Module 1: Relations

Module 1, Section 1

Review of Mathematics 11

Introduction

In this section you will review a number of the concepts and processes encountered at the end of Principles of Math 11. Most of these concepts deal with functions and coordinate geometry. A solid foundation of polynomials and rational functions is necessary to master transformations, which you will encounter in Sections 2 and 3 of this module and also in Modules 2 and 3.

Section 1 — Outline

Lesson 1 Graphing Calculator Review

Lesson 2 Functions and Interval Notation

Lesson 3 Inverse Functions

Lesson 4 Polynomial Functions and Their Graphs

Review

Notes

Lesson 1

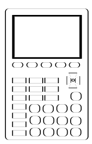
GRAPHING CALCULATOR REVIEW

Outcomes

Upon completing this lesson, you will be able to carry out these operations on a graphing calculator:

- Enter and edit polynomial equations
- Graph the equations and adjust the viewing window
- Solve the equations

Overview



If you used a graphing calculator for Principles of Math 11, then consider this lesson optional. You should do just a few of the exercises to make sure you remember how the graphing and solving functions work.

The directions here are specific to the **TI-83** or **TI-83Plus** models from Texas Instruments. You may be using a different TI model, or one made by Hewlett-Packard, Sharp, Casio, or another manufacturer. Any graphing calculator will get you through the provincial exam (except the HP-48, which is not allowed). But if you use another brand of calculator, you will need to refer to its user manual to find out how to do what these instructions tell you.

Solving Polynomial Equations Using the Graphing Calculator

You have developed considerable skill at finding the rational (which includes integral) roots of given polynomial equations. But the roots of a polynomial equation are *not* necessarily rational. They might well be irrational numbers (non-periodic non-terminating decimals). This possibility can make the algebraic solution of the equation very tedious. The graphing calculator can simplify the process.

A word of caution as we begin: enter values and follow the steps *slowly* and *carefully*. The calculator has no tolerance for entry errors, no matter how small.

Also, negative numbers must be signed using the negation [(-)]

button just to the left of the ENTER button, not the subtract – operation button; otherwise you'll get a "syntax error" message. (On other brands of calculators, the (-) key may be a ± or ±/– key.)

In this lesson, we use two sizes of hyphens to distinguish between the negation key and the subtract or minus key, just as the TI calculators do. For negation we use a short hyphen [-] and for subtraction we use a longer one [–]. (After this lesson, we'll use just the longer dash in all equations; you will know the rule by then for choosing the correct key.)

Q: What do I do when the calculator says "Syntax error"?

A: Choose option 2 "Goto" from your screen. The blinking cursor will go directly to the error you made so that you can fix it.

As you go through the following examples, perform the steps on your calculator rather than just reading the text. You might want to go over the example a number of times until you feel comfortable with the functions. As with any skill, practice makes perfect.

If you make typing errors at any time, you can always scroll to your error using the four cursor (arrowhead) keys and then:

1) type over,
2) use the DEL key to delete, or
3) use the Insert function (by pressing 2nd and then DEL), and then type more characters in the same space.

Note: Upon first turning on your graphing calculator, you should see a blank display—if you don't, press CLEAR. In this mode, your graphing calculator functions as any scientific calculator does, thus enabling you to solve such equations as 2 + 2 or sin 25.

Example 1

Solve the equation $3x^3 – 13x^2 – 10x = –50$

Solution

First we rearrange the equation (on paper) so that we have zero on one side of the equation:

$3x^3 – 13x^2 – 10x + 50 = 0$

Turn on your calculator and ensure that the memory is cleared by pressing [2nd], then [+], then scroll down to option "**3:Clear Entries**" using the down arrowhead key and select that option by pressing [ENTER]. Now you will see a confirmation screen, so you press [ENTER] while the cursor is next to the words **Clear entries**. You will see the word **Done**. Press [CLEAR] to get a blank screen.

Shortcuts

You can select the menu options simply by pressing their number, if you prefer not to scroll through the other options.

The [2nd] key is used in the same way as it is used on a regular calculator in that it performs the function shown ***above*** the keys.

The [X,T,θ,n] Key

The [X,T,θ,n] key is the one you use to insert a *variable* into the equation you type. To type "sin θ", you hit the [sin] key, then the [X,T,θ,n] key. Then close the parenthesis. This key also inserts the "x" into polynomial equations.

{Braces}

To type a brace, use the [2nd] key and the corresponding parenthesis. Most users don't bother with braces since "nested" parentheses work just as well. Equations 1 and 2 mean the same thing on a graphing calculator:

1. $\left[3\left(1-\frac{x^2}{3}\right)\right]$
2. $\left(3\left(1-\frac{x^2}{3}\right)\right)$

Equation 1 is easier for humans to read, but equation 2 is easier to enter on the calculator—fewer keys to press.

Step 1: To solve for the roots of this equation, we will solve for the zeros of the corresponding polynomial function $Y_1 = 3x^3 - 13x^2 - 10x + 50$. We begin by typing in the function as follows:

Press [Y=]. This bring up a flashing cursor to the right of **Y_1=** in your display window.

Step 2: At the flashing cursor we begin typing in our function carefully:

3 [X,T,θ,n] [^] 3 [–] 13 [X,T,θ,n] [^] 2 [–] 10 [X,T,θ,n] [+] 50

Although spaces have been used between the above numbers for clarity, don't type spaces on the graphing calculator. Here's how your display should look:

Y_1 =3X^3–13X^2–10

X+50

Y_2=

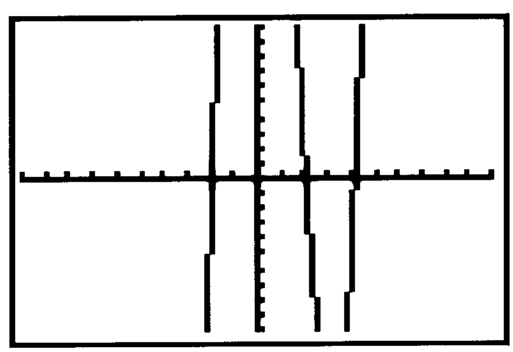
Y_3=

Y_4=

Our font is a little different from that of the graphing calculator but hopefully you get the picture. Notice the use of the [^] key. It indicates to the graphing calculator that the operation is a power/exponent.

Step 3: Now we graph the polynomial function defined in Step 2 by pressing [GRAPH].

You should see a display similar to the one shown here:



Graph 1: $3x^3 - 13x^2 - 10x = -50$

Step 4: We identify the approximate value(s) of the zero(s) by inspection of the graph. Sometimes we need to adjust the viewing window a little so that we can see where/if the x- intercepts or zeros occur, but in this case all ***three*** are visible. A cubic polynomial can have, at most, three Real zeros, so we need not worry that some are not visible.

a) The least zero is in the interval {–3,–1} (i.e., between –3 and –1). A guess might be –2.0.

b) The middle zero is in the interval {2,3} (between 2 and 3). A guess might be 2.5.

c) The greatest zero is in the interval {4,5} (between 4 and 5). A guess might be 4.5.

Step 5: Now we will solve for the actual zeros, one by one. The calculator needs a few details. The TI-83 will expect to receive them in this very specific order:

Function, Variable, Guess, {Lower bound, Upper bound}

At this point, different calculators use different key sequences to solve equations. The remaining steps 6–10 are for the TI-83. If you are using a different calculator, look in the index of its user manual under "Solving equations".

Be sure to:

- use the X,T,θ,n key for X.
- use the minus key within the equation itself.
- use the negation (-) key for the guess and the bounds.

Want to know the math behind the button? Calculators and computers solve functions with some form of Isaac Newton's method. On a graph, Newton's method finds solutions or zeros (x-axis crossing points) like this: using your initial "guess" and the slope of the graph at the point of your guess, it calculates where that slope (a straight line, of course) crosses the x-axis. Then it takes the crossing point as a second guess—it goes to the graph point directly 'above' or 'below' where the first slope crossed the x-axis. It calculates the new slope of the equation at that point, goes to where that new slope crosses the x-axis, and repeats the process.

If a guess is close to a zero or solution—to a point where the graph crosses the x-axis—you can see that the graph's slope from that point will be almost parallel to the graph itself; the slope's crossing point on the x-axis will be close to the crossing point of the graph. Only a few repeats will be needed, before it homes in on the actual crossing point. In reality, graphing calculators perform so many repeats of Newton's method in the time it takes to press one button that your first guess need not be all that close to a true solution. Just make sure that your first guess is clearly closer to one solution (or crossing point) than it is to any other solutions. Better yet, set the Lower and Upper Bound so that they contain only one crossing point. If your guess is sort of midway between two crossing points, you can't control which one Newton's method will find!

Do type in the commas, and ***do not*** use spaces.

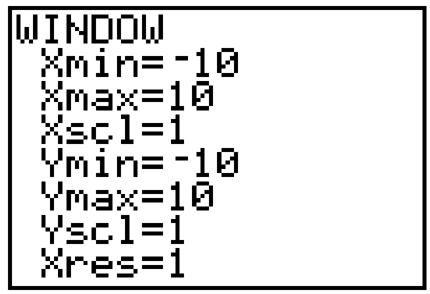
	TI-83
Step 6	Press MATH, scroll down to option **0:Solver…**. Select it with ENTER.
	You should now see: **EQUATION SOLVER** **Eqn:0=** If you do not see EQUATION SOLVER, scroll up. If you see an equation already written in, use the cursor-up (up-arrow) key to place the cursor on the equation. Then use the CLEAR key to remove it.
Step 7	Type the following equation exactly. **3X^3 – 13X^2 – 10X + 50** Then press ENTER Next to the **X=** on the next line, type your first guess: –2.0 (remember to use the negation key, not minus). On the **bound=** line, type your lower and upper bound (for the first guess) between *braces*, like this: {–3, –1} Finally, place your cursor on the **X=** line and press ALPHA SOLVE. (ALPHA is the green key near 2nd, and SOLVE is on the ENTER key)
Step 8	Your answer appears: **X=–1.920589771** which we round to –1.921. If your answer is different, look carefully at your equation for mistakes. Use the cursor keys to locate and correct them.
Step 9	Go back to Step 7. Change your **bound=** line to {2,3} and your **X=** guess to 2.5. Then press ALPHA SOLVE. Ensure you get the answer **2.07815274** which rounds to 2.078. If you got a **BAD GUESS** message, it means your guess is outside your bounds.
Step 10	Go back to Step 7 and use your third guess of 4.5 and bounds of 4 and 5. Then press ALPHA SOLVE. Ensure you get the answer **4.175770562** which rounds to 4.176.

So the three solutions for the equation $3x^3 - 13x^2 - 10x = -50$ from least to greatest are: 1.921, 2.078, and 4.176.

Adjusting the Viewing Window

Finally, we adjust the “viewing window” on the calculator so that more of the graph is visible. Recall that the graph you saw on your calculator (see page 9) went off the top and bottom of the screen—you could not see it all.

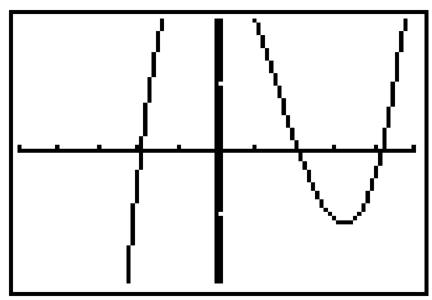
To fix that, you adjust the Viewing Window. Press the WINDOW button at the top of the keyboard. You see the list of values at the right. From the top, these values tell you that the X-axis in your window ranges from –10 to +10, with a tic-mark for every number. The same is true for the Y-axis.



Co-ordinate values for the Standard Viewing Window on the TI-83.

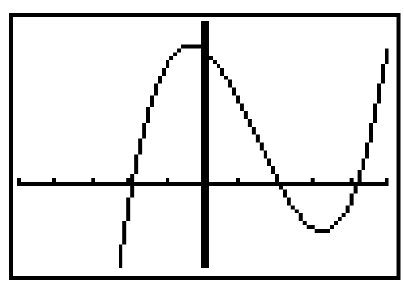
For your graph to “look right,” adjust the scale so that there are fewer values displayed along the X-axis (from –5 to +5, say) and more displayed along the Y-axis.

As a first guess, use your cursor to set Xmin = –5 and Xmax = 5. Likewise, set Ymin = –30 and Ymax = 30. Then press GRAPH again. Now the graph looks like the one at right.



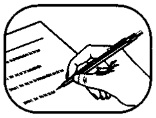
The graph with the WINDOW parameters at Ymin = –30 and Ymax = 30

This is better, but we're still missing the upper loop of the graph. Our final adjustment is to change Ymax to +60. That yields the more appropriate display to the lower right. We could change the Xmin value from –5 to –3 for a more balanced look, but what we have is good enough.



X [–5, 5] Y [–30, 60]

When you answer graphing-calculator questions on the Provincial Exam, you will hand-sketch the graph in your calculator's viewing window, then write in the Xmin and Xmax, and Ymin and Ymax values, which you set for your window. The example at right shows the correct way to write your window settings.



Guided Practice

Solve each of the equations below using your graphing calculator and showing the following detail:

Sketch the display as you see it.

For *each* of the possible solutions:

a) state your "guess".

b) indicate the upper and lower bounds for x using interval notation. For example {5,7} would indicate that x falls between 5 and 7.

c) state the actual solutions, correct to three decimal places.

Your graph sketch goes here.

1. $x^3 - x^2 - 12x = -3$

X [,] Y[,]

2. $-x^3 + 2x^2 - x + 1 = 0$

X [,] Y[,]

3. $x^3 + 6x^2 + 3x - 5 = 0$

X [,] Y[,]

4. $x^3 - 3x^2 = 9x - 9$

X [,] Y[,]

5. $0.25x^3 - 0.5x^2 - 6x - 2 = 0$

X [,] Y[,]

Check your answers in the Module 1 Answer Key.

Lesson 2

Functions and Interval Notation

Outcomes

Upon completing this lesson, you will be able to:

- identify the domain and range of various functions using set notation and interval notation
- find the x- and y-intercepts of any function
- perform operations on functions

Overview

The concepts of domain and range are necessary to describe functions. Interval notation is a most convenient way to describe domain and range. We will also review combinations of functions and composition of functions.

Definitions

A ***function*** is a relation where each x-value has only one y-value

For any function, $y = f(x)$, the ***domain*** is the set of possible x-values, and the ***range*** is the set of possible y-values.

We can evaluate a function at a particular point by substituting either numbers or algebraic constants into the $f(x)$ expression and simplifying the result.

Example 1

If $f(x) = x^3 - 2x$, find $f(0)$, $f(-2)$, and $f(a)$

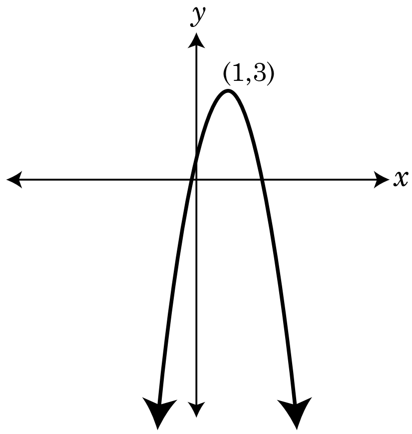
$f(0) = 0^3 - 2(0) = 0$

$f(-2) = (-2)^3 - 2(-2) = -8 + 4 = -4$

$f(a) = a^3 - 2a$

Example 2

State the domain and range of the function $p(x) = -2(x - 1)^2 + 3$



There is no restriction on the values that x can take.

Domain = $\Re$

The parabola has a maximum value at (1,3)

Range = $\{y \mid y \leq 3, y \in \Re\}$

Turn to Appendix 1 for some background information on how to read set notation.

Informal Rules for Finding a Function's Domain and Range

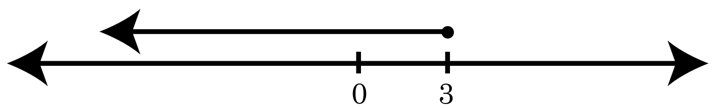
There is no "sure-fire formula" to determine the extent of a function along the X-axis (its domain) or the Y-axis (its range). The domain and range are often obvious if you just look at the graph; here are some guidelines for finding the domain and range of a function by looking at the equation, not the graph.

1. Linear functions, $f(x) = mx + b$, always have infinite domain and range, in both directions (i.e., out to $-\infty$ and also to $+\infty$). The only exception is when $m = 0$; then $y = b$ and the domain remains infinite but the range is simply b.
2. Parabolic functions, $f(x) = a(x-h)2 + b$, have infinite domain but a limited range—the range goes to infinity in one direction on the Y-axis, but not in the other direction. (This rule applies to any even-powered function: $x2$, $x4$, $x6$ and so on.) If the exact range boundary is not obvious from the equation, your graphing calculator can identify it for you; this is taught in Lesson 4.
3. In square-root functions (with x somewhere under the root sign), both the domain and range are infinite in one direction but not in the other.
4. In any function where x appears in the denominator of a fraction, watch out for specific values of x where the denominator becomes zero. At those points the equation has no meaning (and the graph shoots off to infinity along an asymptote). That x-value must be excluded from the domain, and the corresponding $f(x)$, or y value (if there is one), must also be excluded from the range.

You may find these guidelines helpful as you begin Math 12 and work through the course. Most students find that domains and ranges become obvious enough that they don't need these guidelines for very long.

Example 3—Interval Notation

One way of reading the set $\{y \mid y \leq 3, y \in \Re\}$ is “All the real numbers between $-\infty$ and 3.” On a number line, it would look like this:



We can write this interval from $-\infty$ up to and including 3 as $(-\infty, 3]$. The “(” means that the set *doesn’t include* $-\infty$ (because infinity is unreachable) and the “]” means that the point 3 is *included* in the set.

So we see that another way of writing Range = $\{y \mid y \leq 3, y \in \Re\}$ is Range = $(-\infty, 3]$.

In this way we can rewrite $\Re$ as $(-\infty,\infty)$.

$\{x \mid -4 < x \leq 3\}$ as $(-4,3]$

$\{x \mid 10 \leq x \leq 12\}$ as $[10,12]$

$\{x \mid x \geq -2\}$ as $[-2,\infty]$

An interval where the end points are both included is called a **closed interval** and shown as [ ].

An interval where both end points are not included is called an **open interval** and shown as ( ).

An interval where only one end point is included is called a **half-open interval** and shown as either [ ) or ( ].

We can form the union of two intervals in the same way that we form the union of two sets. Remember that $\cup$ is the symbol for union.

$\{x \mid x < -5, x \in \Re\} \cup \{x \mid 0 < x < 4, x \in \Re\}$ can be written as $(-\infty,-5)\cup(0,4)$.

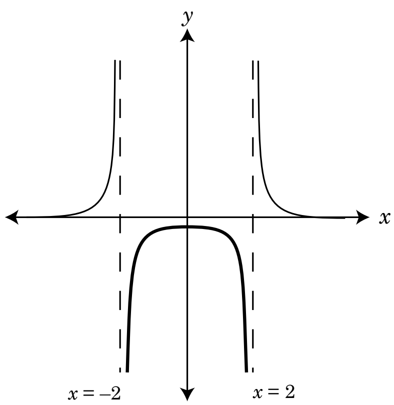
Most rational functions have restrictions because the denominator of a function cannot be zero. The domain and range both have restrictions. Interval notation is a convenient way to express a restricted range or domain.

Example 4

The rational function $f(x)=\dfrac{1}{(x^2-4)}$ has asymptotes at $x = \pm 2$, and $y = 0$. It has a y-intercept at $y = -1/4$, but no x-intercepts.

Domain = $\{x \mid x \neq -2, 2\}$. In interval notation this would be $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$, which is very awkward.

Range = $\{y \mid y \leq -1/4 \text{ or } y > 0\}$. In interval notation, this would be $(-\infty,-1/4)\cup(0,\infty)$.



Note: Remember, an asymptote is a line that a curve approaches to infinity.

Combination of Functions

Two functions can be combined arithmetically by +, –, × or ÷. The normal rules about addition, subtraction, etc. apply.

For example, if $f(x)=\dfrac{1}{2x-1}$ and $g(x)=\sqrt{3x-2}$, then we can create new combined functions by simple arithmetic like this:

$$f(x)+g(x)=(f+g)(x)=\frac{1}{2x-1}+\sqrt{3x-2}$$

Note: There are two ways to show the addition of functions:
$f(x) + g(x)$ or $(f+g)(x)$.

Similarly, $(f-g)(x)=\dfrac{1}{2x-1}-\sqrt{3x-2}$

$$(f\times g)(x)=\frac{1}{2x-1}\times\sqrt{3x-2}=\frac{\sqrt{3x-2}}{2x-1}$$

$$(f\div g)(x)=\frac{1}{2x-1}\div\sqrt{3x-2}=\frac{1}{(2x-1)\sqrt{3x-2}}$$

Rule: When functions are combined arithmetically like that, the domain of the result is the *intersection* of domains from the two functions—it's the set of all points that belong in *both* the original domains. The same is true of the combined range—it's the *intersection* of the two separate ranges.

That intersection-of-domains rule applies for all four arithmetic operations, between any two functions. But the division operation (f÷g) has an additional rule: the combined domain and range cannot include any value that makes the new denominator go to zero.

Note: Whenever this course uses a plain square root sign, it refers only to the positive square root. This definition is common in most modern mathematics.

Examples: If you solve $x = \sqrt{4}$, then the answer is $x = 2$ but not $x = -2$. If it wants the negative root, this course will ask you to solve $x = -\sqrt{4}$. If this course asks you for both roots, it will ask you to solve $x = \pm\sqrt{4}$.

Example 5

Using the above two functions f and g, find the domain and range of $f+g$, $f-g$, $f\times g$, and $f\div g$.

Solution

By inspection, domain of $f = \{x \mid x \neq \frac{1}{2}\}$ and the range of $f = \{y \mid y \neq 0\}$. $x = \frac{1}{2}$ and $y = 0$ are not allowed because f cannot have a zero in the denominator.

Similarly, domain of $g = [\frac{2}{3},\infty)$

Range of $g = [0,\infty)$

The square root of a negative number is not real so g cannot be less than $\frac{2}{3}$. But the value $\frac{2}{3}$ is in the domain of g.

Now for the intersections. Remember that $\cap$ is the symbol for intersection:

Domain of $(f+g)$ = domain of $(f-g)$ = domain of $(f\times g)$ = $\{x \mid x \neq \frac{1}{2}\} \cap [\frac{2}{3},\infty) = [\frac{2}{3},\infty)$.

Range of $(f+g)$ = range of $(f-g)$ = range of $(f\times g)$ = $[y \mid y \neq 0\} \cup [0,\infty)$ = $[0,\infty)$.

The domain of $(f\div g) = (\frac{2}{3},\infty)$ instead of $[\frac{2}{3},\infty)$. That's because *g(x)* is in the denominator and $g(\frac{2}{3}) = 0$. So $\frac{2}{3}$ must be deleted from the combined domain.

The range of $(f\div g)$ is $(0,\infty)$, just as it is for $f+g$, $f-g$, and $(f\times g)$. The value 0 was excluded from the range of f already, so it's not going to appear in the range of the combined function.

Sometimes, when you write the range or domain of a combined function, it may be simpler to leave the intersection symbol $\cap$ in your answer.

Finding Intercepts

For more complex functions, we often need to know the x- and y-intercepts in order to find the domain and range.

Rule: To find the y-intercept, set $x = 0$. To find the x-intercept, set $y = 0$.

Example 6

Find the x- and y-intercepts for the function, $f(x) = x^2(x + 1)(x - 2)$

Solution

y-intercept: Set $x = 0$

$f(0) = 0^2(0 + 1)(0 - 2) = 0$

x-intercepts: Set $y = 0$

Solve: $x^2(x + 1)(x - 2) = 0$

$x^2 = 0$, or $x + 1 = 0$, or $x - 2 = 0$

$\therefore x = 0, -1, \text{ or } 2$

Composition of Functions

A composition of two functions is when they are arranged so that one is a function of the other.

Composition is not the same as "combination" using arithmetic operations between functions!

The composition is written as $(f \circ g)(x) = f(g(x))$ or as $(g \circ f)(x) = g(f(x))$, either with a small hollow circle for the operation, or with one function nested inside the other. In this course, $f(g(x))$ is the usual notation but we'll start by using both forms.

Example 7

If $f(x) = x - 3$ and $g(x) = 2x + 1$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$(f \circ g)(x)$	$= f(g(x))$	
	$= f(2x + 1)$	Substitute formula for $g(x)$
	$= (2x + 1) - 3$	Apply formula for $f(x)$
	$= 2x - 2$	Simplify
$(g \circ f)(x)$	$= g(f(x))$	
	$= g(x - 3)$	Substitute formula for $f(x)$
	$= 2(x - 3) + 1$	Apply formula for $g(x)$
	$= 2x - 5$	Simplify

Notes to remember:

1. As Example 7 suggests, $(g \circ f)(x) \neq (f \circ g)(x)$ except in special cases.
2. For $(f \circ g)(x) = f(g(x))$, the range of g becomes the domain of f.

Example 8

If $f(x) = \frac{3}{x}$ and $h(x) = 2(x + 1)$, write an equation for $(f \circ h)(x)$.

Specify the domain and range.

Solution

$(f \circ h)(x) = f(h(x))$

$$= \frac{3}{2(x+1)}$$

To find the restrictions on the domain, remember that the denominator cannot be zero.

$$2(x+1) \neq 0$$
$$x \neq -1$$

To find the restrictions in the range, write the function as

$y = \frac{3}{2(x+1)}$, rearrange and solve for y.

$$2y(x+1) = 3$$
$$2xy + 2y = 3$$
$$2xy = 3 - 2y$$
$$x = \frac{(3-2y)}{2y}$$

Restriction: $y \neq 0$

Domain of $(f \circ h) = \{x \mid x \neq -1\}$

Range of $(f \circ h) = \{y \mid y \neq 0\}$

Guided Practice

1. Given that $f(x) = 4x^2 - x + 3$ and $g(x) = 1 - 2x$, find:

   a) $f(0)$
   b) $g(0)$
   c) $f(-2)$
   d) $g(^1/_4)$
   e) $f(a)$
   f) $g(b - 1)$
   g) $(f+g)(x)$
   h) $(g - f)(x)$
   i) $(f \times g)(-2)$
   j) $(f \div g)(0)$
   k) $(g \div f)(b - 1)$

2. Determine the x- and y-intercepts for the following functions:

   a) $f(x) = 2x^2 - 8$
   b) $g(x) = \sqrt{2x+5}$
   c) $k(x) = 5 - x$

3. Using the information from your answers to question 2, write the domain and range for each function using:

   i) set notation
   ii) interval notation

4. Given that $p(x) = \sqrt{x-4}$ and $q(x) = 3x + 1$:

   a) determine each of the following:
      i) $p(q(x))$
      ii) $p(q(3))$
      iii) $q(p(a))$
   b) find the domain and range of:
      i) $p(q(x))$
      ii) $q(p(x))$

Check your answers in the Module 1 Answer Key.

Lesson 3
Inverse Functions

Outcomes

Upon completing this lesson, you will be able to:

- verify that two functions are inverses of each other
- identify one-to-one functions
- find inverse functions

Overview

Many of the important new functions you will learn about in *Principles of Mathematics 12* are the inverses of other functions. The concept of an inverse is essential to solving mathematics problems.

Definition

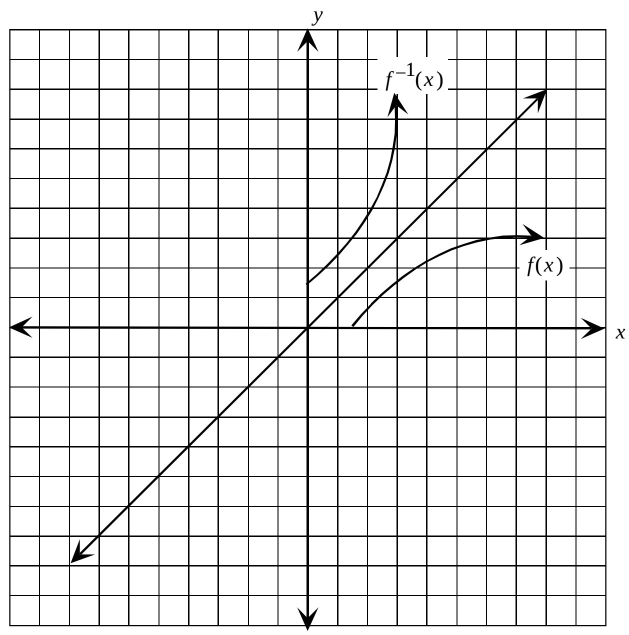
Inverse Function: Let f and g be two functions such that $f(g(x)) = x$ for every x in the domain of g and $g(f(x)) = x$ for every x in the domain of f.

Then the function g is the inverse of the function f, denoted by f^{-1}. Thus $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. The domain of f must be equal to the range of f^{-1} and vice versa.

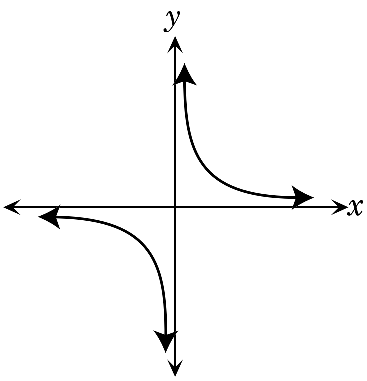
The graphs of f and f^{-1} are related to each other in this way. **If the point (a, b) lies on the graph f, then the point (b, a) lies on the graph of f^{-1} and vice versa.** This means that the graph of f is a reflection of the graph of f^{-1} in the line $y = x$.

From Principles of Mathematics 11, you may remember that:

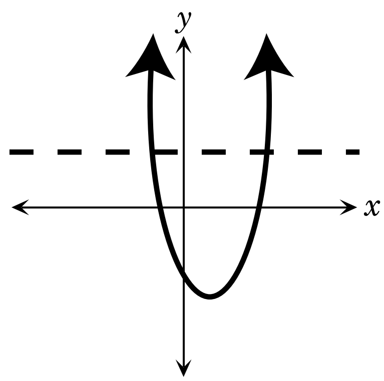
1. The inverse of a function is its reflection in the line $y = x$. Each point in the inverse function is the same distance away from the line, but on opposite sides of the line. Every point (a,b) is transformed to (b,a).



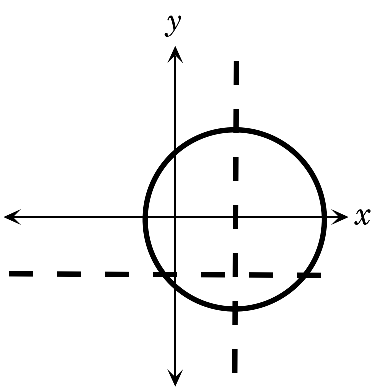
2. For an inverse function to exist, the original function must be one-to-one. Every x-value must have only one y-value and vice versa. Thus, the graph of the function must pass both the vertical and horizontal line tests. Passing the *vertical line test* means that the original function is truly a function; passing the *horizontal line test* means that the inverse will also be a function. (Failing the vertical line test means that we have a relation, but not a function, like the circle relation in the illustration on the next page.)



One-to-one function.
Has an inverse function.



Not one-to-one.
Horizontal line cuts
the graph at two points.
Inverse is not a function.



Not a function. Vertical line cuts the graph at two points. Inverse will not be a function either because a horizontal line does the same.

Sometimes in Principles of Mathematics 12, we will get around this restriction by considering only a portion of the original function say, a piece of the graph which passes both line tests, even if the whole graph passes only the vertical test.

3. In the inverse function, range and domain get reversed:
   The domain of $f(x)$ becomes the range of $f^{-1}(x)$.
   The range of $f(x)$ becomes the domain of $f^{-1}(x)$.

4. To find the inverse of a function $y = f(x)$, replace x and y with each other and solve for y.

Example 1

Find the inverse of $f(x) = 2x - 5$

$x = 2y - 5$ replace x and y with each other

$2y = x + 5$ solve for y

$$y = \frac{x+5}{2}$$

$$f^{-1}(x) = \frac{x+5}{2}$$

Check that your answer is correct by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

$$f\left(f^{-1}(x)\right) = \frac{2(x+5)}{2} - 5 = x$$

$$f^{-1}\left(f(x)\right) = \frac{2x-5+5}{2} = \frac{2x}{2} = x$$

$\therefore f^{-1}(x) = \dfrac{x+5}{2}$ is indeed the inverse of $f(x) = 2x - 5$

Example 2

a) Find the inverse of the function $f(x) = \dfrac{1}{(x+2)}$. The domain of $f(x)$ is $[0,\infty)$.

b) State the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $y = \dfrac{1}{x+2}$ replace $f(x)$ by y

$x = \dfrac{1}{y+2}$ swap x and y

$x(y+2) = 1$ solve for y

$$y = \frac{1}{x} - 2$$

$$\therefore f^{-1}(x) = \frac{1}{x} - 2$$

Check:

$$f\left(f^{-1}(x)\right)=\frac{1}{\left(\frac{1}{x}-2\right)+2}$$
$$=\frac{1}{\frac{1}{x}}$$
$$=x$$

$$f^{-1}\left(f(x)\right)=\frac{1}{\frac{1}{x+2}}-2$$
$$=x+2-2$$
$$=x$$

b) domain of f = $[0,\infty)$ (given)

range of f = $(0,\frac{1}{2}]$ (the maximum value of f is $\frac{1}{2}$ when $x = 0$)

domain of f^{-1} = range of f = $(0,\frac{1}{2}]$

range of f^{-1} = domain of f = $[0,\infty)$

Example 3

Given $f(x)=\dfrac{2x}{x+1}$, find $f^{-1}(x)$

Step 1	$y=\dfrac{2x}{x+1}$	Replace $f(x)$ with y to make manipulation easier
Step 2	$x=\dfrac{2y}{y+1}$	Switch x and y variables
Step 3	$x(y+1)=2y$	To isolate y, you need to eliminate fractions by multiplying both sides by $(y + 1)$
Step 4	$xy+x=2y$	Expand bracket
Step 5	$xy-2y=-x$	Collect terms with "y" variable on one side
Step 6	$y(x-2)=-2$	Factor left hand side
Step 7	$y=\dfrac{-x}{x-2}$	Divide both sides by $(x-2)$

Guided Practice

1. For each of the following functions, f:

   i) find its inverse, f^{-1}

   ii) check your answer by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

   iii) find the domain and range of f and f^{-1}

   a) $f(x) = \dfrac{x}{3}$

   b) $f(x) = x^2 - 2$

   c) $f(x) = \sqrt{3x - 2}$

   d) $f(x) = \dfrac{1}{x-3},\ x \geq 0$

2. Find $f^{-1}(x)$ given

   $f(x) = \dfrac{x+1}{3x}$

Check your answers in the Module 1 Answer Key.

Notes

Lesson 4

Polynomial Functions and Their Graphs

Outcomes

Upon completing this lesson, you will be able to:

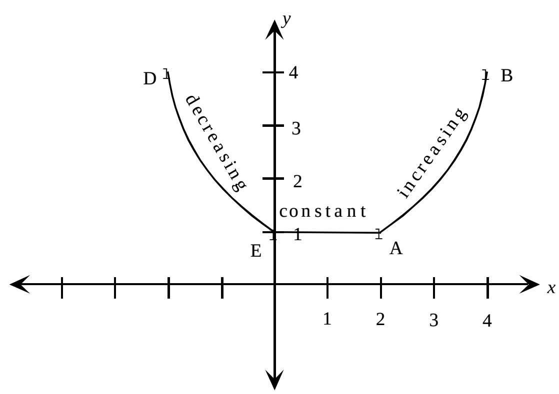
- identify a polynomial function
- relate its factors to its zeroes
- graph a polynomial function

Overview

A ***polynomial function*** is an expression that can be written in he form $a_n x^n + a_{n-1} x^{n-1} + \dots a_2 x^2 + a_1 x + a_0$ where n is a non-negative integer. In the above polynomial, each of the $a_n x$ parts is called a ***term***. Terms are always added together (or subtracted) in polynomials—never multiplied. The a_i values are called ***coefficients*** of the terms.

Notes

1. The graph of a polynomial function is continuous. This means the graph has no breaks—you could sketch the graph without lifting your pencil from the paper.
2. The graph of a polynomial function has only smooth turns. The graph of *f* has, at most, $(n - 1)$ turning points. Turning points are points at which the graph changes from increasing (as we move to the right) to decreasing or vice versa.



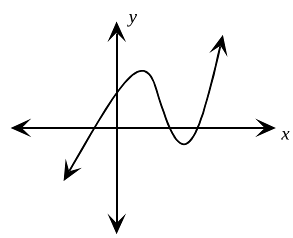
The function shown above is ***decreasing*** in the interval from D to E, it remains ***constant*** from E to A, and it is ***increasing*** in the interval from A to B. (Incidentally, because of its sharp corners and its flat section, it cannot be the graph of a polynomial function.)

For the graphs that you investigated, the cubic equation will have at most (3 – 1) turns, or two turns. For the graphs that you investigated, the quartic equations will have at most (4 – 1) turns, or 3 turns.

3. a) When the degree, n, of a polynomial is ***odd*** (i.e., of degree 1, 3, 5, ...):

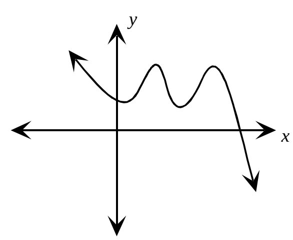
   If the leading coefficient is positive (> 0), then the graph falls to the left and rises to the right.

   Another way to express this is the graph rises from the third quadrant.



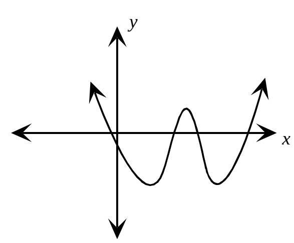
   If the leading coefficient is negative (< 0), then the graph rises to the left and falls to the right.

   Or it falls from the second quadrant.

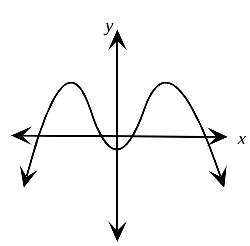


   b) When the degree, n, of a polynomial is ***even*** (i.e., of degree 2, 4, 6, ...):

   If the leading coefficient is positive (> 0), then the graph rises to the left and right, or "opens up".

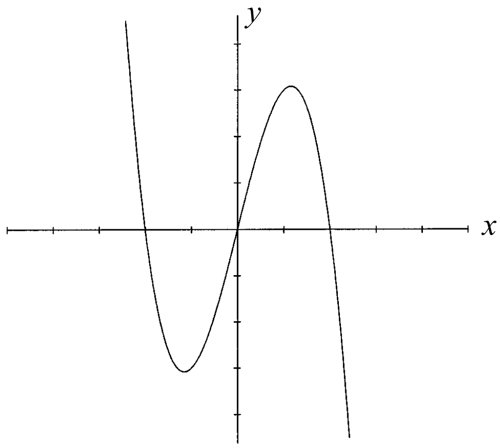


   If the leading coefficient is negative (< 0), then the graph falls to the left and right, or "opens down".

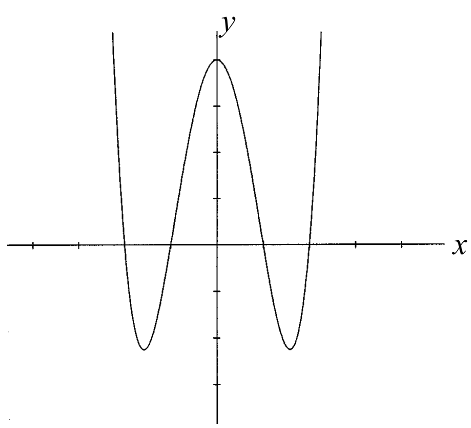


4. The function, *f*, has at most *n* real roots. If you have a cubic function, you can expect three roots at most.

   When you have a quartic function, you can expect it to have at most four roots, and so on.



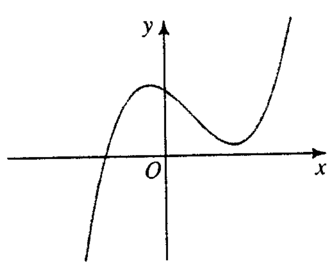
$f(x) = -x^3 + 4x$
cubic—at most three roots



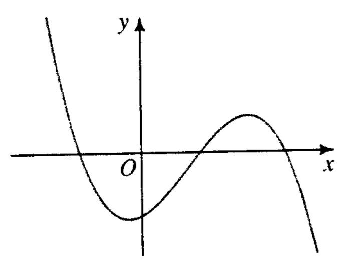
$f(x) = x^4 - 5x^2 + 4$
quartic—at most four roots

5. In general, the graph of a cubic function is shaped like a "sideways S" as shown.

   Graphs of $f(x) = ax^3 + bx^2 + cx + d$:



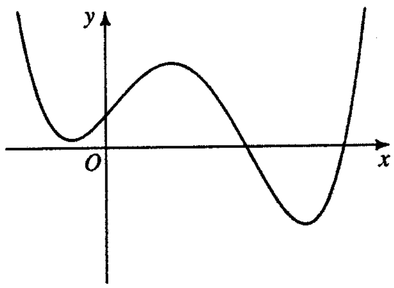
$a > 0$



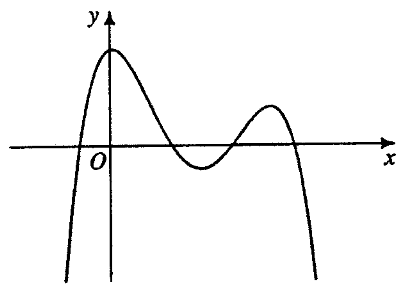
$a < 0$

In general, the graph of a quartic equation has a “W shape” or an “M shape.”

Graphs of $f(x) = ax^4 + bx^3 + cx^2 + dx + e$:



$a > 0$



$a < 0$

6. If a polynomial $f(x)$ has a squared factor such as $(x - c)^2$, then $x = c$ is a ***double root*** of $f(x) = 0$. In this case, the graph of $y = f(x)$ is tangent to the x-axis at $x = c$, as shown in Figures 1, 2, and 3.

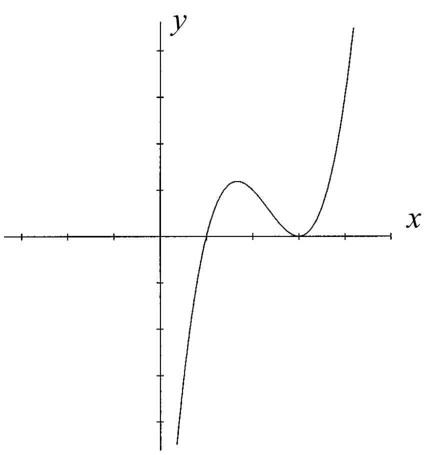


Figure 1
Cubic
$y = (x - 1)(x - 3)^2$

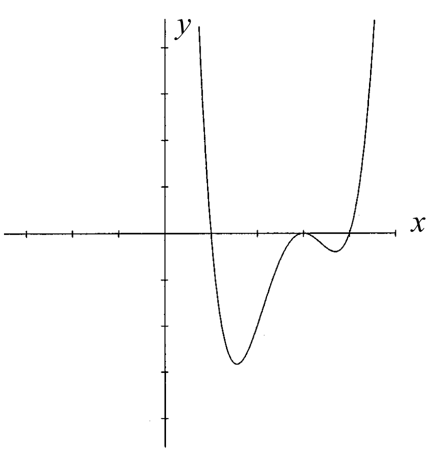


Figure 2
Quartic
$y = (x - 1)(x - 3)^2(x - 4)$

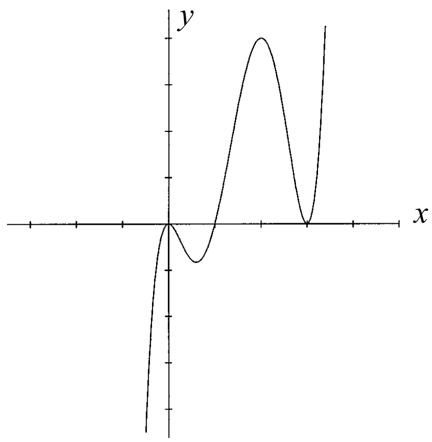


Figure 3
Quintic (5th degree)
$y = x^2(x - 1)(x - 3)^2$
[which is a more efficient way of writing
$y = (x - 0)^2 (x - 1)(x - 3)^2$]

If a polynomial $P(x)$ has a cubed factor such as $(x - c)^3$, then $x = c$ is a ***triple root*** of $P(x) = 0$. In this case, the graph of $y = P(x)$ flattens out (or plateaus) around $(c, 0)$ and crosses the x-axis at this point, as shown in Figures 4, 5, and 6.

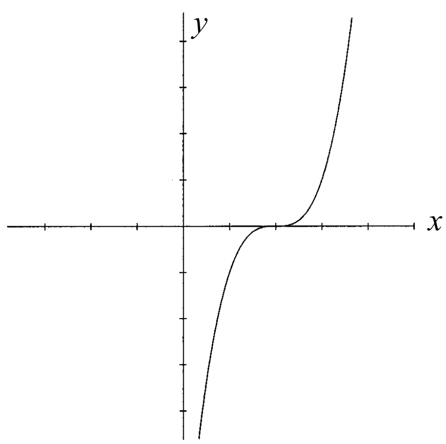


Figure 4
Cubic
$y = (x - 2)^3$

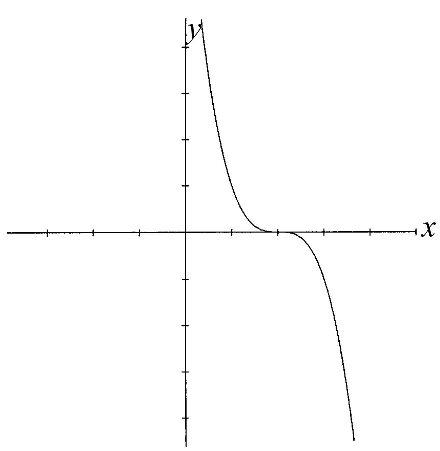


Figure 5
Cubic
$y = -(x - 2)^3$

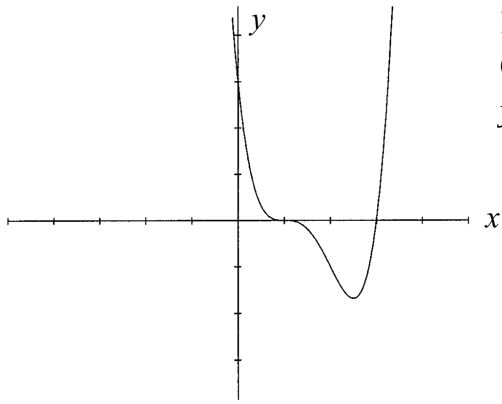


Figure 6
Quartic
$y = (x - 1)^3(x - 3)$

7. Polynomial functions may have relative maxima, relative minima, absolute maxima or absolute minima or a combination. See the above Figure 6 for an example of absolute and relative minima. The absolute minimum at (2.25, –1.75) and a relative minimum at (1,0). Absolute and relative maxima would exist if the graph opened down.

Example 1

Sketch the graph of the function, $f(x) = x^2(x + 1)(x - 2)$. Find the maximum and minimum points.

Solution

Find the y-intercept. Set $x = 0$.

$$f(0) = 0^2(0 + 1)(0 - 2) = 0$$

Therefore (0,0) is the y-intercept for this function.

Find the x-intercepts. Set $y = 0$

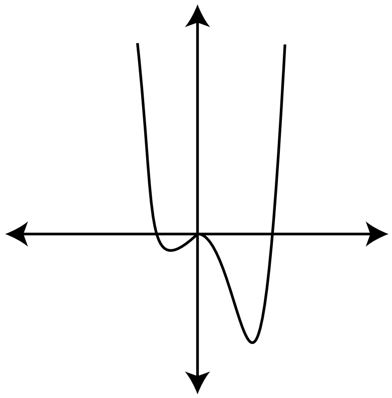
$$x^2(x + 1)(x - 2) = 0$$

$$x^2 = 0 \text{ or } x + 1 = 0 \text{ or } x - 2 = 0$$

$$x = \{-1, 0, 2\}$$

The term with the highest power is x^4 (found by multiplying together the x terms in the equation: $x^2 \times x \times x = x^4$), so the function is a quartic, or fourth order, polynomial.

The coefficient of x^4 is positive, so the graph “opens up.” x^2 is a double root, so the graph touches the x-axis at $x = 0$.



The interval (–1,0) between the roots is less than the interval (0,2), so the graph goes down farther between 0 and 2. Note that we have a relative minimum in the interval (–1,0), a relative maximum at (0,0), and an absolute minimum in the interval (0,2). To determine range, we need to find the value of the absolute minimum. We can do this using the "minimum" choice from the CALC menu of a graphing calculator.

Press Y= . At the flashing cursor, carefully type

X,T,θ x^2 (X,T,θ + 1)(X,T,θ – 2)

Your display should look like this:

Y_1 $x^2(x+1)(x-2)$
Y_2
Y_3
etc.

Press GRAPH to see your display.

Press WINDOW to obtain the **WINDOW** or **WINDOW FORMAT** menu. Change the **Ymin** setting to –5 in order to better see the minimum value.

To calculate the minimum value, press 2nd TRACE to get the CALC ulate menu. Scroll down to 3:minimum or press 3. You will notice the blinking cursor on the graph (if not, use left arrow key to activate). Use the arrows to move the cursor to the left of the lowest minimum value, and press ENTER . Use the right arrow to move the cursor to the right of the lowest minimum value and press ENTER again. Move the cursor back close to the minimum as your guess.

Press [ENTER] again.

The minimum occurs at (+1.443001, –2.833422).

Therefore range = [–2.833,∞), or $\{y \mid y \geq -2.833]$. We use a closed interval because the graph reaches down to approximately –2.833.

Example 2

Inspect the following polynomial functions and determine their basic shape and orientation.

Determine the domain and range of each.

a) $f(x) = x^3 - 4x$

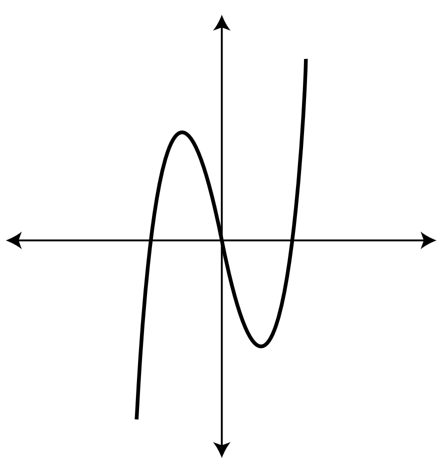
b) $g(x) = -x^6 + 4x^2$

c) $k(x) = -x^5 + 4x^3 - 6$

Hint: The absolute minimum or absolute maximum always defines the range.

Solutions

a) $f(x)$ is a third order (cubic) polynomial, with a positive leading coefficient. The graph rises from Quadrant III in an S shape. Furthermore, when factored, $f(x) = x(x - 2)(x + 2)$, we can easily find its zeroes.



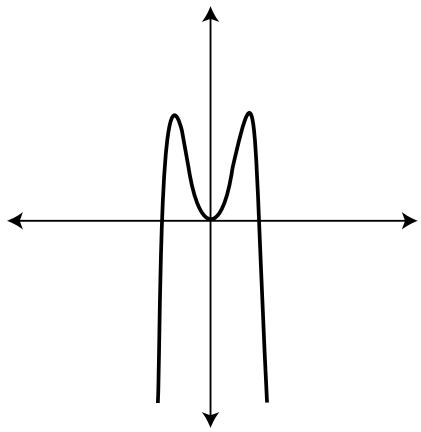
Solving $x(x - 2)(x + 2) = 0$, we get $x = -2$, 0, or 2. Because it has 3 roots,
it has 3 – 1 = 2 turns.

domain = $\Re$ and range $\Re$

b) g is a sixth order polynomial, with a negative leading coefficient, so it opens down.

When factored, we see that $g(x) = -x^2(x^2 + 2)(x^2 - 2)$
x^2 is a repeated factor
$x^2 + 2$ has no real factors

$x^2 - 2 = 0$ yields $x^2 = 2$, so $x = \pm\sqrt{2}$

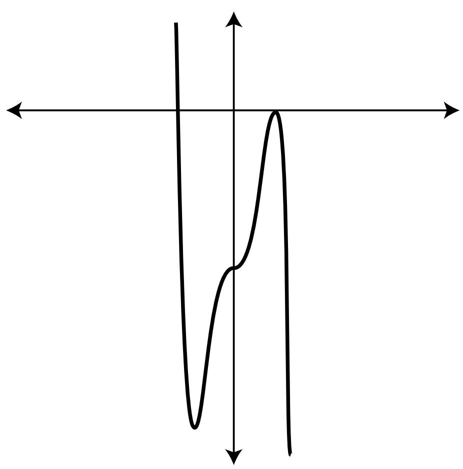


The maximum number of turns is 6 – 1 = 5, but because of the $x^2 + 2$ factor we only get 3 turns.
domain = ℜ

Using the graphing calculator as in
Example 1, we get
range = (–∞, 3.0792014]

c) *k* is a fifth order polynomial, or quintic, which means it is an odd polynomial. It has a negative leading coefficient, so it falls from Quadrant II. It has at most 5 – 1= 4 turns.

domain = ℜ
range = ℜ



When we plot it using the graphing calculator, we find it has only one maximum and one minimum, with a kink near the *y*-axis called a stationary point (which you don't have to remember).

The local maximum is located (1.54591952,
–0.0510976).

In Lesson 1 to solve an equation with the graphing calculator, we used the "0:Solver" function from the MATH menu and typed in the equation to be solved. That method works, but we could miss a solution for a higher-order polynomial because there can be several solutions close to each other.

To find the zero (x-intercept) of a *graphed* equation, press 2nd CALC, then **2:zero**. Just as you did for a minimum or maximum, mark the left bound, right bound, and a first guess for each crossing point on the x-axis. (Include only one crossing point between your left and right bounds.) Then write down the displayed x-value as the solution. Repeat for each crossing point on the x-axis.

In this case c) you'll find a zero at –2.146348, which you round to –2.146. If you try the same thing for the relative maximum, near x = 1.5, which appears to touch the x-axis, you'll get an error message. That's because the graph does not quite reach the x-axis there; the y-value reaches –0.051 but not 0.

You can solve equations using either MATH **0:Solver** or CALC **2:zero**. Because the **2:zero** function works off a graph, it gives you a better chance of noticing and reporting every solution.

Guided Practice

1. Rewrite each polynomial in descending order. Determine:

   i) the degree of the polynomial

   ii) the maximum number of turns in the graph

   a) $f(x) = 8x - x^4$

   b) $g(x) = x^3 + 5x^4 - 2x^2$

   c) $h(x) = -3x^5 - x^2 + x$

2. State the left-right behavior of the graphs in question 1, as to whether they open up, open down, rise from Quadrant III, or fall from Quadrant II.

3. Match each polynomials function with the correct graph.

   a) $f(x) = -3x + 5$

   b) $f(x) = x^2 - 2x$

   c) $f(x) = -2x^2 - 9x - 9$

   d) $f(x) = 3x^3 - 9x + 1$

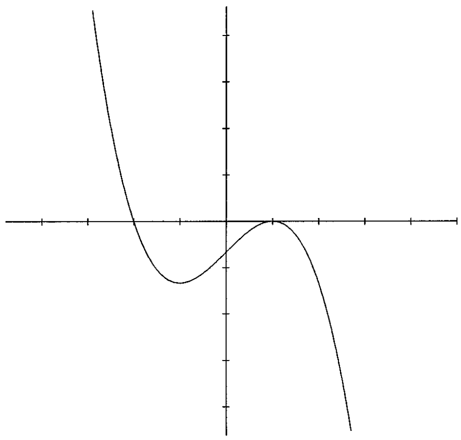
   e) $f(x) = -\frac{1}{3}x^3 + x - \frac{2}{3}$

   f) $f(x) = -\frac{1}{4}x^4 + 2x^2$

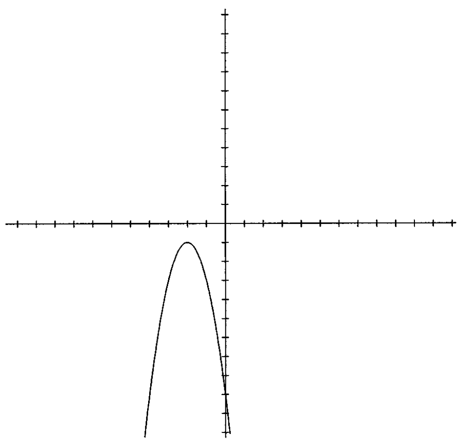
   g) $f(x) = 3x^4 + 4x^3$

   h) $f(x) = x^5 - 5x^3 + 4x$

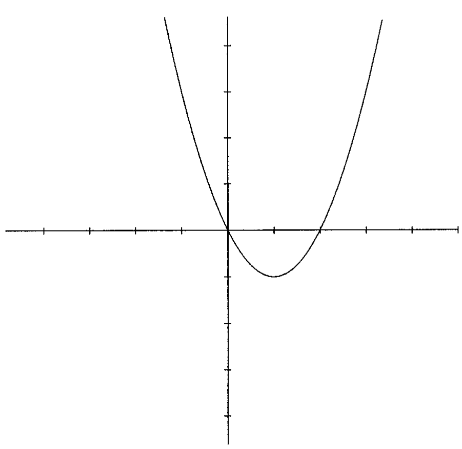
i)



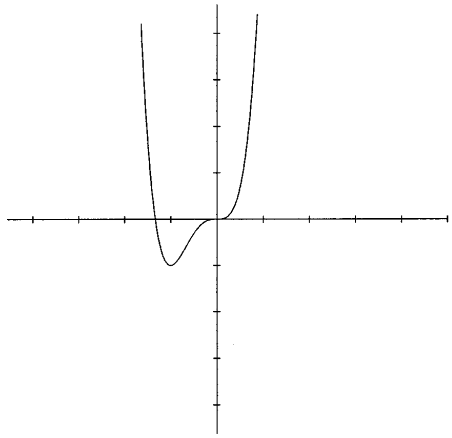
ii)



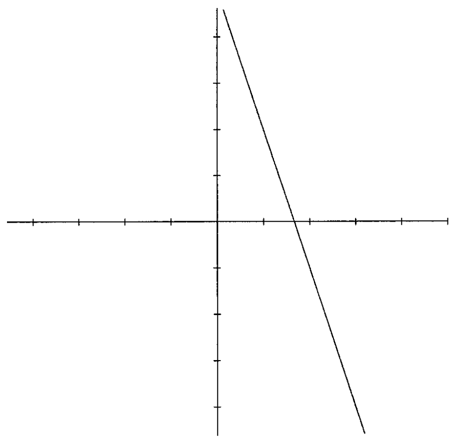
iii)



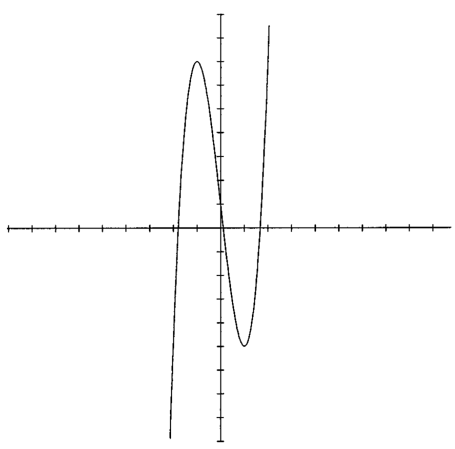
iv)



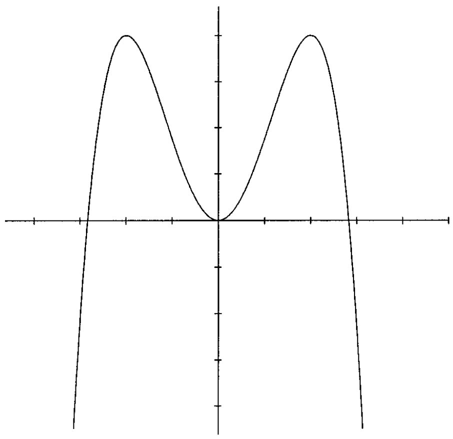
v)



vi)



vii)



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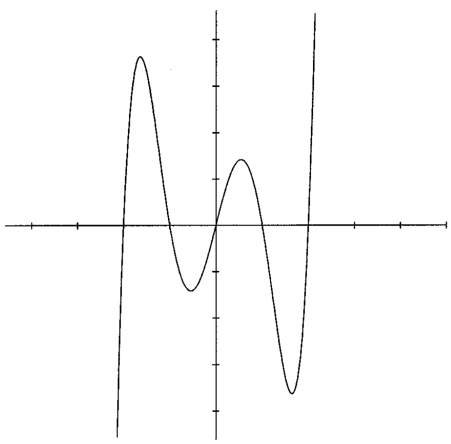


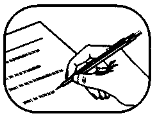
4. Find *(i)* the zeros of the following functions, *(ii)* the maximum number of turns, *(iii)* the left-right behaviour, *(iv)* the domain and range, and *(v)* sketch the graph of the function.

   a) $f(x) = 4x - x^3$

   b) $g(x) = (x-2)^4$

   c) $h(x) = x^3 + x^2 - 6x$

Check your answers in the Module 1 Answer Key.



Review

1. Solve the following equations using your graphing calculator.

   a) $x^3 + 2x^2 - 4x = 7$

   b) $2x^4 + 3x^2 = 4$

   c) $x^5 - 4x^3 = x^2 - 3x$

   d) $0.4x^3 - 0.78x^2 + 2.4x + 7 = 0$

2. If $f(x) = 3x^2 + 3x - 1$ and $g(x) = \sqrt{2x+1}$, find:

   a) $f(2)$

   b) $g(-1)$

   c) $(f+g)(4)$

   d) $(f \div g)(4)$

   e) $f(a+4)$

   f) $(g \circ f)(1)$

   g) $(f \circ g)(x)$

   h) $f(f^{-1}(x))$

   i) $(f \times g)(t+2)$

   j) $g^{-1}(x)$

3. *i*) Determine the x- and y-intercepts of these functions.
   ii) Find the domain and range.

   a) $f(x) = \sqrt{1-2x}$

   b) $g(x) = 2(x+2)^2 - 4$

   c) $h(x) = (x+2)^2(x-4)(x-2)$

4. Find the inverse of the following functions.

   a) $f(x) = 5x - 2$

   b) $g(x) = (x+1)^2 - 3$

   c) $h(x) = \dfrac{1}{x^2 - 1}$

5. Sketch the graphs of the following functions. Label the intercepts and give the domain and range of each.

   a) $f(x) = x^3 + x^2 - 2$

   b) $g(x) = -2(x^2 - 9)(x+2)(2x-1)$

Check your answers In the Module 1 Answer Key.

Now do the section assignment which follows this section.

PRINCIPLES OF MATHEMATICS 12

Section Assignment 1.1

General Instructions for Assignments

These instructions apply to all the section assignments but will not be reprinted each time. Remember them for future sections.

(1) Treat this assignment as a test, so do not refer to your module or notes or other materials. A scientific calculator and graphing calculator are permitted.

(2) Where questions require computations or have several steps, showing these can result in part marks for some exercises. Steps must be neat and well-organized, however, or the instructor will only consider the answer.

(3) Always read the question carefully to ensure you answer what is asked. Often unnecessary work is done because a question has not been read correctly.

(4) Always clearly underline your final answer so that it is not confused with your work.

Section Assignment 1.1

Review of Mathematics 11

Total Value: 40 marks
(Mark values in margins)

1. If $h(x) = \frac{3}{2x+1}$, find:

(1) a) $h(2)$

(1) b) $h\left(-\frac{1}{2}\right)$

c) $h(a-1)$ *(1)*

d) $h^{-1}(x)$ *(2)*

(4) 2. For the the function $g(x) = \sqrt{2x+3}$ find:

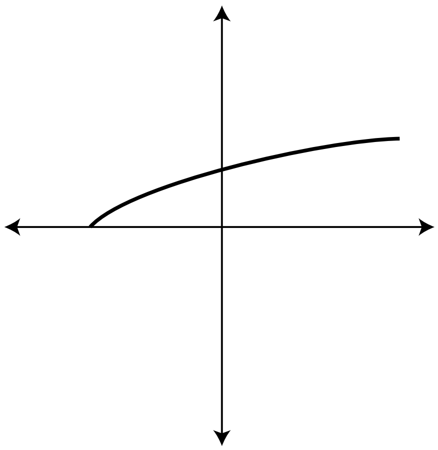
a) the domain of g

b) the range of g

c) the y-intercept

d) the x-intercept

3. For the the function $q(x)$ below, sketch the graph of $q^{-1}(x)$: *(1)*



4. If $f(x) = x^2 - 2$ and $g(x) = \dfrac{1}{3x}$ $(x \neq 0)$, find:

a) $(f+g)(x)$ *(1)*

b) $(f \times g)(-2)$ *(1)*

(1) c) $f(g(x))$

(1) d) $g(f(-1))$

(2) e) $f^{-1}(x)$ where $x \leq 0$

(5) 5. Without using your graphing calculator, sketch the graph of $g(x) = -2(x - 1)^2(x + 3)$. Label the intercepts and give the domain and range.

6. Given $f(x)=\dfrac{x-4}{x+1}$, find. $f^{-1}(x)$ *(4)*

7. Find the maximum value of the function $f(x) = -x^4 - 3x^2 + x + 7$ and use it to determine the range of f. *(5)*

8. Solve using your graphing calculator:

(1) a) $2x^4 - 3x^3 = x$

(2) b) $\dfrac{x^2 - 2x - 4}{2x^2 + x - 3} = x$

9. Given: $f(x) = 2 + x$

$g(x) = \sqrt{x - 7}$

$h(x) = 2x^2 - 5$

Determine:

a) $g(h(x))$ *(1)*

b) $f(h(3))$ *(1)*

c) $h(g(f(-1)))$ *(2)*

Total: 37 marks