin from the top of the column, so from the first row from the top; then select a red seed (× 100,

32 In many modern dictionaries of the Quechua language, the word "*Hunu*" is translated as "*one million*", following the translation made by González Holguín[§HOL] in 1608. In Tk-yupana I preferred to follow the interpretation of the linguist *Pilares Casas*, which seems more probable and the value that he attributes is "*ten thousand*"; even more after the reading of the vocabulary of Domingo de Santo Tomas[§SAT] which translates to "*Huno or chunga Guaranga*" with "*ten thousand in number*".[N/A]

hundreds, we have deliberately missed the tens that will be added later to show the additivity of the numeral system) and click on two gaps in the second row of the last column; then select the white seed (× 10, tens) and click on nine boxes of the third row from the top of the same column; then select the yellow seed (× 1000, thousands) and click on a square of the fourth row from the top of the last column, and you are done (Illustration 55).

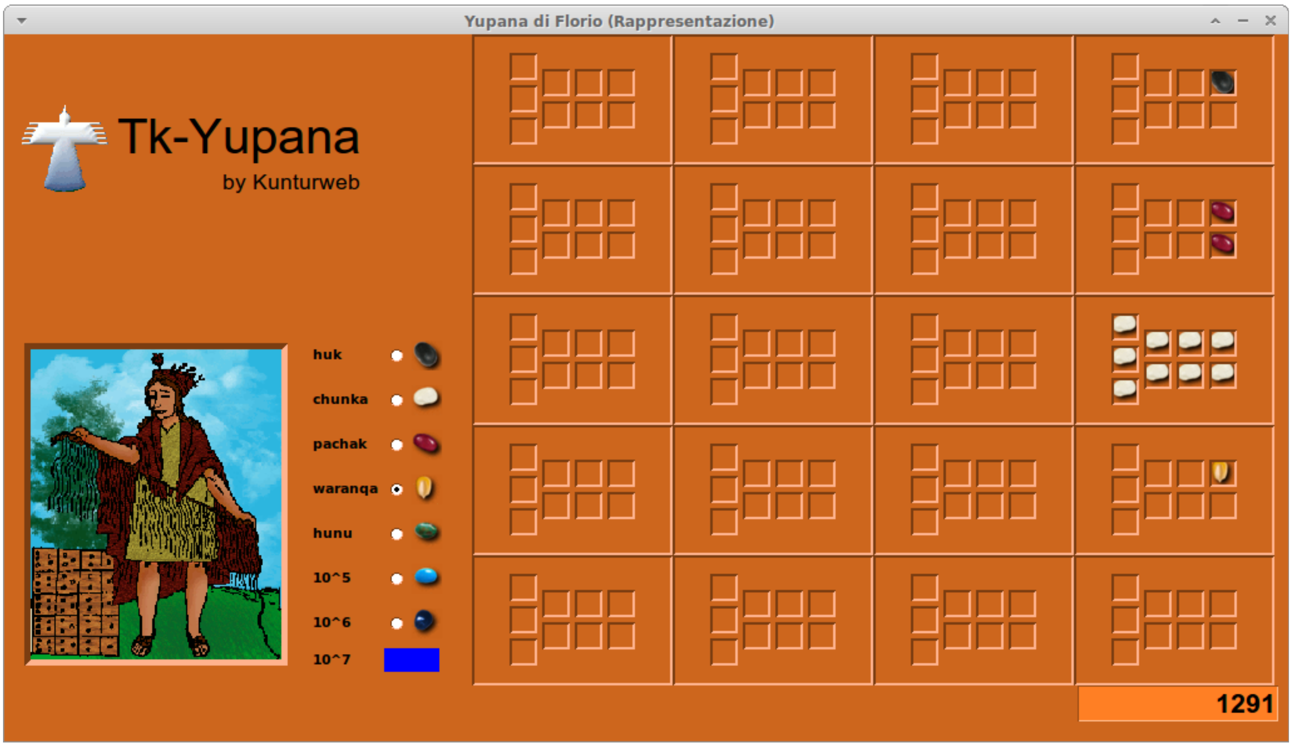
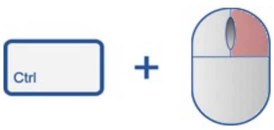


Illustration 55: Entering the number 1,291

To clear a seed (and decrease the number of the corresponding amount) you should click with the right button of the mouse on the corresponding seed/gap by holding down the *control key* (CTRL).



While holding down the CTRL key and clicking with the right mouse button on a gap, a seed is removed from the Yupana

3.11.2 - Addition

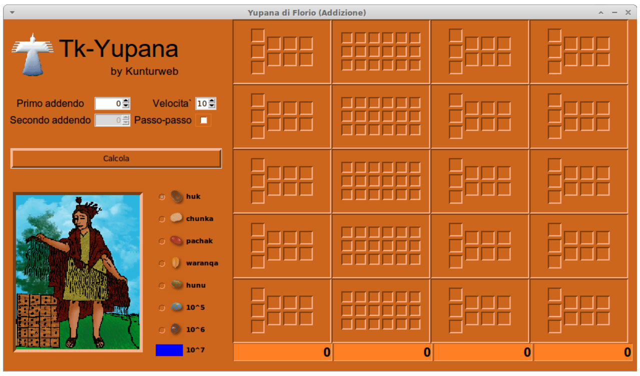


Illustration 56: Yupana by Florio: addition

This window allows you to add two numbers. The numbers are inserted through the panel on the upper left of the window, which involves the insertion of the first and second addend, as well as the possibility to select the speed with which the yupana is filled (see Illustration 56).

Entering the 1st term

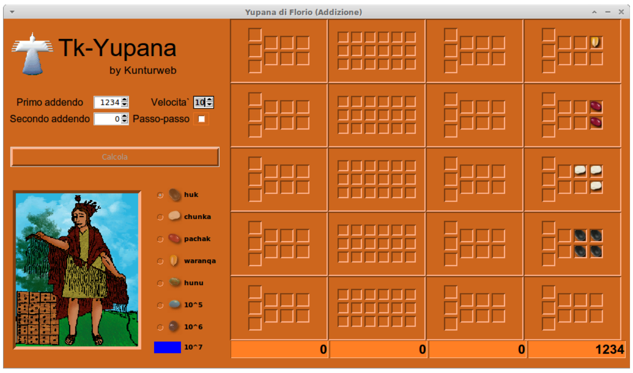


Illustration 57: Entering number 1,234

You can enter the first term using the field's arrows, or by entering the number directly using the keypad. Once you leave the field, the second term is enabled (otherwise disabled) and the first number is represented in the yupana with coloured seeds according to the rules described in the previous paragraph.

Entering the 2nd term

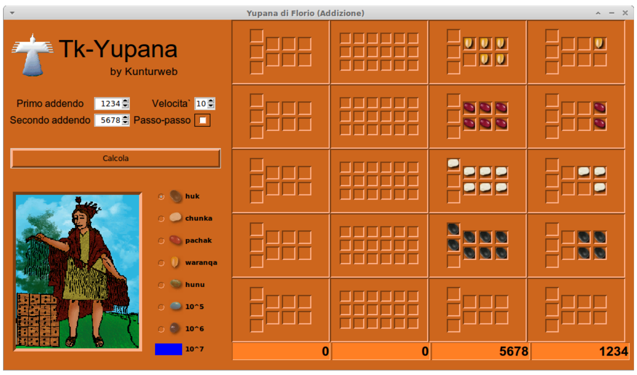


Illustration 58: Entering 2nd term: 5,678

Once you have entered the first term (and only then) you can enter the second term, using the arrows or by directly entering the field by typing on the keyboard. When you leave the field, the button "Calculate" is enabled.

Sum of its parts

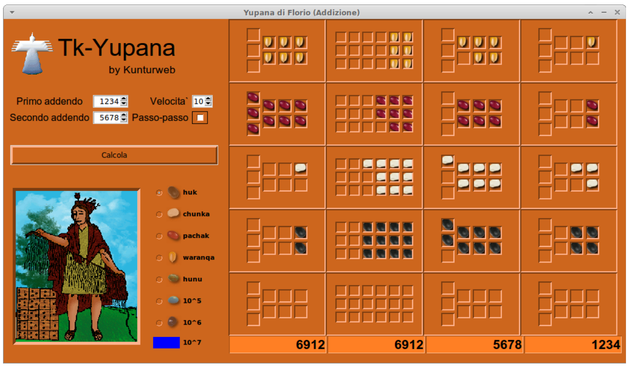


Illustration 59: Addition of the terms

By clicking on the "Calculate" button. the addition of the two numbers previously entered begins. If the speed is set to 0, the calculation is done instantly and the result displayed on the yupana (Illustration 59). If the speed is set with a value greater than zero, the filling of the table is done step by step and with increasing speed according to the number entered in field speed.

3.11.3 - Multiplication

The implementation of the multiplication by Cinzia Florio performed in Tk-yupana does not allow you to enter all the numbers nor the multiplicand nor the multiplier. This is not due to a program error, or by the limitations of the numeral system, but is simply due to a programmer choice, that wanted to represent the table drawn by Poma de Ayala. With regards to the multiplicand see note 16. With regard to the multiplier, the limits depend on both the number of columns (only 2) and the number of values (0-9) attributable to the same.

To understand the functioning of the Florio's multiplication algorithm and use tk-yupana properly, it must be quite clear that this is a multiplier table, then an aid in the calculation of a multiplication of two terms: **M** × **m**, where **M** is the multiplicand and **m** the multiplier (see paragraph 2.6.3 -)

When activated the multiplication window, the yupana by *Poma de Ayala* is shown in a vertical position, with five rows and four columns: the two central columns correspond to a multiplier divided into 3 + 5[33] (see Illustration 60). The size of the table is different depending on the resolution of the screen and is able to operate even on a net book with resolution 800x480[34]

33 It is recalled that according to the interpretation of the author, the yupana represented by Poma de Ayala is a drawing of the calculation carried out (in particular 32 × 5), and then the table would be used as a multiplying tool in which the column one (the right one) was used to represent the multiplicand, while the columns two and three (the central ones) were used to represent the multiplier decomposed into two addends. The fourth and last column (the left one) was used to derive the result. [N/A]

34 Tk-yupana has been tested on an Asus Eee PC 701 4G with Linux Xandros operating system. [N/A]

In this window you can perform the following actions:

- access the drop-down menu in the upper right corner (icon of condor)
- select a different value for the seed to be inserted in the table
- put a seed in the table

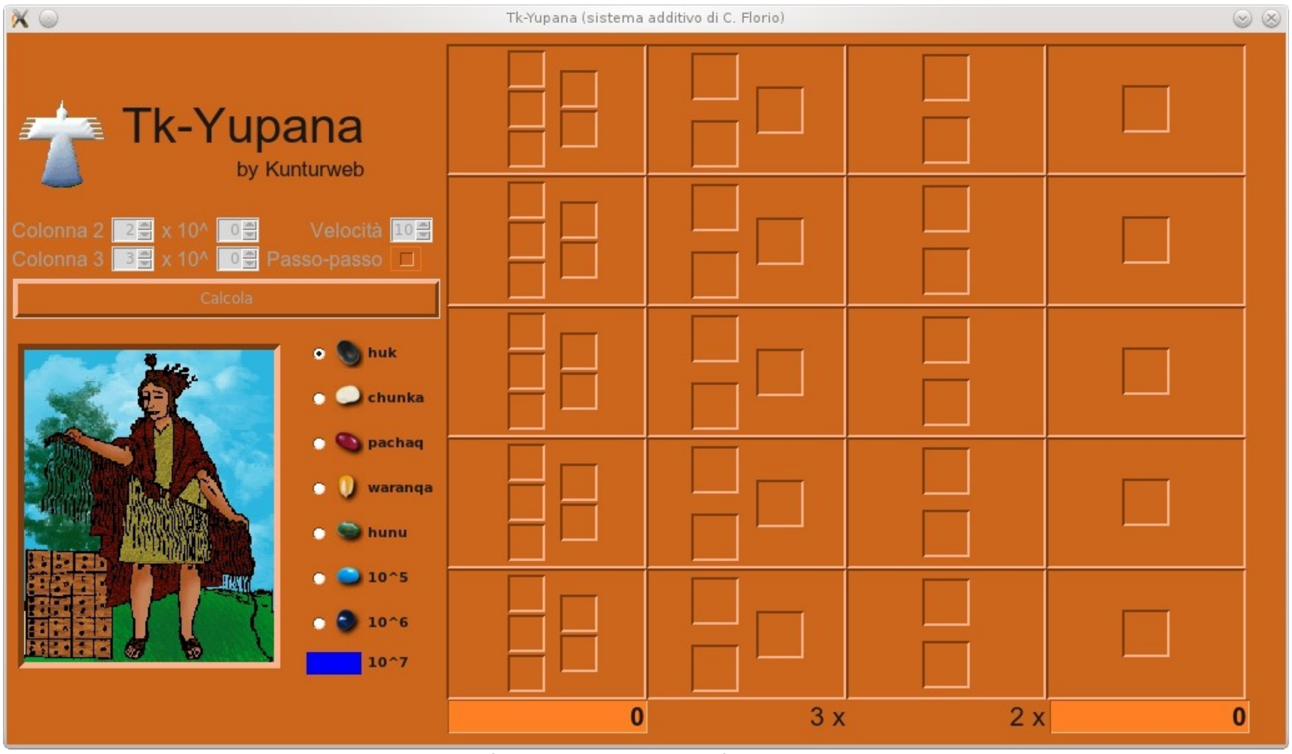


Illustration 60: Yupana by Florio with the default decomposition of the multiplier (3+2)x ...

Note that at the base of the column two (second column from the right) is shows the caption "2 ×" indicating the multiplication factor of the first term, while at the bottom of column three (the third column from the right) is shown the caption "3 × ", corresponding to the multiplication factor of the second addend. The default configuration shown is 3 +2 and can be changed later.

Entering the multiplicand

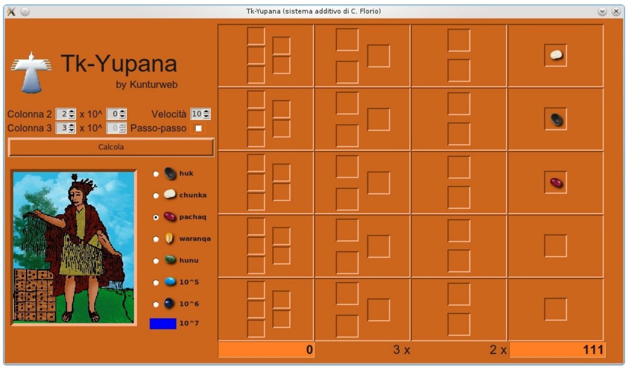
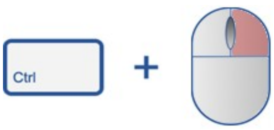


Illustration 61: Entering number 111

To insert a multiplying proceed as in paragraph 3.11.1 - . For example, to enter the number 111 = 1 × 10² + 1 × 10¹ + 1 × 10⁰ we select the red seed (× 100) and click the 3rd box (from the top) in the right column; then we select the white colour (× 10) and click on the first box from the top of the right column and finally we select the black colour (× 1) and click on the second box from the top of one column (Illustration 61). Note that we enter the tens in the 1st row just to show that the numeral system is additive and does not depend on the position of the seeds in the yupana. Once you have entered at least one seed in a box of column 1, then the section of the mask that concerned the multiplier is activated(to set the multiplier, see section 2.6.3 -).

To clear a seed from the table (and decrease the number of the corresponding amount) you have to click with the ***right button of the mouse*** on the corresponding seed/box by holding down the ***Control key*** (CTRL).



While holding down the CTRL key and clicking with the right mouse button on a cell, the corresponding seed is removed from the Yupana

Entering the multiplier

Once inserted the multiplicand **M**, it is possible to change the breakdown of the multiplier **m** as sum of two terms, different from 3 and 2, acting on the numbers in the upper left corner relative to the columns 2 and 3 (Illustration 62).

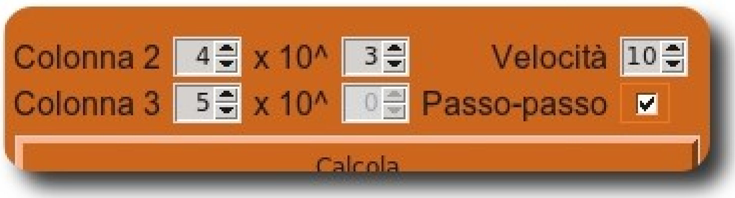


Illustration 62: Fields for the decomposition of the multiplier

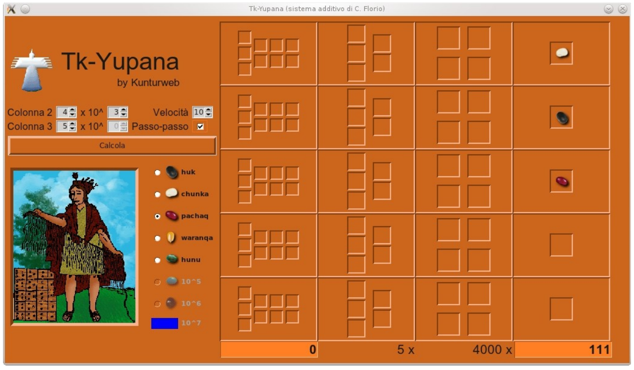


Illustration 63: Example of decomposition of the multiplier 4005 in 4000 + 5

Changing those numbers, will change accordingly the number of boxes displayed in columns 2 and 3 respectively and the same multiplicative factors (factors that appear under the columns themselves). In this regard, see the Illustration 63, in which the multiplier 4,005 is divided in two: 4,000 + 5

The format is: Column **X** **A** 10^ **S**, where:

- **X** = number of the column
- **A** = number of seeds that can be inserted in column X
- 10^ = symbol for exponentiation of the number 10
- **S** = power of number 10 (weight to be assigned to column X)

Note that by increasing the variable **S** from *column 2*, progressively decrease the possibility of selecting **high** values of the seeds (see Illustration 63 in which some seeds are disabled). This has been introduced to avoid errors of the program due to a "breakthrough" of the upper limit of 10^7. The rule is that the greatest exponent **S** for *column 2* depends on the maximum value of the power **J** introduced in the multiplicand and vice versa. The sum of **S** and **J** must not exceed the upper limit of **7**.

By now you can not change the exponent of column 3. In the future, I will introduce also this feature.

The options "Speed" and "Step-by-step"

Near the group of fields relating to the multiplier, are present:

- A chek-box with the label: Step-by-Step
- A check-box with the label: "Speed"

By checking the box *Step-by-step* calculation will be performed a step at a time: first the multiplication of column 1 to column 2, then the multiplication of column 1 to column 3, then the summations and simultaneously displaying the result.

If the box is not selected, the calculation is performed without pause.

The speed selector is used to change the speed at which the boxes are filled: 10 corresponds to "instantaneous", while values below 10 correspond to values of the speed gradually decreasing.

Calculation of product

Once filled the multiplicand, the multiplier and decided how to set the calculation parameters you can start the calculation by pressing the "Calculate" button.

If the option *step-by-step* is not selected, the calculation is carried out without interruption and the result (product) shown right below column 4 (Illustration 66), otherwise the button will change the word "Calculate" in “*First multiplication (column 2)*” and waits after a button is pressed.

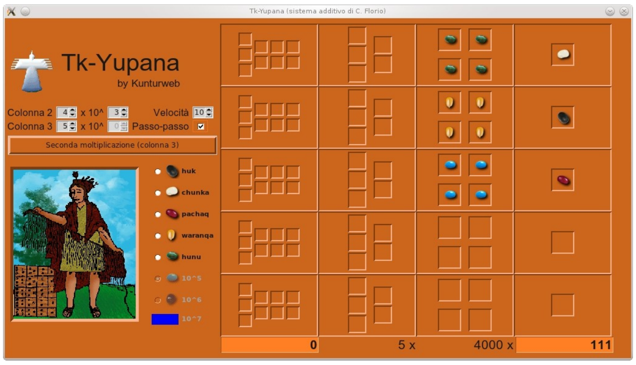


Illustration 64: 111 x 4005: First multiplication (multiply the first column by the second)

Pressing the button again the second column is filled according to the values selected and the words of the button changes to “*Second multiplication (column 3)*”. The program waits for a button to be pressed again (Illustration 64)

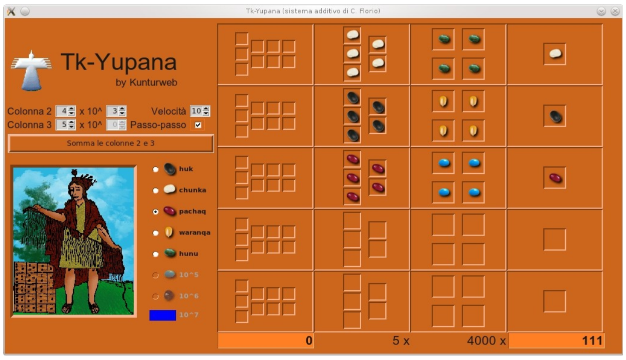


Illustration 65: 111 x 4005: Second multiplication (multiply the first column by the third)

Pressing the button again the third column is filled according to the values selected and the words of the button changes to “*Addition of columns 2 and 3*”. The program waits for a button to be pressed again (Illustration 65)

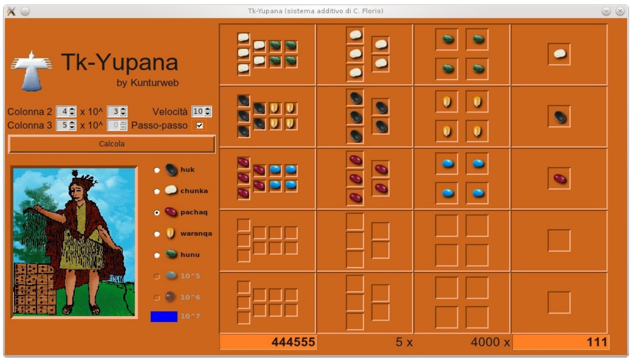


Illustration 66: 111 x 4,005: sum of columns 2 and 3, and display the result

A last press of the button allows you to fill in column 4, the sum of columns 2 and 3, and to display the product (Illustration 66).

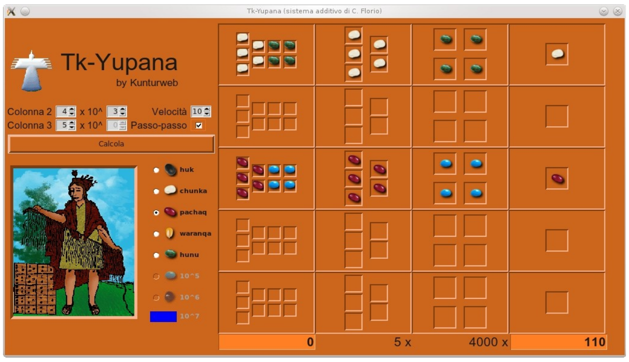
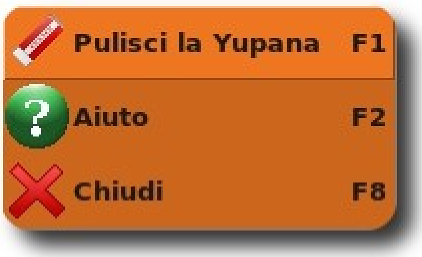


Illustration 67: Deleting an entire row by removing the seed in column 1

You can also remove an entire row by pressing the CTRL key and clicking with the right button of the mouse on the seed of *column 1*. The total is reset and the calculation must be repeated (Illustration 67).

3.11.4 - Menu functions



Clicking on the condor (upper left picture) a drop-down menu is displayed, from which you can select the following functions:

1. **Clean the Yupana**: remove all the seeds from the table and clears the counter of the total (**F1**).
2. **Help**: Displays a help file (**F12**)
3. **Close**: Closes the window (**F8**).

4 - License

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5 - Bibliography

These are the articles/sites from which all information about the functions of the program implemented have been retrieved.

CODE	REFERENCES	URL
[§ACO]	ACOSTA José De, **Historia Natural y Moral de las Indias**, 1590, Libro VI cap XVIII – De los memoriales y cuentas que usaron los Indios del Perú	http://to.ly/jdJO
[§CHI]	CHIRINOS RIVERA Andres, 2008, Quipus del Tahuantinsuyo	
[§DEP]	DE PASQUALE Nicolino, 2001, **Il volo del condor**, Pescara Informa, ottobre 2001	http://to.ly/jdJW
[§FLO]	FLORIO Cinzia, 2010, **Ruote di Acosta: elementi decorativi, strumenti mnemonici o mezzi di scrittura?**, in Eliana Guagliano (curatrice), Letterature Americane ed atre Arti, Atti del XXXI Convegno Internazionale di Americanistica (Salerno), Oèdipus Editore, Salerno cinziaflorio@tiscali.it	
[§FLR]	FLORIO Cinzia, 2009, **Incontri e disincontri nella individuazione di una relazione matematica nella Yupana di Guaman Poma de Ayala** e **Incontri e disincontri tra Europa e America**, Salerno, 14-15 maggio e 10-12 Dicembre 2008 - Oédipus Editore, Salerno cinziaflorio@tiscali.it	http://to.ly/jdK4
[§GIL]	GILDORF Thomas E., **Ethnomathematics of the Inkas**, Department of Mathematics, University of North Dakota	http://to.ly/jdKa
[§HOL]	GONZÁLEZ HOLGUÍN Diego, 1608, **Vocabulario de la Lengua General de todo Peru**	http://to.ly/jdKN
[§LAU]	LAURENCICH Laura, 2007, **La yupana: abaco e ordinatore dello spazio inca o esempio di sincretismo culturale?**, Atti del XXVIII Convegno Internazionale di Americanistica . Quaderni di Thule. Argo Editrice.	
[§LES]	LEONARD Molly, SHAKIBAN Cheri, **The Incan Abacus: A Curious Counting Device**, Journal of Mathematics and Culture, Nov. 2010, 5 (2), University of St. Thomas, MN, USA	http://to.ly/jdOd
[§MOV]	MORA Lyda, VALERO Nydia, **La Yupana como Herramienta**	http://to.ly/jdL8

Pedagógica en la Primaria, Universidad Pedagógica Nacional

[§PAR] PAREJA Diego, 1986, **Instrumentos Prehispánicos de cálculo: el quipu y la yupana**, Revista Integracion, Dep. De Matematicas UIS, Vol. 4, No. 1, enero-junio 1986 http://to.ly/jdL5

[§PIL] PILARES CASAS Guido A., 2005, **Los sistemas numéricos del Quechua y el Aimara** - Dirección Nacional de Educación Bilingüe - Ministerio de Educación - Lima, Junio 2005 http://to.ly/jdLi

[§POM] POMA DE AYALA Felipe Guaman, 1615, **El Primer Nueva Corónica y Buen Gobierno**, Royal Library di Copenhagen http://to.ly/jdLr

[§RDP] RADICATI DI PRIMEGLIO, Carlos, **El sistema contable de los Incas**

[§SAT] SANTO TOMÁS Domingo De, 1560, **Lexicon o Vocabulario de la lengua general del Peru** http://to.ly/jdLx

[§VEL] VELASCO Juan De, 1841-44, **Historia del Reino de Quito en la América Meridional**, Quito. http://to.ly/jdLz