Automated analysis of security protocols with global state (Full version)

Steve Kremer

INRIA Nancy - Grand'Est & Loria, France

Robert Künnemann*

Department of Computer Science, TU Darmstadt, Germany

Abstract

Security APIs, key servers and protocols that need to keep the status of transactions, require to maintain a global, non-monotonic state, e.g., in the form of a database or register. However, most existing automated verification tools do not support the analysis of such stateful security protocols – sometimes because of fundamental reasons, such as the encoding of the protocol as Horn clauses, which are inherently monotonic. A notable exception is the recent tamarin prover which allows specifying protocols as multiset rewrite (msr) rules, a formalism expressive enough to encode state. As multiset rewriting is a "low-level" specification language with no direct support for concurrent message passing, encoding protocols correctly is a difficult and error-prone process.

We propose a process calculus which is a variant of the applied pi calculus with constructs for manipulation of a global state by processes running in parallel. We show that this language can be translated to msr rules whilst preserving all security properties expressible in a dedicated first-order logic for security properties. The translation has been implemented in a prototype tool which uses the tamarin prover as a backend. We apply the tool to several case studies among which a simplified fragment of PKCS#11, the Yubikey security token, and an optimistic contract signing protocol.

1 Introduction

Automated analysis of security protocols has been extremely successful. Using automated tools, flaws have been for instance discovered in the Google Single Sign On Protocol [5], in commercial security tokens implementing the PKCS#11 standard [10], and one may also recall Lowe's attack [21] on the Needham-Schroeder public key protocol 17 years after its publication. While efficient tools such as ProVerif [7], AVISPA [4] or Maude-NPA [14] exist, these tools fail to analyze protocols that require *non-monotonic global state*, i.e., some database, register or memory location that can be read and altered by different parallel threads. In particular ProVerif, one of the most efficient and widely used protocol analysis tools, relies on an abstraction that encodes protocols in first-order Horn clauses. This abstraction is well suited for the monotonic knowledge of an attacker (who never forgets), makes the tool extremely efficient for verifying an unbounded number of protocol sessions and allows to build on existing techniques for Horn clause resolution. However, Horn clauses are inherently monotonic: once a fact is true it cannot be set to false anymore. As a result, even though ProVerif's input language, a variant of the applied pi calculus [2], allows a priori encodings of a global memory, the abstractions performed by ProVerif introduce false attacks. In the ProVerif user manual [8, Section 6.3.3] such an encoding of memory cells and its limitations are indeed explicitly discussed: *"Due to the abstractions performed by ProVerif, such a cell is treated in an approximate way: all values written in the cell are considered as a set, and when one reads the cell, ProVerif just guarantees that the obtained value is one*

[*] Most of this work was carried out when the author was affiliated to INRIA Paris - Rocquencourt, France

of the written values (not necessarily the last one, and not necessarily one written before the read)." Some work [3, 22, 12] has nevertheless used ingenious encodings of mutable state in Horn clauses, but these encodings have limitations that we discuss below.

A prominent example where non-monotonic global state appears are security APIs, such as the RSA PKCS#11 standard [23], IBM's CCA [11] or the trusted platform module (TPM) [27]. They have been known to be vulnerable to logical attacks for some time [20, 9] and formal analysis has shown to be a valuable tool to identify attacks and find secure configurations. One promising paradigm for analyzing security APIs is to regard them as a participant in a protocol and use existing analysis tools. However, Herzog [18] already identified not accounting for mutable global state as a major barrier to the application of security protocol analysis tools to verify security APIs. Apart from security APIs many other protocols need to maintain databases: key servers need to store the status of keys, in optimistic contract signing protocols a trusted party maintains the status of a contract, RFID protocols maintain the status of tags and more generally websites may need to store the current status of transactions.

Our contributions We propose a tool for analyzing protocols that may involve non-monotonic global state, relying on Schmidt *et al.*'s tamarin tool [25, 26] as a backend. We designed a new process calculus that extends the applied pi calculus by defining, in addition to the usual constructs for specifying concurrent processes, constructs for explicitly manipulating global state. This calculus serves as the tool's input language. The heart of our tool is a translation from this extended applied pi calculus to a set of multiset rewrite rules that can then be analyzed by tamarin which we use as a backend. We prove the correctness of this translation and show that it preserves all properties expressible in a dedicated first order logic for expressing security properties. As a result, relying on the tamarin prover, we can analyze protocols without bounding the number of sessions, nor making any abstractions. Moreover it allows to model a wide range of cryptographic primitives by the means of equational theories. As the underlying verification problem is undecidable, tamarin may not terminate. However, it offers an interactive mode with a GUI which allows to manually guide the tool in its proof. Our specification language includes support for private channels, global state and locking mechanisms (which are crucial to write meaningful programs in which concurrent threads manipulate a common memory). The translation has been carefully engineered in order to favor termination by tamarin. We illustrate the tool on several case studies: a simple security API in the style of PKCS#11, a complex case study of the Yubikey security device, as well as several examples analyzed by other tools that aim at analyzing stateful protocols. In all of these case studies we were able to avoid restrictions that were necessary in previous works.

Related work The most closely related work is the StatVerif tool by Arapinis *et al.* [3]. They propose an extension of the applied pi calculus, similar to ours, which is translated to Horn clauses and analyzed by the ProVerif tool. Their translation is sound but allows for false attacks, limiting the scope of protocols that can be analyzed. Moreover, StatVerif can only handle a finite number of memory cells: when analyzing an optimistic contract signing protocol this appeared to be a limitation and only the status of a single contract was modeled, providing a manual proof to justify the correctness of this abstraction. Finally, StatVerif is limited to the verification of secrecy properties. As illustrated by the Yubikey case study, our work is more general and we are able to analyze complex injective correspondance properties.

Mödersheim [22] proposed a language with support for sets together with an abstraction where all objects that belong to the same sets are identified. His language, which is an extension of the low level AVISPA intermediate format, is compiled into Horn clauses that are then analyzed, e. g., using ProVerif. His approach is tightly linked to this particular abstraction limiting the scope of applicability. Mödersheim also discusses the need for a more high-level specification level which we provide in this work.

There has also been work tailored to particular applications. In [13], Delaune *et al.* show by a dedicated hand proof that for analyzing PKCS#11 one may bound the message size. Their analysis still requires to artificially bound the number of keys. Similarly in spirit, Delaune *et al.* [12] give a dedicated result for analyzing protocols based on the TPM and its registers. However, the number of reboots (which reinitialize registers) needs to be limited.

Guttman [17] also extended the strand space model by adding support for state. While the protocol execution is modeled using the classical strand spaces model, state is modeled by a multiset of facts, and manipulated by multiset rewrite rules. The extended model has been used for analyzing by hand an optimistic contract signing protocol. As of now, protocol analysis in the strand space model with state has not been mechanized yet.

In the goal of relating different approaches for protocol analysis Bistarelli *et al.* [6] also proposed a translation from a process algebra to multiset rewriting: they do however not consider private channels, have no support for global state and assume that processes have a particular structure. These limitations significantly simplify the translation and its correctness proof. Moreover their work does not include any tool support for automated verification.

Obviously any protocol that we are able to analyze can be directly analyzed by the tamarin prover [25, 26] as the rules produced by our translation could have been given directly as an input to tamarin. Indeed, tamarin has already been used for analyzing a model of the Yubikey device [19], the case studies presented with Mödersheim's abstraction, as well as those presented with StatVerif. It is furthermore able to reproduce the aforementioned results on PKCS#11 [13] and the TPM [12] – moreover, it does so without bounding the number of keys, security devices, reboots, etc. Contrary to ProVerif, tamarin sometimes requires additional *typing lemmas* which are used to guide the proof. These lemmas need to be written by hand (but are proved automatically). In our case studies we also needed to provide a few such lemmas manually. In our opinion, an important disadvantage of tamarin is that protocols are modeled as a set of multiset rewrite rules. This representations is very low level and far away from actual protocol implementations, making it very difficult to model a protocol adequately. Encoding private channels, nested replications and locking mechanisms directly as multiset rewrite rules is a tricky and error prone task. As a result we observed that, in practice, the protocol models tend to be simplified. For instance, locking mechanisms are often omitted, modeling protocol steps as a single rule and making them effectively atomic. Such more abstract models may obscure issues in concurrent protocol steps and increase the risk of implicitly excluding attacks in the model that are well possible in a real implementation, e. g., race conditions. Using a more high-level specification language, such as our process calculus, arguably eases protocol specification and overcomes some of these risks.

2 Preliminaries

Terms and equational theories As usual in symbolic protocol analysis we model messages by abstract terms. Therefore we define an order-sorted term algebra with the sort *msg* and two incomparable subsorts *pub* and *fresh*. For each of these subsorts we assume a countably infinite set of names, FN for fresh names and PN for public names. Fresh names will be used to model cryptographic keys and nonces while public names model publicly known values. We furthermore assume a countably infinite set of variables for each sort s, $\mathcal{V}_s$ and let $\mathcal{V}$ be the union of the set of variables for all sorts. We write $u : s$ when the name or variable u is of sort s. Let Σ be a signature, i.e., a set of function symbols, each with an arity. We write f/n when function symbol f is of arity n. We denote by $\mathcal{T}_\Sigma$ the set of well-sorted terms built over Σ, PN, FN and $\mathcal{V}$. For a term t we denote by $names(t)$, respectively $vars(t)$ the set of names, respectively variables, appearing in t. The set of ground terms, i.e., terms without variables, is denoted by $\mathcal{M}_\Sigma$. When Σ is fixed or clear from the context we often omit it and simply write $\mathcal{T}$ for $\mathcal{T}_\Sigma$ and $\mathcal{M}$ for $\mathcal{M}_\Sigma$.

We equip the term algebra with an equational theory E, that is a finite set of equations of the form $M = N$ where $M, N \in \mathcal{T}$. From the equational theory we define the binary relation $=_E$ on terms, which is the smallest equivalence relation containing equations in E that is closed under application of function symbols, bijective renaming of names and substitution of variables by terms of the same sort. Furthermore, we require E to distinguish different fresh names, i. e., $\forall a, b \in FN : a \neq b \Rightarrow a \neq_E b$.

Example. Symmetric encryption can be modelled using a signature

$$\Sigma = \{\, senc/2, sdec/2, encCor/2, true/0 \,\}$$

and an equational theory defined by

$$\mathit{sdec}(\mathit{senc}(m,k),k) = m \quad \mathit{encCor}(\mathit{senc}(x,y),y) = \mathit{true}$$

The last equation allows to check whether a term can be correctly decrypted with a certain key.

For the rest of the paper we assume that E refers to some fixed equational theory and that the signature and equational theory always contain symbols and equations for pairing and projection, i.e., $\{\langle .,. \rangle, \mathsf{fst}, \mathsf{snd}\} \subseteq \Sigma$ and equations $\mathsf{fst}(\langle x,y \rangle) = x$ and $\mathsf{snd}(\langle x,y \rangle) = y$ are in E. We will sometimes use $\langle x_1, x_2, \ldots, x_n \rangle$ as a shortcut for $\langle x_1, \langle x_2, \langle \ldots, \langle x_{n-1}, x_n \rangle \ldots \rangle$.

We also use the usual notion of positions for terms. A position p is a sequence of positive integers and $t|_p$ denotes the subterm of t at position p.

Facts We also assume an unsorted signature Σ_{fact}, disjoint from Σ. The set of *facts* is defined as

$$\mathcal{F} := \{F(t_1,\ldots,t_k) \mid t_i \in \mathcal{T}_\Sigma, F \in \Sigma_{\mathit{fact}} \text{ of arity } k\}.$$

Facts will be used both to annotate protocols, by the means of events, and for defining multiset rewrite rules. We partition the signature Σ_{fact} into *linear* and *persistent* fact symbols. We suppose that Σ_{fact} always contains a unary, persistent symbol $!\mathsf{K}$ and a linear, unary symbol Fr. Given a sequence or set of facts S we denote by $\mathit{lfacts}(S)$ the multiset of all linear facts in S and $\mathit{pfacts}(S)$ the set of all persistent facts in S. By notational convention facts whose identifier starts with '!' will be persistent. $\mathcal{G}$ denotes the set of ground facts, i.e., the set of facts that does not contain variables. For a fact f we denote by $\mathit{ginsts}(f)$ the set of ground instances of f. This notation is also lifted to sequences and sets of facts as expected.

Substitutions A substitution σ is a partial function from variables to terms. We suppose that substitutions are well-typed, i.e., they only map variables of sort s to terms of sort s, or of a subsort of s. We denote by $\sigma = \{^{t_1}/_{x_1}, \ldots, ^{t_n}/_{x_n}\}$ the substitution whose domain is $\mathbf{D}(\sigma) = \{x_1, \ldots, x_n\}$ and which maps x_i to t_i. As usual we homomorphically extend σ to apply to terms and facts and use a postfix notation to denote its application, e.g., we write $t\sigma$ for the application of σ to the term t. A substitution σ is grounding for a term t if $t\sigma$ is ground. Given function g we let $g(x) = \perp$ when $x \notin \mathbf{D}(x)$. When $g(x) = \perp$ we say that g is undefined for x. We define the function $f := g[a \mapsto b]$ with $\mathbf{D}(f) = \mathbf{D}(g) \cup \{a\}$ as $f(a) := b$ and $f(x) := g(x)$ for $x \neq a$.

Sets, sequences and multisets We write $\mathbb{N}_n$ for the set $\{1,\ldots,n\}$. Given a set S we denote by S^* the set of finite sequences of elements from S and by $S^\#$ the set of finite multisets of elements from S. We use the superscript $^\#$ to annotate usual multiset operation, e.g. $S_1 \cup^\# S_2$ denotes the multiset union of multisets S_1, S_2. Given a multiset S we denote by $\mathit{set}(S)$ the set of elements in S. The sequence consisting of elements $e_1, \ldots, e_n$ will be denoted by $[e_1, \ldots, e_n]$ and the empty sequence is denoted by $[]$. We denote by $|S|$ the length, i.e., the number of elements of the sequence. We use $\cdot$ for the operation of adding an element either to the start or to the end, e.g., $e_1 \cdot [e_2, e_3] = [e_1, e_2, e_3] = [e_1, e_2] \cdot e_3$. Given a sequence S, we denote by $\mathit{idx}(S)$ the set of positions in S, i.e., $\mathbb{N}_n$ when S has n elements, and for $i \in \mathit{idx}(S)$ S_i denotes the ith element of the sequence. Set membership modulo E is denoted by $\in_E$ and defined as $e \in_E S$ if $\exists e' \in S.\ e' =_E e$. $\subset_E$ and $=_E$ are defined for sets in a similar way. Application of substitutions are lifted to sets, sequences and multisets as expected. By abuse of notation we sometimes interpret sequences as sets or multisets; the applied operators should make the implicit cast clear.

3 A cryptographic pi calculus with explicit state

3.1 Syntax and informal semantics

Our calculus is a variant of the applied pi calculus [2]. In addition to the usual operators for concurrency, replication, communication and name creation, it offers several constructs for reading and updating an explicit global state. The grammar for processes is described in Figure 1.

$\langle M,N \rangle$ ::= $x, y, z \in \mathcal{V}$
  | $p \in PN$
  | $n \in FN$
  | $f(M_1,\ldots,M_n)$ ($f \in \Sigma$ of arity n)

$\langle P,Q \rangle$ ::= 0
  | $P \mid Q$
  | $!\, P$
  | $\nu n;\, P$
  | out(M, N); P
  | in(M, N); P
  | if M=N then P [else Q]
  | event F ; P ($F \in \mathcal{F}$)
  | insert M,N; P
  | delete M; P
  | lookup M as x in P [else Q]
  | lock M; P
  | unlock M; P
  | $[L]$ –$[A]$→ $[R]; P$ ($L, R, A \in \mathcal{F}^*$)

Figure 1: Syntax

0 denotes the terminal process. $P \mid Q$ is the parallel execution of processes P and Q and $!P$ the replication of P, allowing an unbounded number of sessions in protocol executions. The construct $\nu n; P$ binds the name n in P and models the generation of a fresh, random value. Processes out(M, N); P and in(M, N); P represent the output, respectively input, of message N on channel M. Readers familiar with the applied pi calculus [2] may note that we opted for the possibility of pattern matching in the input construct, rather than merely binding the input to a variable x. The process if M=N then P else Q will execute P if $M =_E N$ and Q otherwise. The event construct is merely used for annotating processes and will be useful for stating security properties. For readability we sometimes omit to write else Q when Q is 0, as well as trailing 0 processes.

The remaining constructs are used for manipulating state and are new compared to the applied pi calculus. We offer two different mechanisms for state. The first construct is *functional* and allows to associate a value to a key. The construct insert M,N binds the value N to a key M. Successive inserts allow to change this binding. The delete M operation simply "undefines" the mapping for the key M. The lookup M as x in P else Q allows to retrieve the value associated to M, binding it to the variable x in P. If the mapping is undefined for M the process behaves as Q. The lock and unlock constructs allow to gain exclusive access to a resource M. This is essential for writing protocols where parallel processes may read and update a common memory. We additionally offer another kind of global state in form of a multiset of ground facts, as opposed to the previously introduced functional store. This multiset can be altered using the construct $[L]$ –$[A]$→ $[R]; P$, which tries to match each fact in the sequence L to facts in the current multiset and, if successful, adds the corresponding instance of facts R to the store. The facts A are used as annotations in a similar way to events. The purpose of this construct is to provide access to the underlying notion of state in tamarin, but we stress that it is distinct from the previously introduced functional state, and its use is only advised to expert users. We allow this "low-level" form of state manipulation in addition to the *functional* state, as it offers a great flexibility and has shown useful in one of our case studies. This style of state manipulation is similar to the state extension in the strand space model [17] and the underlying specification language of the tamarin tool [25, 26]. Note that, even though those stores are distinct (which is a restriction imposed by our translation), data can be moved from one to another, for example as follows: lookup 'store1' as x in [] –[ ]→ [store2(x)].

In the following example, which will serve as our running example, we model a security API that, even though much simplified, illustrates the most salient issues that occur in the analysis of security

APIs such as PKCS#11 [13, 10, 15] .

Example. We consider a security device that allows the creation of keys in its secure memory. The user can access the device via an API. If he creates a key, he obtains a handle, which he can use to let the device perform operations on his behalf. For each handle the device also stores an attribute which defines what operations are permitted for this handle. The goal is that the user can never gain knowledge of the key, as the user's machine might be compromised. We model the device by the following process (we use $\mathsf{out}(m)$ as a shortcut for $\mathsf{out}(c,m)$ for a public channel c):

$$!P_{new} \mid !P_{set} \mid !P_{dec} \mid !P_{wrap}, \text{ where}$$

```
$P_{new} := \nu h;\ \nu k;$ event $\mathrm{NewKey}(h,k);$
        insert $\langle$'key'$,h\rangle,k;$
        insert $\langle$'att'$,h\rangle,$'dec'$;$ out$(h)$
```

In the first line, the device creates a new handle h and a key k and, by the means of the event $\mathrm{NewKey}(h,k)$, logs the creation of this key. It then stores the key that belongs to the handle by associating the pair $\langle$'key'$,h\rangle$ to the value of the key k. In the next line, $\langle$'att'$,h\rangle$ is associated to a public constant 'dec'. Intuitively, we use the public constants 'key' and 'att' to distinguish two databases. The process

```
$P_{set} :=$ in$(h);$ insert $\langle$'att'$,h\rangle$, 'wrap'
```

allows the attacker to change the attribute of a key from the initial value 'dec' to another value 'wrap'. If a handle has the 'dec' attribute set, it can be used for decryption:

```
$P_{dec} :=$ in$(\langle h,c\rangle);$ lookup $\langle$'att'$,h\rangle$ as $a$ in
        if $a=$'dec' then
            lookup $\langle$'key'$,h\rangle$ as $k$ in
                if $encCor(c,k)=true$ then
                    event $\mathrm{DecUsing}(k,sdec(c,k));$
                    out$(sdec(c,k))$
```

The first lookup stores the value associated to $\langle$'att'$,h\rangle$ in a. The value is compared against 'dec'. If the comparison and another lookup for the associated key value k succeeds, we check whether decryption succeeds and, if so, output the plaintext.

If a key has the 'wrap' attribute set, it might be used to encrypt the value of a second key:

```
$P_{wrap} :=$ in$(\langle h_1,h_2\rangle);$ lookup $\langle$'att'$,h_1\rangle$ as $a_1$ in
            if $a_1=$'wrap' then
                lookup $\langle$'key'$,h_1\rangle$ as $k_1$ in
                    lookup $\langle$'key'$,\ h_2\rangle$ as $k_2$ in
                        event $\mathrm{Wrap}(k_1,k_2);$
                        out$(senc(k_2,k_1))$
```

The bound names of a process are those that are bound by νn. We suppose that all names of sort *fresh* appearing in the process are under the scope of such a binder. Free names must be of sort *pub*. A variable x can be bound in three ways: *(i)* by the construct lookup M as x, or *(ii)* $x \in vars(N)$ in the construct $\mathsf{in}(M,N)$ and x is not under the scope of a previous binder, *(iii)* $x \in vars(L)$ in the construct $[L] -[A]\rightarrow [R]$ and x is not under the scope of a previous binder. While the construct lookup M as x always acts as a binder, the input and $[L] -[A]\rightarrow [R]$ constructs do not rebind an already bound variable but perform pattern matching. For instance in the process

$$P = \mathsf{in}(\mathsf{c},f(x));\ \mathsf{in}(\mathsf{c},g(x))$$

x is bound by the first input and pattern matched in the second. It might seem odd that lookup acts as a binder, while input does not. We justify this decision as follows: as P_{dec} and P_{wrap} in the previous

$$\frac{a \in \mathit{FN} \cup \mathit{PN} \quad a \notin \tilde{n}}{\nu\tilde{n}.\sigma \vdash a}\textsc{Dname} \qquad \frac{\nu\tilde{n}.\sigma \vdash t \quad t =_E t'}{\nu\tilde{n}.\sigma \vdash t'}\textsc{DEq}$$

$$\frac{x \in \mathbf{D}(\sigma)}{\nu\tilde{n}.\sigma \vdash x\sigma}\textsc{DFrame} \qquad \frac{\nu\tilde{n}.\sigma \vdash t_1 \cdots \nu\tilde{n}.\sigma \vdash t_n \quad f \in \Sigma^k}{\nu\tilde{n}.\sigma \vdash f(t_1,\ldots,t_n)}\textsc{DAppl}$$

Figure 2: Deduction rules.

$$\frac{\dfrac{\dfrac{x_1 \in \mathbf{D}(\sigma)}{\nu\tilde{n}.\sigma \vdash \mathit{senc}(k_2,k_1)} \quad \dfrac{x_2 \in \mathbf{D}(\sigma)}{\nu\tilde{n}.\sigma \vdash k_1}}{\nu\tilde{n}.\sigma \vdash \mathit{sdec}(\mathit{senc}(k_2,k_1),k_1)} \quad \mathit{sdec}(\mathit{senc}(k_2,k_1),k_1) =_E k_2}{\nu\tilde{n}.\sigma \vdash k_2}$$

Figure 3: Proof tree witnessing that $\nu\tilde{n}.\sigma \vdash k_2$

example show, lookups appear often after input was received. If lookup were to use pattern matching, the following process

$$P = \mathsf{in}(c,x);\ \mathsf{lookup}\ \text{‘store’}\ \mathsf{as}\ x\ \mathsf{in}\ P'$$

might unexpectedly perform a check if ‘store’ contains the message given by the adversary, instead of binding the content of ‘store’ to x, due to an undetected clash in the naming of variables.

A process is ground if it does not contain any free variables. We denote by $P\sigma$ the application of the homomorphic extension of the substitution σ to P. As usual we suppose that the substitution only applies to free variables. We sometimes interpret the syntax tree of a process as a term and write $P|_p$ to refer to the subprocess of P at position p (where $|$, if and lookup are interpreted as binary symbols, all other constructs as unary).

3.2 Semantics

Frames and deduction Before giving the formal semantics of our calculus we introduce the notions of frame and deduction. A *frame* consists of a set of fresh names $\tilde{n}$ and a substitution σ and is written $\nu\tilde{n}.\sigma$. Intuitively a frame represents the sequence of messages that have been observed by an adversary during a protocol execution and secrets $\tilde{n}$ generated by the protocol, a priori unknown to the adversary. Deduction models the capacity of the adversary to compute new messages from the observed ones.

Definition 1 (Deduction). *We define the deduction relation $\nu\tilde{n}.\sigma \vdash t$ as the smallest relation between frames and terms defined by the deduction rules in Figure 2.*

Example. If one key is used to wrap a second key, then, if the intruder learns the first key, he can deduce the second. For $\tilde{n} = k_1, k_2$ and $\sigma = \{ {}^{\mathit{senc}(k_2,k_1)}/_{x_1}, {}^{k_1}/_{x_2} \}$, $\nu\tilde{n}.\sigma \vdash k_2$, as witnessed by the proof tree given in Figure 3.

Operational semantics We can now define the operational semantics of our calculus. The semantics is defined by a labelled transition relation between process configurations. A *process configuration* is a 6-tuple $(\mathcal{E}, \mathcal{S}, \mathcal{S}^{\mathrm{MS}}, \mathcal{P}, \sigma, \mathcal{L})$ where

- $\mathcal{E} \subseteq \mathit{FN}$ is the set of fresh names generated by the processes;
- $\mathcal{S} : \mathcal{M}_\Sigma \to \mathcal{M}_\Sigma$ is a partial function modeling the functional store;
- $\mathcal{S}^{\mathrm{MS}} \subseteq \mathcal{G}^{\#}$ is a multiset of ground facts and models the multiset of stored facts;
- $\mathcal{P}$ is a multiset of ground processes representing the processes executed in parallel;
- σ is a ground substitution modeling the messages output to the environment;

Standard operations:

$$
\begin{array}{rcl}
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{0\},\sigma,\mathcal{L}) & \longrightarrow & (\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P},\sigma,\mathcal{L}) \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{P|Q\},\sigma,\mathcal{L}) & \longrightarrow & (\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{P,Q\},\sigma,\mathcal{L}) \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{!P\},\sigma,\mathcal{L}) & \longrightarrow & (\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{!P,P\},\sigma,\mathcal{L}) \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{\nu a;P\},\sigma,\mathcal{L}) & \longrightarrow & (\mathcal{E}\cup\{a'\},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{P\{a'/a\}\},\sigma,\mathcal{L}) \\
 & & \hfill\text{if } a' \text{ is fresh} \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P},\sigma,\mathcal{L}) & \xrightarrow{K(M)} & (\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P},\sigma,\mathcal{L}) \quad \text{if } \nu\mathcal{E}.\sigma\vdash M \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{\mathrm{out}(M,N);P\},\sigma,\mathcal{L}) & \xrightarrow{K(M)} & (\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{P\},\sigma\cup\{{}^{N}/_{x}\},\mathcal{L}) \\
 & & \hfill\text{if } x \text{ is fresh and } \nu\mathcal{E}.\sigma\vdash M \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{\mathrm{in}(M,N);P\},\sigma,\mathcal{L}) & \xrightarrow{K(\langle M,N\tau\rangle)} & (\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{P\tau\},\sigma,\mathcal{L}) \\
 & \multicolumn{2}{r}{\text{if } \exists\tau.\ \tau \text{ is grounding for } N,\nu\mathcal{E}.\sigma\vdash M,\nu\mathcal{E}.\sigma\vdash N\tau} \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{\mathrm{out}(M,N);P,\mathrm{in}(M',N');Q\},\sigma,\mathcal{L}) & \longrightarrow & (\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup\{P,Q\tau\},\sigma,\mathcal{L}) \\
 & \multicolumn{2}{r}{\text{if } M=_E M' \text{ and } \exists\tau.\ N=_E N'\tau \text{ and } \tau \text{ grounding for } N'} \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup\{\text{if } M=N \text{ then } P \text{ else } Q\},\sigma,\mathcal{L}) & \longrightarrow & (\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup\{P\},\sigma,\mathcal{L}) \quad \text{if } M=_E N \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup\{\text{if } M=N \text{ then } P \text{ else } Q\},\sigma,\mathcal{L}) & \longrightarrow & (\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup\{Q\},\sigma,\mathcal{L}) \quad \text{if } M\neq_E N \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup\{\mathrm{event}(F);\ P\},\sigma,\mathcal{L}) & \xrightarrow{F} & (\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup\{P\},\sigma,\mathcal{L})
\end{array}
$$

Operations on global state:

$$
\begin{array}{rcl}
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{\text{insert } M,N;\ P\},\sigma,\mathcal{L}) & \longrightarrow & (\mathcal{E},\mathcal{S}[M\mapsto N],\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{P\},\sigma,\mathcal{L}) \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{\text{delete } M;\ P\},\sigma,\mathcal{L}) & \longrightarrow & (\mathcal{E},\mathcal{S}[M\mapsto\bot],\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{P\},\sigma,\mathcal{L}) \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{\text{lookup } M \text{ as } x \text{ in } P \text{ else } Q\ \},\sigma,\mathcal{L}) & \longrightarrow & (\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{P\{V/x\}\},\sigma,\mathcal{L}) \\
 & & \hfill\text{if } \mathcal{S}(N)=_E V \text{ is defined and } N=_E M \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{\text{lookup } M \text{ as } x \text{ in } P \text{ else } Q\ \},\sigma,\mathcal{L}) & \longrightarrow & (\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{Q\},\sigma,\mathcal{L}) \\
 & & \hfill\text{if } \mathcal{S}(N) \text{ is undefined for all } N=_E M \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{\text{lock } M;\ P\},\sigma,\mathcal{L}) & \longrightarrow & (\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{P\},\sigma,\mathcal{L}\cup\{\,M\,\}) \\
 & & \hfill\text{if } M\not\in_E\mathcal{L} \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{\text{unlock } M;\ P\},\sigma,\mathcal{L}) & \longrightarrow & (\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{P\},\sigma,\mathcal{L}\setminus\{\,M' \mid M'=_E M\,\}) \\
(\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}},\mathcal{P}\cup^{\#}\{[l \mathrel{-\!\![a]\!\!\rightarrow} r];\ P\},\sigma,\mathcal{L}) & \xrightarrow{a'} & (\mathcal{E},\mathcal{S},\mathcal{S}^{\mathrm{MS}}\setminus \mathit{lfacts}(l')\cup^{\#} r',\mathcal{P}\cup^{\#}\{\,P\tau\,\},\sigma,\mathcal{L}) \\
\text{if } \exists\tau,l',a',r'. & \multicolumn{2}{l}{\tau \text{ grounding for } l \mathrel{-\!\![a]\!\!\rightarrow} r, l' \mathrel{-\!\![a']\!\!\rightarrow} r' =_E (l \mathrel{-\!\![a]\!\!\rightarrow} r)\tau,} \\
 & \multicolumn{2}{l}{\mathit{lfacts}(l')\subseteq^{\#}\mathcal{S}^{\mathrm{MS}},\mathit{pfacts}(l')\subset\mathcal{S}^{\mathrm{MS}}}
\end{array}
$$

Figure 4: Operational semantics

- $\mathcal{L}\subseteq\mathcal{M}_\Sigma$ is the set of currently acquired locks.

The transition relation is defined by the rules described in Figure 4. Transitions are labelled by sets of ground facts. For readability we omit empty sets and brackets around singletons, i.e., we write $\rightarrow$ for $\xrightarrow{\emptyset}$ and $\xrightarrow{f}$ for $\xrightarrow{\{f\}}$. We write $\rightarrow^*$ for the reflexive, transitive closure of $\rightarrow$ (the transitions that are labelled by the empty sets) and write $\overset{f}{\Rightarrow}$ for $\rightarrow^*\xrightarrow{f}\rightarrow^*$. We can now define the set of traces, i.e., possible executions, that a process admits.

Definition 2 (Traces of P)**.** *Given a ground process P we define the* set of traces of P *as*

$$
\begin{aligned}
\mathit{traces}^{pi}(P) = \Big\{[F_1,\ldots,F_n] \mid\ & (\emptyset,\emptyset,\emptyset,\{P\},\emptyset,\emptyset) \\
& \xRightarrow{F_1} (\mathcal{E}_1,\mathcal{S}_1,\mathcal{S}_1^{\mathrm{MS}},\mathcal{P}_1,\sigma_1,\mathcal{L}_1) \\
& \xRightarrow{F_2} \ldots \xRightarrow{F_n} (\mathcal{E}_n,\mathcal{S}_n,\mathcal{S}_n^{\mathrm{MS}},\mathcal{P}_n,\sigma_n,\mathcal{L}_n)\Big\}
\end{aligned}
$$

Example. In Figure 5 we display the transitions that illustrate how the first key is created on the security device in our running example and witness that $[\mathrm{NewKey}(h',k')]\in\mathit{traces}^{pi}(P)$.

$$\begin{aligned}
&(\emptyset,\emptyset,\emptyset,\{\,\underbrace{!P_{new}, !P_{set} \mid !P_{dec} \mid !P_{wrap}}_{=:\mathcal{P}'}\,\}^{\#},\emptyset,\emptyset) \rightarrow (\emptyset,\emptyset,\emptyset,\{\,P_{new}\,\}^{\#} \cup^{\#} \mathcal{P}',\emptyset,\emptyset)\\
\rightarrow\ &(\emptyset,\emptyset,\emptyset,\{\,\nu h; \nu k; \text{event NewKey}(h,k); \ldots\,\}^{\#} \cup^{\#} \mathcal{P}',\emptyset,\emptyset)\\
\rightarrow^{*}&(\{\,h',k'\,\},\emptyset,\emptyset,\{\,\text{event NewKey}(h',k'); \ldots\,\}^{\#} \cup^{\#} \mathcal{P}',\emptyset,\emptyset)\\
\xrightarrow{\text{NewKey}(h',k')}\ &(\{\,h',k'\,\},\emptyset,\emptyset,\{\,\text{insert } \langle\text{‘key’}, h'\rangle, k'; \ldots\,\}^{\#} \cup^{\#} \mathcal{P}',\emptyset,\emptyset)\\
\rightarrow^{*}&(\{\,h',k'\,\},\mathcal{S},\emptyset,\{\,\text{out}(h');0\,\}^{\#} \cup^{\#} \mathcal{P}',\emptyset,\emptyset) \rightarrow^{*} (\{\,h',k'\,\},\mathcal{S},\emptyset,\mathcal{P}',\{\,{}^{h'}/_{x_1}\,\},\emptyset)\\
&\quad \text{where } \mathcal{S}(\langle\text{‘key’},h'\rangle) = k' \text{ and } \mathcal{S}(\langle\text{‘att’},h'\rangle) = \text{‘dec’}.
\end{aligned}$$

Figure 5: Example of transitions modelling the creation of a key on a PKCS#11-like device

4 Labelled multiset rewriting

We now recall the syntax and semantics of labelled multiset rewriting rules, which constitute the input language of the tamarin tool [25].

Definition 3 (Multiset rewrite rule)**.** *A labelled multiset rewrite rule ri is a triple (l, a, r), $l, a, r \in \mathcal{F}^*$, written $l \;\text{-}[a]\!\rightarrow r$. We call $l = prems(ri)$ the premises, $a = actions(ri)$ the actions, and $r = conclusions(ri)$ the conclusions of the rule.*

Definition 4 (Labelled multiset rewriting system)**.** *A labelled multiset rewriting system is a set of labelled multiset rewrite rules R, such that each rule $l \;\text{-}[a]\!\rightarrow r \in R$ satisfies the following conditions:*

- *l, a, r do not contain fresh names*
- *r does not contain* Fr*-facts*

A labelled multiset rewriting system is called well-formed, if additionally

- *for each $l' \;\text{-}[a']\!\rightarrow r' \in_E ginsts(l \;\text{-}[a]\!\rightarrow r)$ we have that $\cap_{r''=_E r'} names(r'') \cap FN \subseteq \cap_{l''=_E l'} names(l'') \cap FN$.*

We define one distinguished rule Fresh which is the only rule allowed to have Fr-facts on the right-hand side

$$\textsc{Fresh} : [] \;\text{-}[]\!\rightarrow [\mathsf{Fr}(x : \mathit{fresh})]$$

The semantics of the rules is defined by a labelled transition relation.

Definition 5 (Labelled transition relation)**.** *Given a multiset rewriting system R we define the* labeled transition relation *$\rightarrow_R \subseteq \mathcal{G}^{\#} \times \mathcal{P}(\mathcal{G}) \times \mathcal{G}^{\#}$ as*

$$S \xrightarrow{a}_R ((S \setminus^{\#} \mathit{lfacts}(l)) \cup^{\#} r)$$

if and only if $l \;\text{-}[a]\!\rightarrow r \in_E ginsts(R \cup \textsc{Fresh})$, $\mathit{lfacts}(l) \subseteq^{\#} S$ and $\mathit{pfacts}(l) \subseteq S$.

Definition 6 (Executions)**.** *Given a multiset rewriting system R we define its set of executions as*

$$\begin{aligned}
exec^{msr}(R) = \Big\{&\emptyset \xrightarrow{A_1}_R \ldots \xrightarrow{A_n}_R S_n \mid\\
&\forall a, i, j \colon 0 \le i \ne j < n.\\
&(S_{i+1} \setminus^{\#} S_i) = \{\mathsf{Fr}(a)\} \Rightarrow (S_{j+1} \setminus^{\#} S_j) \ne \{\mathsf{Fr}(a)\}\Big\}
\end{aligned}$$

The set of executions consists of transition sequences that respect freshness, i. e., for a given name a the fact $\mathsf{Fr}(a)$ is only added once, or in other words the rule Fresh is at most fired once for each name. We define the set of traces in a similar way as for processes.

Definition 7 (Traces). *The set of traces is defined as*

$$\begin{aligned} traces^{msr}(R) = \Big\{ [A_1, \ldots, A_n] \mid \ & \forall\, 0 \le i \le n.\ A_i \neq \emptyset \\ & and\ \emptyset \overset{A_1}{\Longrightarrow}_R \ldots \overset{A_n}{\Longrightarrow}_R S_n \in exec^{msr}(R) \Big\} \end{aligned}$$

where $\overset{A}{\Longrightarrow}_R$ *is defined as* $\xrightarrow{\emptyset}{}^{*}_{R} \xrightarrow{A}_{R} \xrightarrow{\emptyset}{}^{*}_{R}$.

Note that both for processes and multiset rewrite rules the set of traces is a sequence of sets of facts.

5 Security Properties

In the tamarin tool [25] security properties are described in an expressive two-sorted first-order logic. The sort *temp* is used for time points, $\mathcal{V}_{temp}$ are the temporal variables.

Definition 8 (Trace formulas). *A trace atom is either false* $\bot$*, a term equality* $t_1 \approx t_2$*, a timepoint ordering* $i \lessdot j$*, a timepoint equality* $i \doteq j$*, or an action* $F@i$ *for a fact* $F \in \mathcal{F}$ *and a timepoint* i*. A trace formula is a first-order formula over trace atoms.*

As we will see in our case studies this logic is expressive enough to analyze a variety of security properties, including complex injective correspondence properties.

To define the semantics, let each sort s have a domain $\mathbf{D}(s)$. $\mathbf{D}(temp) = \mathcal{Q}$, $\mathbf{D}(msg) = \mathcal{M}$, $\mathbf{D}(fresh) = FN$, and $\mathbf{D}(pub) = PN$. A function $\theta : \mathcal{V} \to \mathcal{M} \cup \mathcal{Q}$ is a valuation if it respects sorts, that is, $\theta(\mathcal{V}_s) \subset \mathbf{D}(s)$ for all sorts s. If t is a term, $t\theta$ is the application of the homomorphic extension of θ to t.

Definition 9 (Satisfaction relation). *The satisfaction relation* $(tr, \theta) \vDash \varphi$ *between trace tr, valuation* θ *and trace formula* φ *is defined as follows:*

$$\begin{array}{lll} (tr,\theta) \vDash \bot & never & \\ (tr,\theta) \vDash F@i & iff & \theta(i) \in idx(tr)\ and\ F\theta \in_E tr_{\theta(i)} \\ (tr,\theta) \vDash i \lessdot j & iff & \theta(i) < \theta(j) \\ (tr,\theta) \vDash i \doteq j & iff & \theta(i) = \theta(j) \\ (tr,\theta) \vDash t_1 \approx t_2 & iff & t_1\theta =_E t_2\theta \\ (tr,\theta) \vDash \neg\varphi & iff & not\ (tr,\theta) \vDash \varphi \\ (tr,\theta) \vDash \varphi_1 \wedge \varphi_2 & iff & (tr,\theta) \vDash \varphi_1\ and\ (tr,\theta) \vDash \varphi_2 \\ (tr,\theta) \vDash \exists x : s.\varphi & iff & there\ is\ u \in \mathbf{D}(s)\ such\ that \\ & & (tr, \theta[x \mapsto u]) \vDash \varphi \end{array}$$

When φ is a ground formula we sometimes simply write $tr \vDash \varphi$ as the satisfaction of φ is independent of the valuation.

Definition 10 (Validity, satisfiability). *Let* $Tr \subseteq (\mathcal{P}(\mathcal{G}))^*$ *be a set of traces. A trace formula* φ *is said to be* valid *for Tr, written* $Tr \vDash^{\forall} \varphi$*, if for any trace* $tr \in Tr$ *and any valuation* θ *we have that* $(tr, \theta) \vDash \varphi$*.*

A trace formula φ *is said to be* satisfiable *for Tr, written* $Tr \vDash^{\exists} \varphi$*, if there exist a trace* $tr \in Tr$ *and a valuation* θ *such that* $(tr, \theta) \vDash \varphi$*.*

Note that $Tr \vDash^{\forall} \varphi$ iff $Tr \nvDash^{\exists} \neg\varphi$. Given a multiset rewriting system R we say that φ is valid, written $R \vDash^{\forall} \varphi$, if $traces^{msr}(R) \vDash^{\forall} \varphi$. We say that φ is satisfied in R, written $R \vDash^{\exists} \varphi$, if $traces^{msr}(R) \vDash^{\exists} \varphi$. Similarly, given a ground process P we say that φ is valid, written $P \vDash^{\forall} \varphi$, if $traces^{pi}(P) \vDash^{\forall} \varphi$, and that φ is satisfied in P, written $P \vDash^{\exists} \varphi$, if $traces^{pi}(P) \vDash^{\exists} \varphi$.

Example. The following trace formula expresses secrecy of keys generated on the security API, which we introduced in Section 3.

$$\neg(\exists h, k\colon msg,\ i, j\colon temp.\ \mathrm{NewKey}(h,k)@i \wedge \mathrm{K}(k)@j)$$

$$
\begin{array}{rcll}
\mathsf{Out}(x) & -[\ ]\!\!\rightarrow & \mathsf{!K}(x) & \text{(MDOut)}\\
\mathsf{!K}(x) & -[K(x)]\!\!\rightarrow & \mathsf{In}(x) & \text{(MDIn)}\\
 & -[\ ]\!\!\rightarrow & \mathsf{!K}(x : \mathit{pub}) & \text{(MDPub)}\\
\mathsf{Fr}(x : \mathit{fresh}) & -[\ ]\!\!\rightarrow & \mathsf{!K}(x : \mathit{fresh}) & \text{(MDFresh)}\\
\mathsf{!K}(x_1), \ldots, \mathsf{!K}(x_k) & -[\ ]\!\!\rightarrow & \mathsf{!K}(f(x_1, \ldots, x_k)) \text{ for } f \in \Sigma^k & \text{(MDAppl)}
\end{array}
$$

Figure 6: The set of rules MD.

6 A translation from processes into multiset rewrite rules

In this section we define a translation from a process P into a set of multiset rewrite rules $[\![P]\!]$ and a translation on trace formulas such that $P \models^{\forall} \varphi$ if and only if $[\![P]\!] \models^{\forall} [\![\varphi]\!]$. Note that the result also holds for satisfiability, as an immediate consequence. For a rather expressive subset of trace formulas (see [25] for the exact definition of the fragment), checking whether $[\![P]\!] \models^{\forall} [\![\varphi]\!]$ can then be discharged to the tamarin prover that we use as a backend.

6.1 Definition of the translation of processes

To model the adversary's message deduction capabilities, we introduce the set of rules MD defined in Figure 6.

In order for our translation to be correct, we need to make some assumptions on the set of processes we allow. These assumptions are however, as we will see, rather mild and most of them without loss of generality. First we define a set of reserved variables that will be used in our translation and whose use we therefore forbid in the processes.

Definition 11 (Reserved variables and facts)**.** *The set of reserved variables is defined as the set containing the elements n_a for any $a \in FN$ and $lock_l$ for any $l \in \mathbb{N}$.*

The set of reserved facts $\mathcal{F}_{res}$ is defined as the set containing facts $f(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n \in \mathcal{T}$ and $f \in \{$ Init, Insert, Delete, IsIn, IsNotSet, state, Lock, Unlock, Out, Fr, In, Msg, ProtoNonce, Eq, NotEq, Event, InEvent $\}$.

Similar to [3], for our translation to be sound, we require that for each process, there exists an injective mapping assigning to every $\mathsf{unlock}\ t$ in a process a $\mathsf{lock}\ t$ that precedes it in the process' syntax tree. Moreover, given a process $\mathsf{lock}\ t; P$ the corresponding unlock in P may not be under a parallel or replication. These conditions allow us to annotate each corresponding pair $\mathsf{lock}\ t$, $\mathsf{unlock}\ t$ with a unique label l. The annotated version of a process P is denoted $\overline{P}$. The formal definition of $\overline{P}$ is given in Appendix A. In case the annotation fails, i.e., P violates one of the above conditions, the process $\overline{P}$ contains $\bot$.

Definition 12 (well-formed)**.** *A ground process P is well-formed if*

- *no reserved variable nor reserved fact appear in P,*
- *any name and variable in P is bound at most once and*
- *$\overline{P}$ does not contain $\bot$.*
- *For each action $l -[a]\!\!\rightarrow r$ that appears in the process, the following holds: for each $l' -[a']\!\!\rightarrow r' \in_E ginsts(l -[a]\!\!\rightarrow r)$ we have that $\cap_{r''=_E r'} names(r'') \cap FN \subseteq \cap_{l''=_E l'} names(l'') \cap FN$.*

A trace formula φ is well-formed if no reserved variable nor reserved fact appear in φ.

The two first restrictions of well-formed processes are not a loss of generality as processes and formulas can be consistently renamed to avoid reserved variables and α-converted to avoid binding names or variables several times. Also note that the second condition is not necessarily preserved during an execution, e.g. when unfolding a replication, $!P$ and P may bind the same names. We only require this condition to hold on the initial process for our translation to be correct.

The annotation of locks restricts the set of protocols we can translate, but allows us to obtain better verification results, since we can predict which unlock is "supposed" to close a given lock. This additional information is helpful for tamarin's backward reasoning. We think that our locking mechanism captures all practical use cases. Using our calculus' "low-level" multiset manipulation construct, the user is also free to implement locks himself, e.g., as

$$[\mathrm{NotLocked}()] \to []; \mathit{code}; [] \to [\mathrm{NotLocked}()]$$

(In this case the user does not benefit from the optimisation we put into the translation of locks.) Obviously, locks can be modelled both in tamarin's multiset rewriting calculus (this is actually what the translation does) and Mödersheim's set rewriting calculus [22]. However, protocol steps typically consist of a single input, followed by several database lookups, and finally an output. In practice, they tend to be modelled as a single rule, and are therefore atomic. Real implementations are however different, as several entities might be involved, database lookups could be slow, etc. In this case, such simplified models could, e. g., miss race conditions. To the best of our knowledge, StatVerif is the only comparable tool that models locks explicitly and it has stronger restrictions.

Definition 13. *Given a well-formed ground process P we define the labelled multiset rewriting system $[\![P]\!]$ as*

$$\mathrm{MD} \cup \{\textsc{Init}\} \cup [\![\overline{P}, [], []]\!]$$

- *where the rule* $\textsc{Init}$ *is defined as*

$$\textsc{Init} : [] \xrightarrow{[\mathrm{Init}()]} [\mathsf{state}_{[]}()]$$

- $[\![P, p, \tilde{x}]\!]$ *is defined inductively for process P, position $p \in \mathbb{N}^*$ and sequence of variables $\tilde{x}$ in Figure 7.*

- *For a position p of P we define state_p to be persistent if $P|_p = !Q$ for some process Q; otherwise state_p is linear.*

In the definition of $[\![P, p, \tilde{x}]\!]$ we intuitively use the family of facts state_p to indicate that the process is currently at position p in its syntax tree. A fact state_p will indeed be true in an execution of these rules whenever some instance of P_p (i.e. the process defined by the subtree at position p of the syntax tree of P) is in the multiset $\mathcal{P}$ of the process configuration. The translation of the zero-process, parallel and replication operators merely use state_p-facts. For instance $[\![P \mid Q, p, \tilde{x}]\!]$ defines the rule

$$[\mathsf{state}_p(\tilde{x})] \to [\mathsf{state}_{p\cdot 1}(\tilde{x}), \mathsf{state}_{p\cdot 2}(\tilde{x})]$$

which intuitively states that when a process is at position p (modelled by the fact $\mathsf{state}_p(\tilde{x})$ being true) then the process is allowed to move both to P (putting $\mathsf{state}_{p\cdot 1}(\tilde{x})$ to true) and Q (putting $\mathsf{state}_{p\cdot 2}(\tilde{x})$ to true). The translation of $[\![P \mid Q, p, \tilde{x}]\!]$ also contains the set of rules $[\![P, p \cdot 1, \tilde{x}]\!] \cup [\![Q, p \cdot 2, \tilde{x}]\!]$ expressing that after this transition the process may behave as P and Q, i.e., the processes at positions $p \cdot 1$, respectively $p \cdot 2$, in the process tree. Also note that the translation of $!P$ results in a persistent fact as $!P$ always remains in $\mathcal{P}$. The translation of the construct $\nu\, a$ translates the name a into a variable n_a, as msr rules must not contain fresh names. Any instantiation of this rule will substitute n_a by a fresh name, which the Fr-fact in the premise guarantees to be new. This step is annotated with a (reserved) action *ProtoNonce*, used in the proof of correctness to distinguish adversary and protocol nonces. Note that the fact $\mathsf{state}_{p\cdot 1}$ in the conclusion carries n_a, so that the following protocol steps are bound to the fresh name used to instantiate n_a. The first rules of the translation of out and in model the communication between the protocol and the adversary, and vice versa. In the case of out, the adversary must know the channel M, modelled by the fact $\mathsf{In}(M)$ in the rule's premisse, and learns the output message, modelled by the fact $\mathsf{Out}(N)$ in the conclusion. In the case of in, the knowledge of the message N is additionally required and the variables of the input message are added to the parameters of the state fact to reflect that these variables are bound. The second and third rules of the translations of out and in model an internal communication, which is synchronous. For

$$
\begin{aligned}
\llbracket 0, p, \tilde{x} \rrbracket &= \{[\mathsf{state}_p(\tilde{x})] \rightarrow []\} \\
\llbracket P \mid Q, p, \tilde{x} \rrbracket &= \{[\mathsf{state}_p(\tilde{x})] \rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x}), \mathsf{state}_{p\cdot 2}(\tilde{x})]\} \cup \llbracket P, p\cdot 1, \tilde{x} \rrbracket \cup \llbracket Q, p\cdot 2, \tilde{x} \rrbracket \\
\llbracket !P, p, \tilde{x} \rrbracket &= \{[!\mathsf{state}_p(\tilde{x})] \rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x})]\} \cup \llbracket P, p\cdot 1, \tilde{x} \rrbracket \\
\llbracket \nu a; P, p, \tilde{x} \rrbracket &= \ \{[\mathsf{state}_p(\tilde{x}), \mathsf{Fr}(n_a : \mathit{fresh})] \ -[\mathit{ProtoNonce}(n_a : \mathit{fresh})]\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x}, n_a : \mathit{fresh})]\} \\
&\quad \cup \llbracket P, p\cdot 1, (\tilde{x}, n_a : \mathit{fresh}) \rrbracket \\
\llbracket \mathrm{Out}(M, N); P, p, \tilde{x} \rrbracket &= \ \{[\mathsf{state}_p(\tilde{x}), \mathsf{In}(M)] \ -[\mathrm{InEvent}(M)]\!\rightarrow [\mathsf{Out}(N), \mathsf{state}_{p\cdot 1}(\tilde{x})], \\
&\quad\ [\mathsf{state}_p(\tilde{x})] \rightarrow [\mathsf{Msg}(M, N), \mathsf{state}_p^{\mathsf{semi}}(\tilde{x})], \\
&\quad\ [\mathsf{state}_p^{\mathsf{semi}}(\tilde{x}), \mathsf{Ack}(M, N)] \rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x})] \ \} \cup \llbracket P, p\cdot 1, \tilde{x} \rrbracket \\
\llbracket \mathrm{In}(M, N); P, p, \tilde{x} \rrbracket &= \ \{[\mathsf{state}_p(\tilde{x}), \mathsf{In}(\langle M, N\rangle)] \ -[\mathrm{InEvent}(\langle M, N\rangle)]\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x} \cup \mathit{vars}(N))], \\
&\quad\ [\mathsf{state}_p(\tilde{x}), \mathsf{Msg}(M, N)] \rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x} \cup \mathit{vars}(N)), \mathsf{Ack}(M, N)]\} \\
&\quad \cup \llbracket P, p\cdot 1, \tilde{x} \cup \mathit{vars}(N) \rrbracket \\
\llbracket \text{if } M = N \text{ then } P &= \{[\mathsf{state}_p(\tilde{x})] \ -[\quad \mathrm{Eq}(M, N) \quad]\!\rightarrow \quad [\mathsf{state}_{p\cdot 1}(\tilde{x})], \\
\text{else } Q, p, \tilde{x} \rrbracket &\quad\ [\mathsf{state}_p(\tilde{x})] \ -[\mathrm{NotEq}(M, N)]\!\rightarrow \quad [\mathsf{state}_{p\cdot 2}(\tilde{x})]\} \\
&\quad \cup \llbracket P, p\cdot 1, \tilde{x} \rrbracket \cup \llbracket Q, p\cdot 2, \tilde{x} \rrbracket \\
\llbracket \text{event } F; P, p, \tilde{x} \rrbracket &= \{[\mathsf{state}_p(\tilde{x})] \ -[\mathrm{Event}(), F]\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x})]\} \cup \llbracket P, p\cdot 1, \tilde{x} \rrbracket \\
\llbracket \text{insert } s, t; P, p, \tilde{x} \rrbracket &= \{[\mathsf{state}_p(\tilde{x})] \ -[\mathrm{Insert}(s, t)]\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x})]\} \cup \llbracket P, p\cdot 1, \tilde{x} \rrbracket \\
\llbracket \text{delete } s; P, p, \tilde{x} \rrbracket &= \ \{[\mathsf{state}_p(\tilde{x})] \ -[\mathrm{Delete}(s)]\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x})]\} \cup \llbracket P, p\cdot 1, \tilde{x} \rrbracket \\
\llbracket \text{lookup } M \text{ as } v &= \ \{[\mathsf{state}_p(\tilde{x})] \ -[\mathrm{IsIn}(M, v)]\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{M}, v)], \\
\text{in } P \text{ else } Q, p, \tilde{x} \rrbracket &\quad\ [\mathsf{state}_p(\tilde{x})] \ -[\mathrm{IsNotSet}(M)]\!\rightarrow [\mathsf{state}_{p\cdot 2}(\tilde{x})]\} \\
&\quad \cup \llbracket P, p\cdot 1, (\tilde{x}, v) \rrbracket \cup \llbracket Q, p\cdot 2, \tilde{x} \rrbracket \\
\llbracket \text{lock}^l \ s; P, p, \tilde{x} \rrbracket &= \ \{[\mathrm{Fr}(\mathrm{lock}_l), \mathsf{state}_p(\tilde{x})] \ -[\mathrm{Lock}(\mathit{lock}_l, s)]\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x}, \mathit{lock}_l)]\} \\
&\quad \cup \llbracket P, p\cdot 1, \tilde{x} \rrbracket \\
\llbracket \text{unlock}^l \ s; P, p, \tilde{x} \rrbracket &= \{[\mathsf{state}_p(\tilde{x})] \ -[\mathrm{Unlock}(\mathit{lock}_l, s)]\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x})]\} \cup \llbracket P, p\cdot 1, \tilde{x} \rrbracket \\
\llbracket [l \ -[a]\!\rightarrow r]; \mathrm{P}, p, \tilde{x} \rrbracket &= \{[\mathsf{state}_p(\tilde{x}), l] \ -[\mathrm{Event}(), a]\!\rightarrow [r, \mathsf{state}_{p\cdot 1}(\tilde{x} \cup \mathit{vars}(l))]\} \\
&\quad \cup \llbracket P, p\cdot 1, \tilde{x} \cup \mathit{vars}(l) \rrbracket
\end{aligned}
$$

Figure 7: Translation of processes: definition of $\llbracket P, p, \tilde{x} \rrbracket$

$$
\begin{array}{rcl}
[\,] & -\![\mathrm{Init}()]\!\!\rightarrow & [\mathsf{state}_{[]}()] \\
[\mathsf{state}_{[]}()] & -\![\;]\!\!\rightarrow & [!\mathsf{state}_{[1]}()] \\
[!\mathsf{state}_{[1]}(), \mathsf{Fr}(h)] & -\![\;]\!\!\rightarrow & [\mathsf{state}_{[11]}(h)] \\
[\mathsf{state}_{[11]}(h), \mathsf{Fr}(k)] & -\![\;]\!\!\rightarrow & [\mathsf{state}_{[111]}(k,h)] \\
[\mathsf{state}_{[111]}(k,h)] & -\![\mathrm{Event}(), \mathrm{NewKey}(h,k)]\!\!\rightarrow & [\mathsf{state}_{[1111]}(k,h)] \\
[\mathsf{state}_{[1111]}(k,h)] & -\![\mathrm{Insert}(\langle\text{'key'}, h\rangle, k)]\!\!\rightarrow & [\mathsf{state}_{[11111]}(k,h)] \\
[\mathsf{state}_{[11111]}(k,h)] & -\![\mathrm{Insert}(\langle\text{'att'}, h\rangle, \text{'dec'})]\!\!\rightarrow & [\mathsf{state}_{[111111]}(k,h)] \\
[\mathsf{state}_{[111111]}(k,h)] & -\![\;]\!\!\rightarrow & [\mathsf{Out}(h), \mathsf{state}_{[1111111]}(k,h)]
\end{array}
$$

Figure 8: The set of multiset rewrite rules $[\![!P_{new}]\!]$ (omitting the rules in MD)

this reason, when the second rule of the translation of out is fired, the state-fact is substituted by an intermediate, *semi-state* fact, $\mathsf{state}^{\mathsf{semi}}$, reflecting that the sending process can only execute the next step if the message was successfully received. The fact $\mathsf{Msg}(M,N)$ models that a message is present on the synchronous channel. Only with the acknowledgement fact $\mathsf{Ack}(M,N)$, resulting from the second rule of the translation of in, is it possible to advance the execution of the sending process, using the third rule in the translation of out, which transforms the semi-state *and* the acknowledgement of receipt into $\mathsf{state}_{p\cdot 1}(\ldots)$. Only now the next step in the execution of the sending process can be executed. The remaining rules essentially update the position in the state facts and add labels. Some of these labels are used to restrict the set of executions. For instance the label $\mathrm{Eq}(M,N)$ will be used to indicate that we only consider executions in which $M =_E N$. As we will see in the next section these restrictions will be encoded in the trace formula.

Example. Figure 8 illustrates the above translation by presenting the set of msr rules $[\![!P_{new}]\!]$ (omitting the rules in MD already shown in Figure 6).

A graph representation of an example trace, generated by the tamarin tool, is depicted in Figure 9. Every box in this picture stands for the application of a multiset rewrite rule, where the premises are at the top, the conclusions at the bottom, and the actions (if any) in the middle. Every premise needs to have a matching conclusion, visualized by the arrows, to ensure the graph depicts a valid msr execution. (This is a simplification of the dependency graph representation tamarin uses to perform backward-induction [25, 26].) Note that the machine notation for $\mathsf{state}_p()$ predicates omits brackets $[\,]$ in the position p and denotes the empty sequence by '0'. We also note that in the current example $!\mathsf{state}_{[1]}()$ is persistent and can therefore be used multiple times as a premise. As $\mathsf{Fr}(\,)$ facts are generated by the Fresh rule which has an empty premise and action, we omit instances of Fresh and leave those premises, but only those, disconnected.

Remark 1. *One may note that, while for all other operators, the translation produces well-formed multiset rewriting rules (as long as the process is well-formed itself), this is not the case for the translation of the* lookup *operator, i. e., it violates the well-formedness condition from Definition 4. Tamarin's constraint solving algorithm requires all rules, with the exception of* Fresh*, to be well-formed. We show however that, under these specific conditions, the solution procedure is still correct. See Appendix B for the proof.*

6.2 Definition of the translation of trace formulas

We can now define the translation for formulas.

Definition 14. *Given a well-formed trace formula φ we define*

$$[\![\varphi]\!]_\forall := \alpha \Rightarrow \varphi \qquad \textit{and} \qquad [\![\varphi]\!]_\exists := \alpha \wedge \varphi$$

where α is defined in Figure 10.

The formula α uses the actions of the generated rules to filter out executions that we wish to discard:

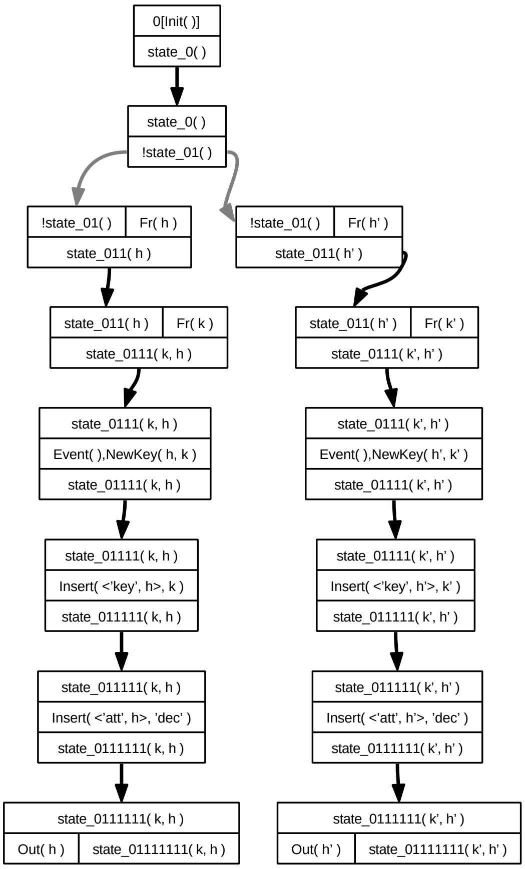

Figure 9: Example trace for the translation of $!P_{new}$.

- α_{init} ensures that the init rule is only fired once.
- α_{eq} and α_{noteq} ensure that we only consider traces where all (dis)equalities hold.
- α_{in} and α_{notin} ensure that a successful lookup was preceded by an insert that was neither revoked nor overwritten while an unsuccessful lookup was either never inserted, or deleted and never reinserted.
- α_{lock} checks that between each two matching locks there must be an unlock. Furthermore, between the first of these locks and the corresponding unlock, there is neither a lock nor an unlock.
- α_{inev} ensures that whenever an instance of MDIn is required to generate an In-fact, it is generated as late as possible, i. e., there is no visible event between the action $K(t)$ produced by MDIn, and a rule that requires $\mathsf{In}(t)$.

We also note that $Tr \vDash^{\forall} [\![\varphi]\!]_{\forall}$ iff $Tr \nvDash^{\exists} [\![\neg\varphi]\!]_{\exists}$.

The axioms in the translation of the formula are designed to work hand in hand with the translation of the process into rules. They express the correctness of traces with respect to our calculus' semantics, but are also meant to guide tamarin's constraint solving algorithm. α_{in} and α_{notin} illustrate what kind of axioms work well: when a node with the action IsIn is created, by definition of the translation, this corresponds to a lookup command. The existential translates into a graph constraint that postulates an insert node for the value fetched by the lookup, and three formulas assuring that *a*) this insert node appears before the lookup, *b*) is uniquely defined, i. e., it is the last insert to the corresponding key, and *c*) there is no delete in between. Due to these conditions, α_{notin} only adds one Insert node per IsIn node – the case where an axiom postulates a node, which itself allows for postulating yet another node needs to be avoided, as tamarin runs into loops otherwise. Similarly, a naïve way of implementing locks using an axiom would postulate that every lock is preceeded by an unlock and no lock or unlock

$\alpha := \alpha_{init} \wedge \alpha_{eq} \wedge \alpha_{noteq} \wedge \alpha_{in} \wedge \alpha_{notin} \wedge \alpha_{lock} \wedge \alpha_{inev}$ and

$$
\begin{aligned}
\alpha_{init} :=& \forall i, j. && \mathrm{Init}()@i \wedge \mathrm{Init}()@j \implies i = j \\
\alpha_{eq} :=& \forall x, y, i. && \mathrm{Eq}(x, y)@i \implies x \approx y \\
\alpha_{noteq} :=& \forall x, y, i. && \mathrm{NotEq}(x, y)@i \implies \neg(x \approx y) \\
\alpha_{in} :=& \forall x, y, t_3. && \mathrm{IsIn}(x, y)@t_3 \implies \exists t_2.\ \mathrm{Insert}(x, y)@t_2 \wedge t_2 \lessdot t_3 \\
&&& \qquad\qquad \wedge \forall t_1, y.\ \mathrm{Insert}(x, y)@t_1 \implies (t_1 \lessdot t_2 \vee t_1 \doteq t_2 \vee t_3 \lessdot t_1) \\
&&& \qquad\qquad \wedge \forall t_1. \quad \mathrm{Delete}(x)@t_1 \implies (t_1 \lessdot t_2 \vee t_3 \lessdot t_1) \\
\alpha_{notin} :=& \forall x, y, t_3. && \mathrm{IsNotSet}(x)@t_3 \implies (\forall t_1, y.\ \mathrm{Insert}(x, y)@t_1 \implies t_3 \lessdot t_1) \vee \\
&&& \qquad\qquad (\exists t_1.\ \mathrm{Delete}(x)@t_1 \wedge t_1 \lessdot t_3 \\
&&& \qquad\qquad\quad \wedge \forall t_2, y.\ (\mathrm{Insert}(x, y)@t_2 \wedge t_2 \lessdot t_3) \implies t_2 \lessdot t_1) \\
\alpha_{lock} :=& \forall x, l, l', i, j. && \mathrm{Lock}(l, x)@i \wedge \mathrm{Lock}(l', x)@j \wedge i \lessdot j \\
&&& \qquad \implies \exists k.\ \mathrm{Unlock}(l, x)@k \wedge i \lessdot k \wedge k \lessdot j \\
&&& \qquad\qquad \wedge (\forall l', m.\ \mathrm{Lock}(l', x)@m \implies \neg(i \lessdot m \wedge m \lessdot k)) \\
&&& \qquad\qquad \wedge (\forall l', m.\ \mathrm{Unlock}(l', x)@m \implies \neg(i \lessdot m \wedge m \lessdot k)) \\
\alpha_{inev} :=& \forall t, i. && \mathrm{InEvent}(t)@i \implies \exists j.\ \mathrm{K}(t)@j \wedge (\forall k.\ \mathrm{Event}()@k \implies (k \lessdot j \vee i \lessdot k)) \\
&&& \qquad\qquad \wedge (\forall k, t'.\ \mathrm{K}(t')@k \implies (k \lessdot j \vee i \lessdot k \vee k \approx j))
\end{aligned}
$$

Figure 10: Definition of α.

in between, unless it is the first lock. This again would cause tamarin to loop, because an unlock is typically preceeded by yet another lock. The axiom α_{lock} avoids this caveat because it only applies to pairs of locks carrying the same annotations.

We will outline how α_{lock} is applied during the constraint solving procedure:

1. If there are two locks for the same term and with possibly different annotations, an unlock for the first of those locks is postulated, more precisely, an unlock with the same term, the same annotation and no lock or unlock for the same term in-between. The axiom itself contains only one case, so the only case distinction that takes place is over which rule produces the matching Unlock-action. However, due to the annotation, all but one are refuted immediately in the next step. Note further that α_{lock} postulates only a single node, namely the node with the action Unlock.

2. Due to the annotation, the fact $\mathsf{state}_p(\ldots)$ contains the fresh name that instantiates the annotation variable. Let $a : \mathit{fresh}$ be this fresh name. Every fact $\mathsf{state}_{p'}(\ldots)$ for some position p' that is a prefix of p and a suffix of the position of the corresponding lock contains this fresh name. Furthermore, every rule instantiation that is an ancestor of a node in the dependency graph corresponds to the execution of a command that is an ancestor in the process tree. Therefore, the backward search eventually reaches the matching lock, including the annotation, which is determined to be a, and hence appears in the Fr-premise.

3. Because of the Fr-premise, any existing subgraph that already contains the first of the two original locks would be merged with the subgraph resulting from the backwards search that we described in the previous step, as otherwise $\mathsf{Fr}(a)$ would be added at two different points in the execution.

4. The result is a sequence of nodes from the first lock to the corresponding unlock, and graph constraints restricting the second lock to not take place between the first lock and the unlock. We note that the axiom α_{lock} is only instantiated once per pair of locks, since it requires that $i \lessdot j$, thereby fixing their order.

In summary, the annotation helps distinguishing which unlock is expected between to locks, vastly improving the speed of the backward search. This optimisation, however, required us to put restrictions on the locks.

6.3 Correctness of the translation

The correctness of our translation is stated by the following theorem.

Theorem 1. *Given a well-formed ground process P and a well-formed trace formula φ we have that*

$$\mathit{traces}^{pi}(P) \vDash^{\star} \varphi \text{ iff } \mathit{traces}^{msr}(\llbracket P \rrbracket) \vDash^{\star} \llbracket \varphi \rrbracket_{\star}$$

where $\star$ is either $\forall$ or $\exists$.

We here give an overview of the main propositions and lemmas needed to prove Theorem 1. To show the result we need two additional definitions. We first define an operation that allows to restrict a set of traces to those that satisfy the trace formula α as defined in Definition 14.

Definition 15. *Let α be the trace formula as defined in Definition 14 and Tr a set of traces. We define*

$$\mathit{filter}(Tr) := \{tr \in Tr \mid \forall\theta.(tr, \theta) \vDash \alpha\}$$

The following proposition states that if a set of traces satisfies the translated formula then the filtered traces satisfy the original formula.

Proposition 1. *Let Tr be a set of traces and φ a trace formula. We have that*

$$Tr \vDash^{\star} \llbracket \varphi \rrbracket_{\star} \text{ iff } \mathit{filter}(Tr) \vDash^{\star} \varphi$$

where $\star$ is either $\forall$ or $\exists$.

The proof (detailed in Appendix) follows directly from the definitions. Next we define the *hiding* operation which removes all reserved facts from a trace.

Definition 16 (*hide*)**.** *Given a trace tr and a set of facts F we inductively define $\mathit{hide}([]) = []$ and*

$$\mathit{hide}(F \cdot tr) := \begin{cases} \mathit{hide}(tr) & \text{if } F \subseteq \mathcal{F}_{res} \\ (F \setminus \mathcal{F}_{res}) \cdot \mathit{hide}(tr) & \text{otherwise} \end{cases}$$

Given a set of traces Tr we define $\mathit{hide}(Tr) = \{\mathit{hide}(t) \mid t \in Tr\}$.

As expected well-formed formulas that do not contain reserved facts evaluate the same whether reserved facts are hidden or not.

Proposition 2. *Let Tr be a set of traces and φ a well-formed trace formula. We have that*

$$Tr \vDash^{\star} \varphi \text{ iff } \mathit{hide}(Tr) \vDash^{\star} \varphi$$

where $\star$ is either $\forall$ or $\exists$.

We can now state our main lemma which is relating the set of traces of a process P and the set of traces of its translation into multiset rewrite rules (proven in the full version).

Lemma 1. *Let P be a well-formed ground process. We have that*

$$\mathit{traces}^{pi}(P) = \mathit{hide}(\mathit{filter}(\mathit{traces}^{msr}(\llbracket P \rrbracket))).$$

Our main theorem can now be proven by applying Lemma 1, Proposition 3 and Proposition 1.

Proof of Theorem 1.

$$\begin{array}{lll} & \mathit{traces}^{pi}(P) \vDash^{\star} \varphi & \\ \Leftrightarrow & \mathit{hide}(\mathit{filter}(\mathit{traces}^{msr}(\llbracket P \rrbracket))) \vDash^{\star} \varphi & \text{by Lemma 1} \\ \Leftrightarrow & \mathit{filter}(\mathit{traces}^{msr}(\llbracket P \rrbracket)) \vDash^{\star} \varphi & \text{by Proposition 3} \\ \Leftrightarrow & \mathit{traces}^{msr}(\llbracket P \rrbracket) \vDash^{\star} \llbracket \varphi \rrbracket_{\star} & \text{by Proposition 1} \end{array}$$

□

7 Case studies

In this section we briefly overview some case studies we performed. These case studies include a simple security API similar to PKCS#11 [23], the Yubikey security token, the optimistic contract signing protocol by Garay, Jakobsson and MacKenzie (GJM) [16] and a few other examples discussed in Arapinis et al. [3] and Mödersheim [22]. The results are summarized in Figure 11. For each case study we provide the number of typing lemmas that were needed by the tamarin prover and whether manual guidance of the tool was required. In case no manual guidance is required we also give execution times. We do not detail all the formal models of the protocols and properties that we studied, and sometimes present slightly simplified versions. All files of our prototype implementation and our case studies are available at `http://sapic.gforge.inria.fr/`.

Example	Typing Lemmas	Automated Run*
Security API à la PKCS#11	1	yes (51s)
Yubikey Protocol [19, 28]	3	no
GJM protocol [3, 16]	0	yes (36s)
Mödersheim's example (locks/inserts) [22]	0	no**
Mödersheim's example (embedded msr rules) [22]	0	yes (1s)
Security Device [3]	1	yes (21s)
Needham-Schroeder-Lowe [21]	1	yes (5s)

* (Running times on Intel Core2 Duo 2.66Ghz with 4GB RAM)
** (little interaction: 7 manual rule selections)

Figure 11: Case studies.

7.1 Security API à la PKCS#11

This example illustrates how our modelling might be useful for the analysis of Security APIs in the style of the PKCS#11 standard [23]. We expect studying a complete model of PKCS#11, such as in [13], to be a straightforward extension of this example. In addition to the processes presented in the running example in Section 3 the actual case study models the following two operations: *(i) encryption:* given a handle and a plain-text, the user can request an encryption under the key the handle points to. *(ii) unwrap* given a ciphertext $senc(k_2, k_1)$, and a handle h_1, the user can request the ciphertext to be *unwrapped*, i.e. decrypted, under the key pointed to by h_1. If decryption is successful the result is stored on the device, and a handle pointing to k_2 is returned. Moreover, contrary to the running example, at creation time keys are assigned the attribute ‘init’, from which they can move to either ‘wrap’, or ‘unwrap’, see the following snippet:

```
1 in(⟨‘set_dec’,h⟩); lock ⟨‘att’,h⟩;
2   lookup ⟨‘att’,h⟩ as a in
3     if a=‘init’ then
4       insert ⟨‘att’,h⟩,‘dec’; unlock ⟨‘att’,h⟩
```

Note that, in contrast to the running example, it is necessary to encapsulate the state changes between lock and unlock. Otherwise an adversary can stop the execution after line 3, set the attribute to ‘wrap’ in a concurrent process and produce a wrapping. After resuming operation at line 4, he can set the key's attribute to ‘dec’, even though the attribute is set to ‘wrap’. Hence, the attacker is allowed to decrypt the wrapping he has produced and can obtain the key. Such subtleties can produce attacks that our modeling allows to detect. If locking is handled correctly, we show secrecy of keys produced on the device, proving the property introduced in Example 5. If locks are removed the attack described before is found.

7.2 Yubikey

The Yubikey [28] is a small hardware device designed to authenticate a user against network-based services. Manufactured by Yubico, a Swedish company, the Yubikey itself is a low cost ($25), thumb-sized USB device. In its typical configuration, it generates one-time passwords based on encryptions of a secret value, a running counter and some random values using a unique AES-128 key contained in the device. The Yubikey authentication server accepts a one-time password only if it decrypts under the correct AES key to a valid secret value containing a counter larger than the last counter accepted. The counter is thus used as a means to prevent replay attacks. To date, over a million Yubikeys have been shipped to more than 30,000 customers including governments, universities and enterprises, e.g. Google, Microsoft, Agfa and Symantec [29].

Besides the counter values used in the one-time password, the Yubikey stores three additional pieces of information: the public id *pid* that is used to identify the Yubikey, a secret id *secretid* that is transmitted as part of the one-time password and only known to the server and the Yubikey, as well as the AES key k, which is also shared with the server. The following process $P_{Yubikey}$ models a single Yubikey, as well as its initial configuration, where an entry in the server's database for the public id *pid* is created. This entry contains a tuple consisting of the Yubikey's secret id, AES key, and an initial counter value.

```
P_Yubikey =
 ν k; ν pid; ν secretid ;
  insert ⟨'Server',pid⟩, ⟨ secretid , k, 'zero'⟩;
  insert ⟨'Yubikey', pid⟩,'zero'+'one';
 out(pid);
 !P_Plugin | !P_ButtonPress
```

Here, the processes $!P_{Plugin}$ and $!P_{ButtonPress}$ model the Yubikey being unplugged and plugged in again (possibly on a different computer), and the emission of the one-time password. We will only discuss $P_{ButtonPress}$ here. When the user presses the button on the Yubikey, the device outputs a one-time password consisting of a counter tc, the secret id *secretid* and additional randomness *npr* encrypted using the AES key k.

```
P_ButtonPress =
 lock pid;
   lookup ⟨'Yubikey',pid⟩ as tc in
      insert ⟨'Yubikey',pid⟩, tc + 'one';
     ν nonce; ν npr;
     event YubiPress(pid, secretid ,k, tc);
     out(⟨pid,nonce,senc(⟨ secretid , tc ,npr⟩,k)⟩);
 unlock pid
```

The one-time password $senc(\langle secretid, tc, npr\rangle, k)$ can be used to authenticate against a server that shares the same secret key, which we model in the process P_{Server}. The process receives the encrypted one-time password along with the public id *pid* of a Yubikey and a *nonce* that is part of the protocol, but is irrelevant for the authentication of the Yubikey on the server.

The server looks up the secret id and the AES key associated to the public id, i. e., to the Yubikey sending the request, as well as the last recorded counter value *otc*. If the key and secret id used in the request match the values retrieved from the database, then the event $\text{Smaller}(otc, tc)$ is logged along with the event $\text{Login}(pid, k, tc)$, which marks a successful login of the Yubikey *pid* with key k for the counter value tc. Afterwards, the old tuple $\langle secretid, k, otc\rangle$ is replaced by $\langle secretid, k, tc\rangle$, to update the latest counter value received.

```
P_Server =
! in(⟨pid,nonce,senc(⟨ secretid , tc ,npr⟩,k)⟩);
 lock pid;
 lookup ⟨'Server',pid⟩ as tuple in
   if fst(tuple)=secretid then
```

```
    if  $fst(snd(tuple))=k$ then
      event  Smaller($snd(snd(tuple))$, $tc$)
      event  Login($pid,k,tc$);
       insert  $\langle$'Server',$pid\rangle$, $\langle$ $secretid,k,tc\rangle$;
unlock  $pid$
```

Note that, in our modelling, the server keeps one lock per public id, which means that it is possible to have several active instances of the server thread in parallel as long as all requests concern different Yubikeys.

An important part of the modelling of the protocol is to determine whether one counter value is smaller than another. To this end, our modelling employs a feature added to the development version of tamarin as of October 2012, a union operator $\cup^{\#}$ for multisets of message terms. The operator is denoted with a plus sign ("+"). We model the counter as a multiset only consisting of the symbols "one" and "zero". The multiplicity of 'one' in the multiset is the value of the counter. A counter value is considered smaller than another one, if the first multiset is included in the second. A test $a < b$ is included by adding the event Smaller(a, b) and an axiom that requires that a is a subset of b:

$$\begin{aligned}\alpha_{Smaller} :=&\forall i : temp, a, b : msg.\ \text{Smaller}(a, b)@i\\ &\Rightarrow \exists z : msg.\ a + z = b\end{aligned}$$

We incorporate this axiom into the security properties just like in Definition 14. Intuitively, we are only interested in traces where a is indeed smaller than b.

The process we analyse models a single authentication server (that may run arbitrary many threads) and an arbitrary number of Yubikeys, i. e., $P_{Server} \mid !P_{Yubikey}$. Among other properties, we show by the means of an injective correspondence property that an attacker that controls the network cannot perform replay attacks, and that each successful login was preceded by a user "pressing the button", formally:

$$\begin{aligned}&\forall\, pid, k, x, t_2.\text{Login}(pid, k, x)@t_2 \Rightarrow\\ &\qquad \exists sid, t_1.\text{YubiPress}(pid, sid, k, x)@t_1 \wedge t_1 \lessdot t_2\\ &\qquad\quad \wedge \forall t_3.\text{Login}(pid, k, x)@t_3 \Rightarrow t_3 = t_2\end{aligned}$$

Besides injective correspondence, we show the absence of replay attacks and the property that a successful login invalidates previously emitted one-time passwords. All three properties follow more or less directly from a stronger invariant, which itself can be proven in 295 steps. To find theses steps, tamarin needs some additional human guidance, which can be provided using the interactive mode. This mode still allows the user to complement his manual efforts with automated backward search. The example files contain the modelling in our calculus, the complete proof, and the manual part of the proof which can be verified by tamarin without interaction.

Our analysis makes three simplifications: First, in P_{Server}, we use pattern matching instead of decryption as demonstrated in the process P_{dec} we introduced in Section 3. Second, we omit the CRC checksum and the time-stamp that are part of the one-time password in the actual protocol, since they do not add to the security of the protocol in the symbolic setting. Third, the Yubikey has actually two counters instead of one, a session counter, and a token counter. We treat the session and token counter on the Yubikey as a single value, which we justify by the fact that the Yubikey either increases the session counter and resets the token counter, or increases only the token counter, thereby implementing a complete lexicographical order on the pair $(session\ counter, token\ counter)$.

A similar analysis has already been performed by Künnemann and Steel, using tamarin's multiset rewriting calculus [19]. However, the model in our new calculus is more fine-grained and we believe more readable. Security-relevant operations like locking and tests on state are written out in detail, resulting in a model that is closer to the real-life operation of such a device. The modeling of the Yubikey takes approximately 38 lines in our calculus, which translates to 49 multiset rewrite rules. The model of [19] contains only four rules, but they are quite complicated, resulting in 23 lines of code. More importantly, the gap between their model and the actual Yubikey protocol is larger – in our calculus, it becomes clear that the server can treat multiple authentication requests in parallel, as long as they do not claim to stem from the same Yubikey. An implementation on the basis of the model from Künnemann and Steel would need to implement a global lock accessible to the authentication server and all Yubikeys. This is however unrealistic, since the Yubikeys may be used at different places around the world, making it unlikely that there exist means of direct communication between them. While a server-side global lock might be conceivable (albeit impractical for performance reasons), a real global lock could not be implemented for the Yubikey as deployed.

7.3 Further Case Studies

We also investigated the case study presented by Mödersheim [22], a key-server example. We encoded two models of this example, one using the insert construct, the other manipulating state using the embedded multiset rewrite rules. For this example the second model turned out to be more natural and more convenient allowing for a direct automated proof without any additional typing lemma.

We furthermore modeled the contract signing protocol by Garay et al. [16] and a simple security device which both served as examples in [3]. In the contract signing protocol a trusted party needs to maintain a database with the current status of all contracts (aborted, resolved, or no decision has been taken). In our calculus the status information is naturally modelled using our insert and lookup constructs. The use of locks is indispensable to avoid the status to be changed between a lookup and an insert. Arapinis et al. [3] showed the crucial property that the same contract can never be both aborted and resolved. However, due to the fact that StatVerif only allows for a finite number of memory cells, they have shown this property for a single contract and provide a manual proof to lift the result to an unbounded number of contracts. We directly prove this property for an unbounded number of contracts. Finally we also illustrate the tool's ability to analyze classical security protocols, by analyzing the Needham Schroeder Lowe protocol [21].

8 Conclusion

We present a process calculus which extends the applied pi calculus with constructs for accessing a global, shared memory together with an encoding of this calculus in labelled msr rules which enables automated verification using the tamarin prover as a backend. Our prototype verification tool, automating this translation, has been successfully used to analyze several case studies. As future work we plan to increase the degree of automation of the tool by automatically generating helping lemmas. To achieve this goal we can exploit the fact that we generate the msr rules, and hence control their form. We also plan to use the tool for more complex case studies including a complete model of PKCS#11 and a study of the TPM 2.0 standard, currently in public review. Finally, we wish to investigate how our constructs for manipulating state can be used to encode loops, needed to model stream protocols such as TESLA.

Acknowledgements The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no 258865, project ProSecure and was supported by CASED (http://www.cased.de).

References

[1] M. Abadi and V. Cortier. Deciding knowledge in security protocols under equational theories. *Theoretical Computer Science*, 387(1-2):2–32, 2006.

[2] M. Abadi and C. Fournet. Mobile values, new names, and secure communication. In *Proc. 28th ACM Symp. on Principles of Programming Languages (POPL'01)*, pages 104–115. ACM Press, 2001.

[3] M. Arapinis, E. Ritter, and M. Ryan. Statverif: Verification of stateful processes. In *Proc. 24th IEEE Computer Security Foundations Symposium (CSF'11)*, pages 33–47. IEEE Press, 2011.

[4] A. Armando, D. A. Basin, Y. Boichut, Y. Chevalier, L. Compagna, J. Cuéllar, P. H. Drielsma, P.-C. Héam, O. Kouchnarenko, J. Mantovani, S. Mödersheim, D. von Oheimb, M. Rusinowitch, J. Santiago, M. Turuani, L. Viganò, and L. Vigneron. The AVISPA tool for the automated validation of internet security protocols and applications. In *Proc. 17th International Conference on Computer Aided Verification (CAV'05)*, LNCS, pages 281–285. Springer, 2005.

[5] A. Armando, R. Carbone, L. Compagna, J. Cuellar, and L. T. Abad. Formal analysis of saml 2.0 web browser single sign-on: Breaking the saml-based single sign-on for google apps. In *Proc. 6th ACM Workshop on Formal Methods in Security Engineering (FMSE'08)*, pages 1–10, 2008.

[6] S. Bistarelli, I. Cervesato, G. Lenzini, and F. Martinelli. Relating multiset rewriting and process algebras for security protocol analysis. *Journal of Computer Security*, 13(1):3–47, 2005.

[7] B. Blanchet. An Efficient Cryptographic Protocol Verifier Based on Prolog Rules. In *Proc. 14th Computer Security Foundations Workshop (CSFW'01)*, pages 82–96. IEEE Press, 2001.

[8] B. Blanchet, B. Smyth, and V. Cheval. *ProVerif 1.88: Automatic Cryptographic Protocol Verifier, User Manual and Tutorial*, 2013.

[9] M. Bond and R. Anderson. API level attacks on embedded systems. *IEEE Computer Magazine*, pages 67–75, October 2001.

[10] M. Bortolozzo, M. Centenaro, R. Focardi, and G. Steel. Attacking and fixing PKCS#11 security tokens. In *Proc. 17th ACM Conference on Computer and Communications Security (CCS'10)*, pages 260–269. ACM Press, 2010.

[11] *CCA Basic Services Reference and Guide*, Oct. 2006. Available online.

[12] S. Delaune, S. Kremer, M. D. Ryan, and G. Steel. Formal analysis of protocols based on TPM state registers. In *Proc. 24th IEEE Computer Security Foundations Symposium (CSF'11)*, pages 66–82. IEEE Press, 2011.

[13] S. Delaune, S. Kremer, and G. Steel. Formal analysis of PKCS#11 and proprietary extensions. *Journal of Computer Security*, 18(6):1211–1245, Nov. 2010.

[14] S. Escobar, C. Meadows, and J. Meseguer. Maude-npa: Cryptographic protocol analysis modulo equational properties. In *Foundations of Security Analysis and Design V*, volume 5705 of *LNCS*, pages 1–50. Springer, 2009.

[15] S. B. Fröschle and N. Sommer. Reasoning with past to prove PKCS#11 keys secure. In *Proc. 7th International Workshop on Formal Aspects in Security and Trust (FAST'10)*, volume 6561 of *LNCS*, pages 96–110, 2010.

[16] J. A. Garay, M. Jakobsson, and P. D. MacKenzie. Abuse-free optimistic contract signing. In *Advances in Cryptology—Crypto'99*, volume 1666 of *LNCS*, pages 449–466. Springer, 1999.

[17] J. D. Guttman. State and progress in strand spaces: Proving fair exchange. *J. Autom. Reasoning*, 48(2):159–195, 2012.

[18] J. Herzog. Applying protocol analysis to security device interfaces. *IEEE Security & Privacy Magazine*, 4(4):84–87, July-Aug 2006.

[19] R. Künnemann and G. Steel. YubiSecure? Formal security analysis results for the Yubikey and YubiHSM. In *Proc. 8th Workshop on Security and Trust Management (STM'12)*, volume 7783 of *LNCS*, pages 257–272, 2012.

[20] D. Longley and S. Rigby. An automatic search for security flaws in key management schemes. *Computers and Security*, 11(1):75–89, March 1992.

[21] G. Lowe. Breaking and fixing the Needham-Schroeder public-key protocol using FDR. In *Proc. 2nd International Workshop on Tools and Algorithms for Construction and Analysis of Systems (TACAS'96)*, volume 1055 of *LNCS*, pages 147–166. Springer, 1996.

[22] S. Mödersheim. Abstraction by set-membership: verifying security protocols and web services with databases. In *Proc. 17th ACM Conference on Computer and Communications Security (CCS'10)*, pages 351–360. ACM, 2010.

[23] RSA Security Inc., v2.20. *PKCS #11: Cryptographic Token Interface Standard.*, June 2004.

[24] B. Schmidt. *Formal Analysis of Key-Exchange Protocols and Physical Protocols.* PhD thesis, ETH Zürich, November 2012.

[25] B. Schmidt, S. Meier, C. Cremers, and D. Basin. Automated analysis of Diffie-Hellman protocols and advanced security properties. In *Proc. 25th IEEE Computer Security Foundations Symposium (CSF'12)*, pages 78–94. IEEE Press, 2012.

[26] B. Schmidt, S. Meier, C. Cremers, and D. Basin. The tamarin prover for the symbolic analysis of security protocols. In *Proc. 25th International Conference on Computer Aided Verification (CAV'13)*, volume 8044 of *LNCS*, pages 696–701. Springer, 2013.

[27] Trusted Computing Group. TPM Specification version 1.2. Parts 1–3, revision 103. `http://www.trustedcomputinggroup.org/resources/tpm_main_specification`, 2007.

[28] Yubico AB, Kungsgatan 37, 111 56 Stockholm Sweden. *The YubiKey Manual - Usage, configuration and introduction of basic concepts (Version 2.2), available at: `http://www.yubico.com/documentation`*, June 2010.

[29] Yubico AB. Yubico customer list, 2013. Accessed: Wed 17 Jul 2013 11:40:50 CEST.

A Definition of the process annotation

Definition 17 (Process annotation)**.** *Given a ground process P we define the annotated ground process $\overline{P}$ as follows:*

$$\begin{aligned}
\overline{0} &:= 0\\
\overline{P|Q} &:= \overline{P}|\overline{Q}\\
\overline{!P} &:= !\overline{P}\\
\begin{array}{r}\overline{\textit{if } t_1 = t_2 \textit{ then } P}\\ \overline{\textit{else } Q}\end{array} &:= \textit{if } t_1 = t_2 \textit{ then } \overline{P} \textit{ else } \overline{Q}\\
\begin{array}{r}\overline{\textit{lookup } M \textit{ as } x}\\ \overline{\textit{in } P \textit{ else } Q}\end{array} &:= \begin{array}{l}\textit{lookup } M \textit{ as } x\\ \textit{in } \overline{P} \textit{ else } \overline{Q}\end{array}\\
\overline{\alpha; P} &:= \alpha; \overline{P}\\
&\quad \textit{where } \alpha \notin \{\, \text{lock } t, \text{unlock } t \colon t \in \mathcal{T}\,\}\\
\overline{\text{lock } t; P} &:= \text{lock}^l\ t; \overline{au(P,t,l)}\\
&\quad \textit{where } l \in \mathbb{N} \textit{ is a fresh label}\\
\overline{\text{unlock}^l\ t; P} &:= \text{unlock}^l\ t; \overline{P}\\
\overline{\text{unlock } t; P} &:= \bot
\end{aligned}$$

where $au(P, t, l)$ annotates the first unlock that has parameter t with the label l, i. e.:

$$\begin{aligned}
au(P|Q,t,l) &:= \bot\\
au(!P,t,l) &:= \bot\\
\begin{array}{r}au(\mathit{if}\ t_1 = t_2\ \mathit{then}\\ P\ \mathit{else}\ Q,t,l)\end{array} &:= \begin{array}{l}\mathit{if}\ t_1 = t_2\ \mathit{then}\ au(P,t,l)\\ \mathit{else}\ au(Q,t,l)\end{array}\\
\begin{array}{r}au(\mathit{lookup}\ M\ \mathit{as}\ x\\ \mathit{in}\ P\ \mathit{else}\ Q,t,l)\end{array} &:= \begin{array}{l}\mathit{lookup}\ M\ \mathit{as}\ x\ \mathit{in}\\ au(P,t,l)\ \mathit{else}\ au(Q,t,l)\end{array}\\
au(\alpha; P,t,l) &:= \alpha; au(P,t,l)\\
&\qquad\qquad \mathit{where}\ \alpha \neq \mathit{unlock}\ t\\
au(\text{unlock}\ t; P,t,l) &:= \text{unlock}^{l}\ t; P\\
au(0,t,l) &:= 0
\end{aligned}$$

B Correctness of tamarin's solution procedure for translated rules

The multiset rewrite system produced by our translation for a well-formed process P could actually contain rewrite rules that are not valid with respect to Definition 4, because they violate the third condition, which is: for each $l' -[a']\!\rightarrow r' \in R \in_E ginsts(l -[a]\!\rightarrow r)$ we have that $\cap_{r''=_E r'} names(r'') \cap FN \subseteq \cap_{l''=_E l'} names(l'') \cap FN$.

This does not hold for rules in $[\![P]\!]_{=p}$ where p is the position of the lookup-operator. The right hand-side of this rule can be instantiated such that, assuming the variable bound by the lookup is named v, this variable v is substituted by a names that does not appear on the left-hand side. In the following, we will show that the results from [25] still hold. In practice, this means that the tamarin-prover can be used for verification, despite the fact that it outputs well-formedness errors for each rule that is a translation of a lock.

We will introduce some notation first. We re-define $[\![P]\!]$ to contain the INIT rule and $[\![\overline{P}, [], []]\!]$, but not MD (which is different to Definition 13). We furthermore define a translation with dummy-facts, denoted $[\![P]\!]^D$, that contains INIT and $[\![\overline{P}, [], []]\!]^D$, which is defined as follows:

Definition 18. *We define $[\![P]\!]^D := \text{INIT} \cup [\![\overline{P}, [], []]\!]^D$, where $[\![\overline{P}, [], []]\!]^D$ is defined just as $[\![\overline{P}, [], []]\!]$, with the exception of two cases, $P = lookup\ M\ as\ v\ in\ P\ else\ Q$ and $P = insert\ s, t; P$, where it is defined as follows:*

$$\begin{aligned}
[\![\mathit{lookup}\ M\ \mathit{as}\ v\ \mathit{in}\ P\ \mathit{else}\ Q, p, \tilde{x}]\!]^D &= \{[\mathsf{state}_p(\tilde{x}), !\mathsf{Dum}(v)] -[\mathrm{IsIn}(M,v)]\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{M}, v)],\\
&\qquad [\mathsf{state}_p(\tilde{x})] -[\mathrm{IsNotSet}(M)]\!\rightarrow [\mathsf{state}_{p\cdot 2}(\tilde{x})]\}\\
&\quad \cup [\![P, p\cdot\ 1, (\tilde{x}, v)]\!]^D \cup [\![Q, p\cdot 2, \tilde{x}]\!]^D\\
[\![\mathit{insert}\ s,t; P, p, \tilde{x}]\!]^D &= \{\ [\mathsf{state}_p(\tilde{x})] -[\mathrm{Insert}(s,t)]\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x}), !\mathsf{Dum}(t)]\ \}\\
&\quad \cup [\![P, p\cdot 1, \tilde{x}]\!]^D
\end{aligned}$$

The only difference between $[\![P]\!]$ and $[\![P]\!]^D$ is therefore, that $[\![P]\!]^D$ produces a permanent fact $!\mathsf{Dum}$ for every value v that appears in an action $\text{insert}(k, v)$, which is a premise to every rule instance with an action $\text{IsIn}(k', v)$. We see that $[\![P]\!]^D$ contains now only valid multiset rewrite rules.

In the following, we would like to show that the tamarin-prover's solution algorithm is correct for $[\![P]\!]$. To this end, we make use of the proof of correctness of tamarin as presented in Benedikt Schmidt's Ph.D. thesis [24]. We will refer to Lemmas, Theorems and Corollaries in this work by their numbers. We will use the notation of this work, to make it easier to the reader to compare our statements against the statements there. In particular, $\overline{trace(execs(R))}$ is $traces^{msr}(R)$ in our notation. We have to show that:

Lemma 2. *For all well-formed process P and guarded trace properties ϕ,*

$$trace(execs([\![P]\!] \cup \text{MD}) \vDash_{\mathcal{DH}_e} \neg\alpha_{in} \vee \phi$$

if and only if

$$\mathit{trace}(\mathit{ndgraphs}(\llbracket P \rrbracket)) \vDash_{ACC} \neg\alpha_{in} \vee \phi.$$

Proof. The proof proceeds similar to the proof to Theorem 3.27. We refer to results in [24], whenever their proofs apply despite the fact that the rules in $\llbracket P \rrbracket$ do not satisfy the third condition of multiset rewrite rules.

$$\begin{aligned}
& \mathit{trace}(\mathit{execs}(\llbracket P \rrbracket \cup \mathrm{MD}) \vDash_{\mathcal{DH}_e} \neg\alpha_{in} \vee \psi & \\
\Leftrightarrow\ & \overline{\mathit{trace}(\mathit{execs}(\llbracket P \rrbracket \cup \mathrm{MD})} \vDash_{\mathcal{DH}_e} \neg\alpha_{in} \vee \phi & \text{(Lemma 3.7 (unaltered))} \\
\Leftrightarrow\ & \overline{\mathit{trace}(\mathit{execs}(\llbracket P \rrbracket \cup \mathrm{MD}))} \downarrow_{\mathcal{RDH}_e} \vDash_{\mathcal{DH}_e} \neg\alpha_{in} \vee \phi & \text{(Definition of } \vDash_{\mathcal{DH}_e}) \\
\Leftrightarrow\ & \overline{\mathit{trace}(\mathit{dgraphs}_{\mathcal{DH}_e}(\llbracket P \rrbracket \cup \mathrm{MD}))} \downarrow_{\mathcal{RDH}_e} \vDash_{\mathcal{DH}_e} \neg\alpha_{in} \vee \phi & \text{(Lemma 3.10 (unaltered))} \\
\Leftrightarrow\ & \overline{\mathit{trace}(\{\mathit{dg} \mid \mathit{dg} \in \mathit{dgraphs}_{\mathcal{ACC}}(\lceil \llbracket P \rrbracket \cup \mathrm{MD} \rceil^{\mathcal{RDH}_e}_{\mathit{insts}})} & \\
& \overline{\wedge \mathit{dg} \downarrow_{\mathcal{RDH}_e} \text{-normal}\})} \vDash_{\mathcal{DH}_e} \neg\alpha_{in} \vee \phi & \text{(Lemma 3.11 (unaltered))} \\
\Leftrightarrow\ & \overline{\mathit{trace}(\mathit{ndgraphs}(\llbracket P \rrbracket))} \vDash_{\mathcal{DH}_e} \neg\alpha_{in} \vee \phi & \text{(Lemma A.12 (*))} \\
\Leftrightarrow\ & \mathit{trace}(\mathit{ndgraphs}(\llbracket P \rrbracket)) \vDash_{\mathcal{ACC}} \neg\alpha_{in} \vee \phi & \text{(Lemma 3.7 and A.20(both unaltered))}
\end{aligned}$$

It is only in Lemma A.12 where the third condition is used: The proof to this lemma applies Lemma A.14, which says that all factors (or their inverses) are known to the adversary. We will quote Lemma A.14 here:

Lemma 3 (Lemma A.14 in [24]). *For all $\mathit{ndg} \in \mathit{ndgraphs}(P)$, conclusions (i, u) in ndg with conclusion fact f and terms $t \in \mathit{afactors}(f)$, there is a conclusion (j, v) in ndg with $j < i$ and conclusion fact $\mathsf{K}^d(m)$ such that $m \in_{ACC} \{t, (t^{-1}) \downarrow_{\mathcal{RBP}_e}\}$.*

If there is $\mathit{ndg} \in \mathit{ndgraphs}(\llbracket P \rrbracket)$, such that $\mathit{trace}(\mathit{ndg}) \vDash_{ACC} \alpha_{in}$, then

$$\begin{aligned}
& \mathit{trace}(\mathit{ndgraphs}(\llbracket P \rrbracket)) \vDash_{\mathcal{ACC}} \neg\alpha_{in} \vee \phi \\
\Leftrightarrow & \forall \mathit{ndg} \in \mathit{ndgraphs}(\llbracket P \rrbracket) \ \text{ s. t. } \mathit{trace}(\mathit{ndg}) \vDash_{ACC} \alpha_{in} \\
& \quad \mathit{trace}(\mathit{ndg}) \vDash_{\mathcal{ACC}} \vDash \phi
\end{aligned}$$

Since for the empty trace, $[] \vDash_{ACC} \alpha_{in}$, we only have to show that Lemma A.14 holds for $\mathit{ndg} \in \mathit{ndgraphs}(\llbracket P \rrbracket)$, such that $\mathit{trace}(\mathit{ndg}) \vDash_{ACC} \alpha_{in}$.

For every $\mathit{ndg} \in \mathit{ndgraphs}(\llbracket P \rrbracket)$, such that $\mathit{trace}(\mathit{ndg}) \vDash_{ACC} \alpha_{in}$, there is a trace equivalent $\mathit{ndg}' \in \mathit{ndgraphs}(\llbracket P \rrbracket^D)$, since the only difference between $\llbracket P \rrbracket$ and $\llbracket P \rrbracket^D$ lies in the dummy conclusion and premises, and α_{in} requires that any v in an action $\mathrm{IsIn}(\mathrm{u}, \mathrm{v})$ appeared previously in an action $\mathrm{Insert}(\mathrm{u}, \mathrm{v})$ (equivalence modulo ACC). Therefore, ndg' has the same K^d-conclusions ndg has, and every conclusion in ndg is a conclusion in ndg'.

We have that Lemma A.14 holds for $\llbracket P \rrbracket^D$, since all rules generated in this translation are valid multiset rewrite rules. Therefore, Lemma A.14 holds for all $\mathit{ndg} \in \mathit{ndgraphs}(\llbracket P \rrbracket)$, such that $\mathit{trace}(\mathit{ndg}) \vDash_{ACC} \alpha_{in}$, too, concluding the proof by showing the marked (*) step.

□

C Proofs of Section 6

Proposition 1. *Let Tr be a set of traces and φ a trace formula. We have that*

$$\mathit{Tr} \vDash^{\star} \llbracket \varphi \rrbracket_{\star} \ \textit{iff} \ \mathit{filter}(\mathit{Tr}) \vDash^{\star} \varphi$$

where $\star$ is either $\forall$ or $\exists$.

Proof. We first show the two directions for the case $\star = \forall$. We start by showing that $Tr \vDash^{\forall} [\![\varphi]\!]$ implies $\mathit{filter}(Tr) \vDash \varphi$.

$$\begin{aligned} Tr \vDash^{\forall} [\![\varphi]\!]_{\forall} &\Rightarrow \mathit{filter}(Tr) \vDash^{\forall} [\![\varphi]\!]_{\forall} && (\text{since } \mathit{filter}(Tr) \subseteq Tr) \\ &\Leftrightarrow \mathit{filter}(Tr) \vDash^{\forall} \alpha \Rightarrow \varphi && (\text{by definition of } [\![\varphi]\!]_{\forall}) \\ &\Leftrightarrow \mathit{filter}(Tr) \vDash^{\forall} \varphi && (\text{since } \mathit{filter}(Tr) \vDash^{\forall} \alpha) \end{aligned}$$

We next show that $\mathit{filter}(Tr) \vDash^{\forall} \varphi$ implies $Tr \vDash^{\forall} [\![\varphi]\!]_{\forall}$.

$$\begin{aligned} \mathit{filter}(Tr) \vDash^{\forall} \varphi &\Rightarrow \mathit{filter}(Tr) \vDash^{\forall} \alpha \wedge \varphi && (\text{since } \mathit{filter}(Tr) \vDash^{\forall} \alpha) \\ &\Leftrightarrow Tr \vDash^{\forall} \neg\alpha \vee (\alpha \wedge \varphi) && (\text{since } \mathit{filter}(Tr) \subseteq Tr \text{ and } (Tr \setminus \mathit{filter}(Tr)) \nvDash^{\forall} \alpha) \\ &\Leftrightarrow Tr \vDash^{\forall} \alpha \Rightarrow \varphi \\ &\Leftrightarrow Tr \vDash^{\forall} [\![\varphi]\!]_{\forall} && (\text{by definition of } [\![\varphi]\!]_{\forall}) \end{aligned}$$

The case of $\star = \exists$ now easily follows:

$$\begin{array}{lll} Tr \vDash^{\exists} [\![\varphi]\!]_{\exists} & \text{iff} & Tr \nvDash^{\forall} [\![\neg\varphi]\!]_{\forall} \\ & \text{iff} & \mathit{filter}(Tr) \nvDash^{\forall} \neg\varphi \\ & \text{iff} & \mathit{filter}(Tr) \vDash^{\exists} \varphi. \end{array}$$

□

Proposition 3. *Let Tr be a set of traces and φ a well-formed trace formula. We have that*

$$Tr \vDash^{\star} \varphi \textit{ iff } \mathit{hide}(Tr) \vDash^{\star} \varphi$$

where $\star$ is either $\forall$ or $\exists$.

Proof. We start with the case $\star = \exists$ and show the stronger statement that for a trace tr

$$\forall\theta.\exists\theta'.\text{ if } (tr, \theta) \vDash \varphi \text{ then } (\mathit{hide}(tr), \theta') \vDash \varphi$$

and

$$\forall\theta.\exists\theta'.\text{ if } (\mathit{hide}(tr), \theta) \vDash \varphi \text{ then } (tr, \theta') \vDash \varphi$$

We will show both statements by a nested induction on $|tr|$ and the structure of the formula. (The underlying well-founded order is the lexicographic ordering of the pairs consisting of the length of the trace and the size of the formula)

If $|tr| = 0$ then $tr = []$ and $tr = \mathit{hide}(tr)$ which allows us to directly conclude letting $\theta' := \theta$.

If $|tr| = n$, we define $\overline{tr}$ and F such that $tr = \overline{tr} \cdot F$. By induction hypothesis we have that

$$\forall\overline{\theta}.\exists\overline{\theta}'.\text{ if } (\overline{tr}, \overline{\theta}) \vDash \varphi \text{ then } (\mathit{hide}(\overline{tr}), \overline{\theta}') \vDash \varphi$$

and

$$\forall\overline{\theta}.\exists\overline{\theta}'.\text{ if } (\mathit{hide}(\overline{tr}), \overline{\theta}) \vDash \varphi \text{ then } (\overline{tr}, \overline{\theta}') \vDash \varphi$$

We proceed by structural induction on φ.

- $\varphi = \bot$, $\varphi = i \lessdot j$, $\varphi = i \doteq j$ or $t_1 \approx t_2$. In these cases we trivially conclude as the truth value of these formulas does not depend on the trace and for both statements we simply let $\theta' := \theta$.
- $\varphi = f@i$. We start with the first statement. Suppose that $(tr, \theta) \vDash f@i$. If $\theta(i) < n$ then we have also that $\overline{tr}, \theta \vDash f@i$. By induction hypothesis, there exists $\overline{\theta}'$ such that $(\overline{tr}, \overline{\theta}') \vDash f@i$. Hence we also have that $(tr, \overline{\theta}') \vDash f@i$ and letting $\theta' := \overline{\theta}'$ allows us to conclude. If $\theta(i) = n$ we know that $f \in tr_n$. As φ is well-formed $f \notin \mathcal{F}_{res}$ and hence $f \in \mathit{hide}(tr)_{n'}$ where $n' = |\mathit{hide}(tr)|$. The proof of the other statement is similar.

- $\varphi = \neg\varphi'$, $\varphi = \varphi_1 \wedge \varphi_2$, or $\varphi = \exists x : s.\varphi'$. We directly conclude by induction hypotheses (on the structure of φ).

From the above statements we easily have that $Tr \vDash^{\exists} \varphi$ iff $hide(Tr) \vDash^{\exists} \varphi$.

The case of $\star = \forall$ now easily follows:

$$Tr \vDash^{\forall} \varphi \text{ iff } Tr \nvDash^{\exists} \neg\varphi \text{ iff } hide(Tr) \nvDash^{\exists} \neg\varphi \text{ iff } hide(Tr) \vDash^{\forall} \varphi$$

□

In order to prove Lemma 1, we need a few additional lemmas.

We say that a set of traces Tr is prefix closed if for all $tr \in Tr$ and for all tr' which is a prefix of tr we have that $tr' \in Tr$.

Lemma 4 (*filter* is prefix-closed)**.** *Let Tr be a set of traces. If Tr is prefix closed then $filter(Tr)$ is prefix closed as well.*

Proof. It is sufficient to show that for any trace $tr = tr' \cdot a$ we have that if $\forall\theta.\ (tr, \theta) \vDash \alpha$ then $\forall\theta.\ (tr', \theta) \vDash \alpha$. This can be shown by inspecting each of the conjuncts of α. □

We next show that the translation with dummy facts defined in Definition 18 produces the same traces as $[\![P]\!]$, excluding traces not consistent with the axioms. For this we define the function d which removes any dummy fact from an execution, i.e.,

$$d(\emptyset \xrightarrow{F_1} S_1 \xrightarrow{F_2} \dots \xrightarrow{F_n} S_n) = \emptyset \xrightarrow{F_1} S'_1 \xrightarrow{F_2} \dots \xrightarrow{F_n} S'_n$$

where $S'_i = S_i \setminus^{\#} \cup_{t \in \mathcal{T}} \mathsf{!Dum}(t)$.

Lemma 5. *Given a ground process P, we have that*

$$filter(exec^{msr}([\![P]\!])) = filter(d(exec^{msr}([\![P]\!]^D \cup \mathrm{MD})))$$

Proof. The only rules in $[\![P]\!]^D$ that differ from $[\![P]\!]$ are translations of insert and lookup. The first one only adds a permanent fact, which by the definition of d, is removed when applying d. The second one requires a fact $\mathsf{!Dum}(t)$, whenever the rule is instantiated such the actions equals $\mathsf{IsIn}(s, t)$ for some s. Since the translation is otherwise the same, we have that

$$filter(d(exec^{msr}([\![P]\!]^D \cup \mathrm{MD}))) \subseteq filter(exec^{msr}([\![P]\!]))$$

For any trace in $filter(d(exec^{msr}([\![p]\!] \cup \mathrm{MD})))$ and any action $\mathsf{IsIn}(s, t)$ in this trace, there is an earlier action $\mathsf{Insert}(s', t')$ such that $s = s'$ and $t = t'$, as otherwise α_{in} would not hold. Therefore the same trace is part of $filter(d(exec^{msr}([\![p]\!]^D \cup \mathrm{MD})))$, as this means that whenever $\mathsf{!Dum}(t)$ is in the premise, $\mathsf{!Dum}(t')$ for $t = t'$ has previously appeared in the conclusion. Since it is a permanent fact, it has not disappeared and therefore

$$filter(d(exec^{msr}([\![P]\!]^D \cup \mathrm{MD}))) \subseteq filter(exec^{msr}([\![P]\!]))$$

□

We slightly abuse notation by defining *filter* on executions to filter out all traces contradicting the axioms, see Definition 15.

Lemma 6. *Let P be a ground process and $\emptyset \xrightarrow{F_1} S_1 \xrightarrow{F_2} \dots \xrightarrow{F_n} S_n \in filter(exec^{msr}([\![P]\!]))$. For all $1 \leq i \leq n$, if $\mathsf{Fr}(a) \in S_i$ and $F(t_1, \dots, t_k) \in S_i$ for any $F \in \Sigma_{fact} \setminus \{\, \mathsf{Fr} \,\}$, then $a \notin \cap_{t =_E t'} names(t')$, for any $t \in \{t_1, \dots, t_k\}$.*

Proof. The translation with the dummy fact introduced in Appendix B will make this proof easier as for $[\![P]\!]^D \cup \mathrm{MD}$, we have that the third condition of Definition 4 holds, namely,

$$\forall l' \ -[a']\!\!\rightarrow r' \in_E \mathit{ginsts}(l \ -[a]\!\!\rightarrow r) : \cap_{r''=_E r'} \mathit{names}(r'') \cap \mathit{FN} \subseteq \cap_{l''=_E l'} \mathit{names}(l'') \cap \mathit{FN} \qquad (1)$$

We will show that the statement holds for all $\emptyset \xrightarrow{F_1} S_1 \xrightarrow{F_2} \dots \xrightarrow{F_n} S_n \in \mathit{filter}(\mathit{exec}^{msr}([\![P]\!]^D \cup \mathrm{MD}))$, which implies the claim by Lemma 5. We proceed by induction on n, the length of the execution.

- Base case, $n = 0$. We have that $S_0 = \emptyset$ and therefore the statement holds trivially.
- Inductive case, $n \geq 1$. We distinguish two cases.
  1. A rule that is not Fresh was applied and there is a fact $F(t_1, \dots, t_k) \in S_n$, such that $F(t_1, \dots, t_k) \notin S_{n-1}$, and $\mathsf{Fr}(a) \in S_n$ such that $a \in \cap_{t_i =_E t'} \mathit{names}(t')$ for some t_i. (If there are no such $F(t_1, \dots, t_k)$ and $\mathsf{Fr}(a)$ we immediately conclude by induction hypothesis.) By Equation 1, $a \in t'_j$ for some $F'(t'_1, \dots, t'_l) \in S_{n-1}$. Since Fresh is the only rule that adds a Fr-fact and $\mathsf{Fr}(a) \in S_n$, it must be that $\mathsf{Fr}(a) \in S_{n-1}$, contradicting the induction hypothesis. Therefore this case is not possible.
  2. The rule Fresh was applied, i. e., $\mathsf{Fr}(a) \in S_n$ and $\mathsf{Fr}(a) \notin S_{n-1}$. If there is no $a \in \cap_{t_i =_E t'} \mathit{names}(t')$ for some t_i, and $F(t_1, \dots, t_k) \in S_n$, then we conclude by induction hypothesis. Otherwise, if there is such a $F(t_1, \dots, t_k) \in S_n$, then, by Equation 1, $a \in t'_j$ for some $F'(t'_1, \dots, t'_l) \in S_i$ for $i < n$. We construct a contradiction to the induction hypothesis by taking the prefix of the execution up to i and appending the instantiation of the Fresh rule to its end. Since $d(\mathit{exec}^{msr}([\![P]\!]^D \cup \mathrm{MD}))$ is prefix closed by Lemma 4 we have that $\emptyset \xrightarrow{F_1} S'_1 \xrightarrow{F_2} \dots \xrightarrow{F_i} S_i \in \mathit{filter}(d(\mathit{exec}^{msr}([\![P]\!]^D \cup \mathrm{MD})))$. Moreover as rule Fresh was applied adding $\mathsf{Fr}(a) \in S_n$ it is also possible to apply the same instance of Fresh to the prefix (by Definition 6) and therefore

     $$\emptyset \xrightarrow{F_1} S'_1 \xrightarrow{F_2} \dots \xrightarrow{F_i} S_i \longrightarrow S_i \cup \{\, \mathsf{Fr}(a) \,\} \in \mathit{filter}(d(\mathit{exec}^{msr}([\![P]\!]^D \cup \mathrm{MD})))$$

     contradicting the induction hypothesis.

□

Lemma 7. *For any frame $\nu\tilde{n}.\sigma$, $t \in \mathcal{M}$ and $a \in \mathit{FN}$, if $a \notin \mathit{st}(t)$, $a \notin \mathit{st}(\sigma)$ and $\nu\tilde{n}.\sigma \vdash t$, then $\nu\tilde{n}, a.\sigma \vdash t$.*

Proof. In [1, Proposition 1] it is shown that $\nu\tilde{n}.\sigma \vdash t$ if and only if $\exists M. \mathit{fn}(M) \cap \tilde{n} = \emptyset$ and $M\sigma =_E t$. Define M' as M renaming a to some fresh name, i.e., not appearing in $\tilde{n}, \sigma, t$. As $a \notin \mathit{st}(\sigma, t)$ and the fact that equational theories are closed under bijective renaming of names we have that $M'\sigma =_E t$ and $\mathit{fn}(M') \cap (\tilde{n}, a) = \emptyset$. Hence $\nu\tilde{n}, a.\sigma \vdash t$. □

Lemma 8. *Let P be a ground process and $\emptyset \xrightarrow{F_1} S_1 \xrightarrow{F_2} \dots \xrightarrow{F_n} S_n \in \mathit{filter}(\mathit{exec}^{msr}([\![P]\!]))$. Let*

$$\tilde{n} = \{a : \mathit{fresh} \mid \mathit{ProtoNonce}(a) \in \bigcup_{1 \leq j \leq n} F_j\},$$

$$\{t_1, \dots, t_m\} = \{t \mid \mathsf{Out}(t) \in_{1 \leq j \leq n} S_j\}.$$

Let $\sigma = \{^{t_1}/_{x_1}, \dots, ^{t_m}/_{x_m}\}$. We have that

1. *if $!\mathsf{K}(t) \in S_n$ then $\nu\tilde{n}.\sigma \vdash t$;*
2. *if $\nu\tilde{n}.\sigma \vdash t$ then there exists S such that*
   - $\emptyset \xrightarrow{F_1} S_1 \xrightarrow{F_2} \dots \xrightarrow{F_n} S_n \longrightarrow^* S \in \mathit{filter}(\mathit{exec}_E^{msr}([\![P]\!]))$,
   - $!\mathsf{K}(t) \in_E S$ *and*
   - $S_n \rightarrow_R^* S$ *for* $R = \{\, \textsc{MDOut}, \textsc{MDPub}, \textsc{MDFresh}, \textsc{MDAppl}, \textsc{Fresh} \,\}$.

Proof. We prove both items separately.

1. The proof proceeds by induction on n, the number of steps of the execution.

Base case: n=0. This case trivially holds as $S_n = \emptyset$.

Inductive case: n>0. By induction we suppose that if $!\mathsf{K}(t) \in S_{n-1}$ then $\nu\tilde{n}'.\sigma' \vdash t$ where $\tilde{n}', \sigma'$ are defined in a similar way as $\tilde{n}, \sigma$ but for the execution of size $n-1$. We proceed by case analysis on the rule used to extend the execution.

- MDOut. Suppose that $\mathsf{Out}(u) \text{ -[ ]}\!\!\rightarrow !\mathsf{K}(u) \in \mathit{ginsts}(\text{MDOut})$ is the rule used to extend the execution. Hence $\mathsf{Out}(u) \in S_{n-1}$ and by definition of σ there exists x such that $x\sigma = u$. We can apply deduction rule DFrame and conclude that $\nu\tilde{n}.\sigma \vdash u$. If $!\mathsf{K}(t) \in S_n$ and $t \neq u$ we conclude by induction hypothesis as $\tilde{n} = \tilde{n}', \sigma = \sigma'$.
- MDPub. Suppose that $\text{-[ ]}\!\!\rightarrow \mathsf{K}(a : \mathit{pub}) \in \mathit{ginsts}(\text{MDPub})$ is the rule used to extend the execution. As names of sort *pub* are never added to $\tilde{n}$ we can apply deduction rule DName and conclude that $\nu\tilde{n}.\sigma \vdash a$. If $K(t) \in S_n$ and $t \neq a$ we conclude by induction hypothesis as $\tilde{n} = \tilde{n}', \sigma = \sigma'$.
- MDFresh. Suppose that $\mathsf{Fr}(a : \mathit{fresh}) \text{ -[ ]}\!\!\rightarrow \mathsf{K}(a : \mathit{fresh}) \in \mathit{ginsts}(\text{MDFresh})$ is the rule used to extend the execution. By definition of an execution we have that $\mathsf{Fr}(a : \mathit{fresh}) \neq (S_{j+1} \setminus S_j)$ for any $j \neq n-1$. Hence $n \notin \tilde{n}$. We can apply deduction rule DName and conclude that $\nu\tilde{n}.\sigma \vdash a$. If $!\mathsf{K}(t) \in S_n$ and $t \neq a$ we conclude by induction hypothesis as $\tilde{n} = \tilde{n}', \sigma = \sigma'$.
- MDAppl. Suppose that $!\mathsf{K}(t_1), \ldots, !\mathsf{K}(t_k) \text{ -[ ]}\!\!\rightarrow !\mathsf{K}(u) \in \mathit{ginsts}(\text{MDAppl})$ is the rule used to extend the execution. We have that $K(t_1), \ldots, K(t_k) \in S_{n-1}$ and $u =_E f(t_1, \ldots, t_k)$. By induction hypothesis, $\nu\tilde{n}'.\sigma' \vdash t_i$ for $1 \leq i \leq k$. As $\tilde{n} = \tilde{n}', \sigma = \sigma'$ we have that $\nu\tilde{n}.\sigma \vdash t_i$ for $1 \leq i \leq k$. We can apply deduction rule DAppl and conclude that $\nu\tilde{n}.\sigma \vdash f(t_1, \ldots, t_k)$. Hence, $\nu\tilde{n}.\sigma \vdash u$ by rule DEq. If $K(t) \in S_n$ and $t \neq f(t_1, \ldots, t_k)$ we conclude by induction hypothesis as $\tilde{n} = \tilde{n}', \sigma = \sigma'$.
- If $S_{n-1} \xrightarrow{\mathit{ProtoNonce}(a)} S_n$ we have that $\mathsf{Fr}(a) \in S_{n-1}$. By Lemma 6, we obtain that if $!\mathsf{K}(t) \in S_{n-1}$ then there exist t' and σ'' such that $t' =_E t$, $\sigma'' =_E \sigma'$ and $a \notin st(t')$ and $a \notin st(\sigma'')$. For each $!\mathsf{K}(u) \in S_n$ there is $!\mathsf{K}(u) \in S_{n-1}$, and by induction hypothesis, $\nu\tilde{n}'.\sigma' \vdash u$. By Lemma 7 and the fact that $\sigma'' =_E \sigma'$ we have that $\nu\tilde{n}', a.\sigma' \vdash u$. As $\tilde{n}', a = \tilde{n}$ and $\sigma' = \sigma$ we conclude.
- All other rules do neither add $!\mathsf{K}(\ )$-facts nor do they change $\tilde{n}$ and may only extend σ. Therefore we conclude by the induction hypothesis.

2. Suppose that $\nu\tilde{n}.\sigma \vdash t$. We proceed by induction on the proof tree witnessing $\nu\tilde{n}.\sigma \vdash t$.

Base case. The proof tree consists of a single node. In this case one of the deduction rules DName or DFrame has been applied.

- DName. We have that $t \notin \tilde{n}$. If $t \in \mathit{PN}$ we use rule MDPub and we have that $S_n \rightarrow S = S_n \cup \{!\mathsf{K}(t)\}$. In case $t \in \mathit{FN}$ we need to consider 3 different cases: *(i)* $!\mathsf{K}(t) \in S_n$ and we immediately conclude (by letting $S = S_n$), *(ii)* $\mathsf{Fr}(t) \in S_n$ and applying rule MDFresh we have that $S_n \rightarrow S = S_n \cup \{!\mathsf{K}(t)\}$, *(iii)* $\mathsf{Fr}(t) \notin S_n$. By inspection of the rules we see that $\mathsf{Fr}(t) \notin S_i$ for any $1 \leq i \leq n$: the only rules that could remove $\mathsf{Fr}(t)$ are MDFresh which would have created the persistent fact $!\mathsf{K}(t)$, or the *ProtoNonce* rules which would however have added t to $\tilde{n}$. Hence, applying successively rules Fresh and MDFresh yields a valid extension of the execution $S_n \rightarrow S_n \cup \{\mathsf{Fr}(t)\} \rightarrow S = S_n \cup \{!\mathsf{K}(t)\}$.
- DFrame. We have that $x\sigma = t$ for some $x \in \mathbf{D}(\sigma)$, that is, $t \in \{t_1, \ldots, t_m\}$. By definition of $\{t_1, \ldots, t_m\}$, $\mathsf{Out}(t) \in S_i$ for some $i \leq n$. If $\mathsf{Out}(t) \in S_n$ we have that $S_n \rightarrow S = (S_n \setminus \{\mathsf{Out}(t)\}) \cup \{!\mathsf{K}(t)\}$ applying rule MDOut. If $\mathsf{Out}(u) \notin S_n$, the fact that the only rule in $[\![P]\!]$ that allows to remove an Out-fact is MDOut, suggests that it was applied before, and thus $!\mathsf{K}(u) \in S$.

Inductive case. We proceed by case distinction on the last deduction rule which was applied.

- DAppl. In this case $t = f(t_1, \dots, t_k)$, such that $f \in \Sigma^k$ and $\nu\tilde{n}\tilde{r}.\sigma \vdash t_i$ for every $i \in \{1, \dots, k\}$. Applying the induction hypothesis we obtain that there are k transition sequences $S_n \to_R^* S^i$ for $1 \leq i \leq k$ which extend the execution such that $t_i \in S^i$. All of them only add $!\mathsf{K}$ facts which are persistent facts. If any two of these extensions remove the same $\mathsf{Out}(t)$-fact or the same $\mathsf{Fr}(a)$-fact it also adds the persistent fact $!\mathsf{K}(t)$, respectively $!\mathsf{K}(a)$, and we simply remove the second occurrence of the transition. Therefore, applying the same rules as for the transitions $S_n \to^* S^i$ (and removing duplicate rules) we have that $S_n \to^* S'$ and $!\mathsf{K}(t_1), \dots, !\mathsf{K}(t_k) \in S'$. Applying rule MDAppl we conclude.
- DEq. By induction hypothesis there exists S as required with $!\mathsf{K}(t') \in_E S$ and $t =_E t'$ which allows us to immediately conclude that $!\mathsf{K}(t) \in_E S$.

□

Lemma 9. *If $\nu\tilde{n}.\sigma \vdash t$, $\tilde{n} =_E \tilde{n}'$, $\sigma =_E \sigma'$ and $t =_E t'$, then $\nu\tilde{n}'.\sigma' \vdash t'$.*

Proof. Assume $\nu\tilde{n}.\sigma \vdash t$. Since an application of DEq can be appended to the leafs of its proof tree, we have $\nu\tilde{n}.\sigma' \vdash t$. Since DEq can be applied to its root, we have $\nu\tilde{n}.\sigma' \vdash t'$. Since $\tilde{n}, \tilde{n}'$ consist only of names, $\tilde{n} = \tilde{n}'$ and thus $\nu\tilde{n}'.\sigma' \vdash t'$. □

To state our next lemma we need two additional definitions.

Definition 19. *Let P be a well-formed ground process and p_t a position in P. We define the set of multiset rewrite rules generated for position p_t of P, denoted $[\![P]\!]_{=p_t}$ as follows:*

$$[\![P]\!]_{=p_t} := [\![P, [], []]\!]_{=p_t}$$

where $[\![\cdot, \cdot, \cdot]\!]_{=p_t}$ is defined in Figure 12.

The next definition will be useful to state that for a process P every fact of the form $\mathsf{state}_p(\tilde{t})$ in a multiset rewrite execution of $[\![P]\!]$ corresponds to an active process in the execution of P which is an instance of the subprocess $P|_p$.

Definition 20. *Let P be a ground process, $\mathcal{P}$ be a multiset of processes and S a multiset of multiset rewrite rules. We write $\mathcal{P} \leftrightarrow_P S$ if there exists a bijection between $\mathcal{P}$ and the multiset $\{\mathsf{state}_p(\tilde{t}) \mid \exists p, \tilde{t}.\ \mathsf{state}_p(\tilde{t}) \in^\# S\}^\#$ such that whenever $Q \in^\# \mathcal{P}$ is mapped to $\mathsf{state}_p(\tilde{t}) \in^\# S$ we have that*

1. *$P|_p\tau = Q\rho$, for some substitution τ and some bijective renaming ρ of fresh, but not bound names in Q, and*
2. *$\exists ri \in_E ginsts([\![P]\!]_{=p})$. $\mathsf{state}_p(\tilde{t}) \in prems(ri)$.*

When $\mathcal{P} \leftrightarrow_P S$, $Q \in^\# \mathcal{P}$ and $\mathsf{state}_p(\tilde{t}) \in^\# S$ we also write $Q \leftrightarrow_P \mathsf{state}_p(\tilde{t})$ if this bijection maps Q to $\mathsf{state}_p(\tilde{t})$.

Remark 2. *Note that $\leftrightarrow_P$ has the following properties (by the fact that it defines a bijection between multisets).*

- *If $\mathcal{P}_1 \leftrightarrow_P S_1$ and $\mathcal{P}_2 \leftrightarrow_P S_2$ then $\mathcal{P}_1 \cup^\# \mathcal{P}_2 \leftrightarrow_P S_1 \cup^\# S_2$.*
- *If $\mathcal{P}_1 \leftrightarrow_P S_1$ and $Q \leftrightarrow_P \mathsf{state}_p(\tilde{t})$ for $Q \in \mathcal{P}_1$ and $\mathsf{state}_p(\tilde{t}) \in S_1$ (i.e. Q and $\mathsf{state}_p(\tilde{t})$ are related by the bijection defined by $\mathcal{P}_1 \leftrightarrow_P S_1$) then $\mathcal{P}_1 \setminus^\# \{Q\} \leftrightarrow_P S_1 \setminus^\# \{\mathsf{state}_p(\tilde{t})\}$.*

Proposition 4. *Let A be a finite set, $<$ a strict total order on A and p a predicate on elements of A. We have that*

$$\begin{array}{rl} \forall i \in A.p(i) & \Leftrightarrow\ \ \forall i \in A.\ p(i) \vee \exists j \in A.\ i < j \wedge \neg p(j) \\ & (\Leftrightarrow\ \ \forall i \in A.\ \neg p(i) \to \exists j \in A.\ i < j \wedge \neg p(j)) \end{array}$$

and

$$\exists i \in A.p(i) \ \ \Leftrightarrow\ \ \exists i \in A.p(i) \wedge \forall j \in A.\ i < j \to \neg p(j)$$

$$
\begin{aligned}
\llbracket 0, p, \tilde{x} \rrbracket_{=p_t} &= \{\, [\mathsf{state}_p(\tilde{x})] \rightarrow [] \,\}_{p \overset{?}{=} p_t} \\
\llbracket P \mid Q, p, \tilde{x} \rrbracket_{=p_t} &= \{\, [\mathsf{state}_p(\tilde{x})] \rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x}), \mathsf{state}_{p\cdot 2}(\tilde{x})] \,\}_{p \overset{?}{=} p_t} \cup \llbracket P, p \cdot 1, \tilde{x} \rrbracket_{=p_t} \cup \llbracket Q, p \cdot 2, \tilde{x} \rrbracket_{=p_t} \\
\llbracket !P, p, \tilde{x} \rrbracket_{=p_t} &= \{\, [!\mathsf{state}_p(\tilde{x})] \rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x})] \,\}_{p \overset{?}{=} p_t} \cup \llbracket P, p \cdot 1, \tilde{x} \rrbracket_{=p_t} \\
\llbracket \nu a; P, p, \tilde{x} \rrbracket_{=p_t} &= \{\, [\mathsf{state}_p(\tilde{x}), \mathsf{Fr}(n_a : \mathit{fresh})] -\![\mathit{ProtoNonce}(n_a : \mathit{fresh})]\!\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x}, n_a : \mathit{fresh})] \,\}_{p \overset{?}{=} p_t} \\
&\quad \cup \llbracket P, p \cdot 1, (\tilde{x}, n_a : \mathit{fresh}) \rrbracket_{=p_t} \\
\llbracket \mathrm{Out}(M, N); P, p, \tilde{x} \rrbracket_{=p_t} &= \{\, [\mathsf{state}_p(\tilde{x}), \mathsf{In}(M)] -\![\mathrm{InEvent}(M)]\!\!\rightarrow [\mathsf{Out}(N), \mathsf{state}_{p\cdot 1}(\tilde{x})], \\
&\qquad [\mathsf{state}_p(\tilde{x})] \rightarrow [\mathsf{Msg}(M, N), \mathsf{state}_p^{\mathsf{semi}}(\tilde{x})], \\
&\qquad [\mathsf{state}_p^{\mathsf{semi}}(\tilde{x}), \mathsf{Ack}(M, N)] \rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x})] \;\}_{p \overset{?}{=} p_t} \cup \llbracket P, p \cdot 1, \tilde{x} \rrbracket_{=p_t} \\
\llbracket \mathrm{In}(M, N); P, p, \tilde{x} \rrbracket_{=p_t} &= \{\, [\mathsf{state}_p(\tilde{x}), \mathsf{In}(\langle M, N \rangle)] -\![\mathrm{InEvent}(\langle M, N \rangle)]\!\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x} \cup \mathit{vars}(N))], \\
&\qquad [\mathsf{state}_p(\tilde{x}), \mathsf{Msg}(M, N)] \rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x} \cup \mathit{vars}(N)), \mathsf{Ack}(M, N)] \,\}_{p \overset{?}{=} p_t} \\
&\quad \cup \llbracket P, p \cdot 1, \tilde{x} \cup \mathit{vars}(N) \rrbracket_{=p_t} \\
\llbracket \text{if } M = N \text{ then } P &= \{\, [\mathsf{state}_p(\tilde{x})] -\![\quad \mathrm{Eq}(M, N) \quad]\!\!\rightarrow \quad [\mathsf{state}_{p\cdot 1}(\tilde{x})], \\
\text{else } Q, p, \tilde{x} \rrbracket_{=p_t} &\qquad [\mathsf{state}_p(\tilde{x})] -\![\mathrm{NotEq}(M, N)]\!\!\rightarrow \quad [\mathsf{state}_{p\cdot 2}(\tilde{x})] \,\}_{p \overset{?}{=} p_t} \\
&\quad \cup \llbracket P, p \cdot 1, \tilde{x} \rrbracket_{=p_t} \cup \llbracket Q, p \cdot 2, \tilde{x} \rrbracket_{=p_t} \\
\llbracket \text{event } F; P, p, \tilde{x} \rrbracket_{=p_t} &= \{\, [\mathsf{state}_p(\tilde{x})] -\![\mathrm{Event}(), F]\!\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x})] \,\}_{p \overset{?}{=} p_t} \cup \llbracket P, p \cdot 1, \tilde{x} \rrbracket_{=p_t} \\
\llbracket \text{insert } s, t; P, p, \tilde{x} \rrbracket_{=p_t} &= \{\, [\mathsf{state}_p(\tilde{x})] -\![\mathrm{Insert}(s, t)]\!\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x})] \,\}_{p \overset{?}{=} p_t} \cup \llbracket P, p \cdot 1, \tilde{x} \rrbracket_{=p_t} \\
\llbracket \text{delete } s; P, p, \tilde{x} \rrbracket_{=p_t} &= \{\, [\mathsf{state}_p(\tilde{x})] -\![\mathrm{Delete}(s)]\!\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x})] \,\}_{p \overset{?}{=} p_t} \cup \llbracket P, p \cdot 1, \tilde{x} \rrbracket_{=p_t} \\
\llbracket \text{lookup } M \text{ as } v &= \{\, [\mathsf{state}_p(\tilde{x})] -\![\mathrm{IsIn}(M, v)]\!\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{M}, v)], \\
\text{in } P \text{ else } Q, p, \tilde{x} \rrbracket_{=p_t} &\qquad [\mathsf{state}_p(\tilde{x})] -\![\mathrm{IsNotSet}(M)]\!\!\rightarrow [\mathsf{state}_{p\cdot 2}(\tilde{x})] \,\}_{p \overset{?}{=} p_t} \\
&\quad \cup \llbracket P, p \cdot 1, (\tilde{x}, v) \rrbracket_{=p_t} \cup \llbracket Q, p \cdot 2, \tilde{x} \rrbracket_{=p_t} \\
\llbracket \text{lock}^l\ s; P, p, \tilde{x} \rrbracket_{=p_t} &= \{\, [\mathrm{Fr}(\mathrm{lock}_l), \mathsf{state}_p(\tilde{x})] -\![\mathrm{Lock}(\mathit{lock}_l, s)]\!\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x}, \mathit{lock}_l)] \,\}_{p \overset{?}{=} p_t} \\
&\quad \cup \llbracket P, p \cdot 1, \tilde{x} \rrbracket_{=p_t} \\
\llbracket \text{unlock}^l\ s; P, p, \tilde{x} \rrbracket_{=p_t} &= \{\, [\mathsf{state}_p(\tilde{x})] -\![\mathrm{Unlock}(\mathit{lock}_l, s)]\!\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x})] \,\}_{p \overset{?}{=} p_t} \cup \llbracket P, p \cdot 1, \tilde{x} \rrbracket_{=p_t} \\
\llbracket [l -\![a]\!\!\rightarrow r];\ \mathrm{P}, p, \tilde{x} \rrbracket_{=p_t} &= \{\, [\mathsf{state}_p(\tilde{x}), l] -\![\mathrm{Event}(), a]\!\!\rightarrow [r, \mathsf{state}_{p\cdot 1}(\tilde{x} \cup \mathit{vars}(l))] \,\}_{p \overset{?}{=} p_t} \\
&\quad \cup \llbracket P, p \cdot 1, \tilde{x} \cup \mathit{vars}(l) \rrbracket
\end{aligned}
$$

Figure 12: Definition of $\llbracket P, p, \tilde{x} \rrbracket_{=p_t}$ where $\{\cdot\}_{a \overset{?}{=} b} = \{\, \cdot \,\}$ if $a = b$ and $\emptyset$ otherwise.

Lemma 10. *Le P be a well-formed ground process. If*

$$(\mathcal{E}_0, \mathcal{S}_0, \mathcal{S}_0^{\mathrm{MS}}, \mathcal{P}_0, \sigma_0, \mathcal{L}_0) \xrightarrow{E_1} (\mathcal{E}_1, \mathcal{S}_1, \mathcal{S}_1^{\mathrm{MS}}, \mathcal{P}_1, \sigma_1, \mathcal{L}_1) \xrightarrow{E_2} \dots \xrightarrow{E_n} (\mathcal{E}_n, \mathcal{S}_n, \mathcal{S}_n^{\mathrm{MS}}, \mathcal{P}_n, \sigma_n, \mathcal{L}_n)$$

where $(\mathcal{E}_0, \mathcal{S}_0, \mathcal{S}_0^{\mathrm{MS}}, \mathcal{P}_0, \sigma_0, \mathcal{L}_0) = (\emptyset, \emptyset, \emptyset, \{ P \}, \emptyset, \emptyset)$ *then there are* $(F_1, S_1), \dots, (F_{n'}, S_{n'})$ *such that*

$$\emptyset \xrightarrow{F_1}_{[\![P]\!]} S_1 \xrightarrow{F_2}_{[\![P]\!]} \dots \xrightarrow{F_{n'}}_{[\![P]\!]} S_{n'} \in \mathit{exec}^{msr}([\![P]\!])$$

and there exists a monotonic, strictly increasing function $f \colon \mathbb{N}_n \to \mathbb{N}_{n'}$ *such that* $f(n) = n'$ *and for all* $i \in \mathbb{N}_n$

1. $\mathcal{E}_i = \{ a \mid \mathit{ProtoNonce}(a) \in \bigcup_{1 \le j \le f(i)} F_j \}$
2. $\forall t \in \mathcal{M}.\, \mathcal{S}_i(t) = \begin{cases} u & \text{if } \exists j \le f(i).\mathrm{Insert}(t,u) \in F_j \\ & \quad \wedge \forall j', u'. j < j' \le f(i) \to \mathrm{Insert}(t,u') \notin_E F_{j'} \wedge \mathrm{Delete}(t) \notin_E F_{j'} \\ \perp & \text{otherwise} \end{cases}$
3. $\mathcal{S}_i^{\mathrm{MS}} = S_{f(i)} \setminus^{\#} \mathcal{F}_{\mathit{res}}$
4. $\mathcal{P}_i \leftrightarrow_P S_{f(i)}$
5. $\{ x\sigma_i \mid x \in \mathbf{D}(\sigma_i) \}^{\#} = \{\mathsf{Out}(t) \in \cup_{k \le f(i)} S_k\}^{\#}$
6. $\mathcal{L}_i =_E \{ t \mid \exists j \le f(i), u.\ \mathrm{Lock}(u,t) \in_E F_j \wedge \forall j < k \le f(i).\mathrm{Unlock}(u,t) \notin_E F_k \}$
7. $[F_1, \dots, F_{n'}] \vDash \alpha$ *where α is defined as in Definition 14.*
8. $\exists k.\ f(i-1) < k \le f(i)$ *and* $E_i = F_k$ *and* $\cup_{f(i-1) < j \ne k \le f(i)} F_j \subseteq \mathcal{F}_{\mathit{res}}$

Proof. We proceed by induction over the number of transitions n.

Base Case. For $n = 0$, we let $f(n) = 1$ and S_1 be the multiset obtained by using the Rule Init:

$$\emptyset \xrightarrow{\mathsf{Init}} \{ \mathsf{state}_{[]}() \}^{\#}$$

Condition 1, Condition 2, Condition 3, Condition 5, Condition 6, Condition 7 and Condition 8 hold trivially. To show that Condition 4 holds, we have to show that $\mathcal{P}_0 \leftrightarrow_P \{ \mathsf{state}_{[]}() \}^{\#}$. Note that $\mathcal{P}_0 = \{ P \}^{\#}$. We choose the bijection such that $P \leftrightarrow_P \mathsf{state}_{[]}()$. For $\tau = \emptyset$ and $\rho = \emptyset$ we have that $P|_{[]}\tau = P\tau = P\rho$. By Definition 19, $[\![P]\!]_{=[]} = [\![P, [], []]\!]_{=[]}$. We see from Figure 12 that for every P we have that $\mathsf{state}_{[]}() \in \mathit{prems}([\![P, [], []]\!]_{=[]})$. Hence, we conclude that there is a ground instance $ri \in_E \mathit{ginsts}([\![P]\!]_{=[]})$ with $\mathsf{state}_{[]}() \in \mathit{prems}(ri)$.

Inductive step. Assume the invariant holds for $n - 1 \ge 0$. We have to show that the lemma holds for n transitions

$$(\mathcal{E}_0, \mathcal{S}_0, \mathcal{S}_0^{\mathrm{MS}}, \mathcal{P}_0, \sigma_0, \mathcal{L}_0) \xrightarrow{E_1} (\mathcal{E}_1, \mathcal{S}_1, \mathcal{S}_1^{\mathrm{MS}}, \mathcal{P}_1, \sigma_1, \mathcal{L}_1) \xrightarrow{E_2} \dots \xrightarrow{E_n} (\mathcal{E}_n, \mathcal{S}_n, \mathcal{S}_n^{\mathrm{MS}}, \mathcal{P}_n, \sigma_n, \mathcal{L}_n)$$

By induction hypothesis, we have that there exists a monotonically increasing function from $\mathbb{N}_{n-1} \to \mathbb{N}_{n'}$ and an execution

$$\emptyset \xrightarrow{F_1}_{[\![P]\!]} S_1 \xrightarrow{F_2}_{[\![P]\!]} \dots \xrightarrow{F_{n'}}_{[\![P]\!]} S_{n'} \in \mathit{exec}^{msr}([\![P]\!])$$

such that Conditions 1 to 8 hold. Let f_p be this function and note that $n' = f_p(n-1)$. Fix a bijection such that $\mathcal{P}_{n-1} \leftrightarrow_P S_{f_p(n-1)}$. We will abuse notation by writing $P \leftrightarrow_P \mathsf{state}_p(\tilde{t})$, if this bijection goes from P to $\mathsf{state}_p(\tilde{t})$.

We now proceed by case distinction over the type of transition from $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}_{n-1}^{\mathrm{MS}}, \mathcal{P}_{n-1}, \sigma_{n-1}, \mathcal{L}_{n-1})$ to $(\mathcal{E}_n, \mathcal{S}_n, \mathcal{S}_n^{\mathrm{MS}}, \mathcal{P}_n, \sigma_n, \mathcal{L}_n)$. We will (unless stated otherwise) extend the previous execution by a number of steps, say s, from $S_{n'}$ to some $S_{n'+s}$, and prove that Conditions 1 to 8 hold for n (since by induction hypothesis, they hold for all $i < n$) and a function $f \colon \mathbb{N}_n \to \mathbb{N}_{n'+s}$ that is defined as follows:

$$f(i) := \begin{cases} f_p(i) & \text{if } i \in \mathbb{N}_{n-1} \\ n' + s & \text{if } i = n \end{cases}$$

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{0\}, \sigma_{n-1}, \mathcal{L}_{n-1}) \;\rightarrow (\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}', \sigma_{n-1}, \mathcal{L}_{n-1})$**.** By induction hypothesis $\mathcal{P}_{n-1} \leftrightarrow_P S_{n'}$. Let p and $\tilde{t}$ be such that $0 \leftrightarrow_P \mathsf{state}_p(\tilde{t})$. By Definition 20, there is a $ri \in ginsts([\![P]\!]_{=p})$ such that $\mathsf{state}_p(\tilde{t})$ is part of its premise. By definition of $[\![P]\!]_{=p}$, we can choose $ri = [\mathsf{state}_p(\tilde{t})] \;\text{--}[\;]\!\rightarrow []$. We can extend the previous execution by one step using ri, therefore:

$$\emptyset \xrightarrow{F_1}_{[\![P]\!]} S_1 \xrightarrow{F_2}_{[\![P]\!]} \ldots \xrightarrow{F_{n'}}_{[\![P]\!]} S_{n'} \rightarrow_{[\![P]\!]} S_{n'+1} \in exec^{msr}([\![P]\!])$$

with $S_{n'+1} = \{\, S_{f(n-1)} \setminus \{\mathsf{state}_p(\tilde{t})\,\}$. It is left to show that Conditions 1 to 8 hold for n.

Condition 1, Condition 2, Condition 3, Condition 5, Condition 6, Condition 7, and Condition 8 hold trivially.

Condition 4 holds because $\mathcal{P}' = \mathcal{P}_{n-1} \setminus^{\#} \{0\}$, $S_{f(n)} = S_{f(n-1)} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t})\,\}^{\#}$, and $0 \leftrightarrow_P \mathsf{state}_p(\tilde{t})$ (see Remark 2).

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{Q|R\}, \sigma_{n-1}, \mathcal{L}_{n-1}) \;\rightarrow (\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}' \cup \{Q, R\},$ $\sigma_{n-1}, \mathcal{L}_{n-1})$**.** By induction hypothesis $\mathcal{P}_{n-1} \leftrightarrow_P S_{n'}$. Let p and $\tilde{t}$ be such that $Q|R \leftrightarrow_P \mathsf{state}_p(\tilde{t})$. By Definition 20, there is a $ri \in ginsts([\![P]\!]_{=p})$ such that $\mathsf{state}_p(\tilde{t})$ is part of its premise. By definition of $[\![P]\!]_{=p}$, we can choose $ri = [\mathsf{state}_p(\tilde{t})] \;\text{--}[\;]\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{t}), \mathsf{state}_{p\cdot 2}(\tilde{t})]$. We can extend the previous execution by one step using ri, therefore:

$$\emptyset \xrightarrow{F_1}_{[\![P]\!]} S_1 \xrightarrow{F_2}_{[\![P]\!]} \ldots \xrightarrow{F_{n'}}_{[\![P]\!]} S_{n'} \rightarrow_{[\![P]\!]} S_{n'+1} \in exec^{msr}([\![P]\!])$$

with $S_{n'+1} = S_{f(n-1)} \setminus \{\, \mathsf{state}_p(\tilde{t})\,\}^{\#} \cup \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}), \mathsf{state}_{p\cdot 2}(\tilde{t})\,\}^{\#}$. It is left to show that Conditions 1 to 8 hold for n.

Condition 1, Condition 2, Condition 3, Condition 5, Condition 6, Condition 7 and Condition 8 hold trivially. We now show that Condition 4 holds.

Condition 4 holds because $\mathcal{P}_n = \mathcal{P}_{n-1} \setminus^{\#} \{Q|R\} \cup^{\#} \{Q, R\}$, $\{Q\} \leftrightarrow_P \{\mathsf{state}_{p\cdot 1}(\tilde{x})\}$ and $\{R\} \leftrightarrow_P \{\mathsf{state}_{p\cdot 2}(\tilde{x})\}$ (by definition of the translation) (see Remark 2).

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{!Q\}, \sigma_{n-1}, \mathcal{L}_{n-1}) \;\rightarrow (\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}' \cup \{!Q, Q\},$ $\sigma_{n-1}, \mathcal{L}_{n-1})$**.** Let p and $\tilde{t}$ such that $!^i Q \leftrightarrow_P \mathsf{state}_p(\tilde{t})$. By Definition 20, there is a $ri \in ginsts([\![P]\!]_{=p})$ such that $\mathsf{state}_p(\tilde{t})$ is part of its premise. By definition of $[\![P]\!]_{=p}$, we can choose $ri = [\mathsf{state}_p(\tilde{t})] \;\text{--}[\;]\!\rightarrow$ $[\mathsf{state}_p(\tilde{t}), \mathsf{state}_{p\cdot 1}(\tilde{t})]$. We can extend the previous execution by 1 step using ri, therefore:

$$\emptyset \xrightarrow{F_1}_{[\![P]\!]} S_1 \xrightarrow{F_2}_{[\![P]\!]} \ldots \xrightarrow{F_{n'}}_{[\![P]\!]} S_{n'} \xrightarrow{(ri)}_{[\![P]\!]} S_{n'+1} \in exec^{msr}([\![P]\!])$$

with $S_{n'+1} = S_{f(n)} \cup^{\#} \{\, \mathsf{state}_{p\cdot 1}(\tilde{t})\,\}^{\#}$. Condition 4 holds because $\mathcal{P}_n = \mathcal{P}_{n-1} \cup^{\#} \{Q\}$ and $\{Q\} \leftrightarrow_P \{\mathsf{state}_{p\cdot 1}(\tilde{t})\}$ (by definition of $[\![P]\!]_{=p}$). Condition 1, Condition 2, Condition 3, Condition 5, Condition 6, Condition 7 and Condition 8 hold trivially.

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{i_n-1}, \mathcal{S}^{\mathrm{MS}}_{i_n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{\, \nu a; Q\,\}, \sigma_{n-1}, \mathcal{L}_{n-1}) \;\rightarrow (\mathcal{E}_{n-1} \cup \{a'\}, \mathcal{S}_{i_n-1}, \mathcal{S}^{\mathrm{MS}}_{i_n-1},$ $\mathcal{P}' \cup \{Q\{^{a'}/_{a}\}\}, \sigma_{n-1}, \mathcal{L}_{n-1})$ **for a fresh** a'**.** Let p and $\tilde{t}$ be such that $\{\nu a; Q\} \leftrightarrow_P \mathsf{state}_p(\tilde{t})$. There is a $ri \in ginsts([\![P]\!]_{=p})$ such that $\mathsf{state}_p(\tilde{t})$ is part of its premise. By definition of $[\![P]\!]_{=p}$, there is a $ri \in ginsts([\![P]\!]_{=p})$, $ri = [\mathsf{state}_p(\tilde{t}), \mathsf{Fr}(a' : \mathit{fresh})] \;\text{--}[\mathit{ProtoNonce}(a' : \mathit{fresh})]\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{t}, a' : \mathit{fresh})]$. Assume there is an $i < n'$ such that $\mathsf{Fr}(a') \in S_i$. If $\mathsf{Fr}(a') \in S_n$, then we can remove the application of the instance of FRESH that added $\mathsf{Fr}(a')$ while still preserving Conditions 1 to 8. If $\mathsf{Fr}(a')$ is consumned at some point, by the definition of $[\![P]\!]$, the transition where it is consumned is annotated either $\mathit{ProtoNonce}(a')$ or $\mathit{Lock}(a', t)$ for some t. In the last case, we can apply a substitution to the execution that substitutes a by a different fresh name that never appears in $\cup_i \leq n' S_i$. The conditions we have by induction hypothesis hold on this execution, too, since $\mathit{Lock} \in \mathcal{F}_{res}$, and therefore Condition 8 is not affected. The first case implies that $a' \in \mathcal{E}_{n-1}$, contradicting the assumption that a' is fresh with respect to the process execution. Therefore, without loss of generality, the previous execution does not contain an $i < n'$ such that $\mathsf{Fr}(a') \in S_i$, and we can extend the previous execution by two steps using the FRESH rule and ri, therefore:

$$\emptyset \xrightarrow{F_1}_{[\![P]\!]} S_1 \xrightarrow{F_2}_{[\![P]\!]} \ldots \xrightarrow{F_{n'}}_{[\![P]\!]} S_{n'} \xrightarrow{(\text{FRESH})}_{[\![P]\!]} S_{n'+1} \xrightarrow{(ri)}_{[\![P]\!]} S_{n'+2} \in exec^{msr}([\![P]\!])$$

with $S_{n'+1} = S_{n'} \cup^{\#} \{ \mathsf{Fr}(a' : \mathit{fresh}) \}^{\#}$ and $S_{n'+2} = S_{f(n)} = S_{n'} \cup^{\#} \{ \mathsf{state}_{p\cdot 1}(\tilde{t}, a' : \mathit{fresh}) \}^{\#}$. We define $f(i) := f_p(i)$ for $i < n$ and $f(n) := f(n-1) + 2$. We now show that Condition 4 holds. As by induction hypothesis $\nu a; Q \leftrightarrow_P \mathsf{state}_{p\cdot 1}(\tilde{t})$ we also have that $P|_p\sigma = \nu a; Q\rho$ for some σ and ρ. Extending ρ with $\{a' \mapsto a\}$ it is easy to see from definition of $[\![P]\!]_{=p}$ that $\{Q\{^{a'}/_{a}\}\} \leftrightarrow_P \{\mathsf{state}_{p\cdot 1}(\tilde{t}, a')\}$. As $\mathcal{P}_n = \mathcal{P}_{n-1} \setminus^{\#} \{\nu a; Q\} \cup^{\#} \{ Q\{^{a'}/_{a}\} \}^{\#}$, we also immediately obtain that $\mathcal{P}_n \leftrightarrow_P S_{f(n)}$. Since a' is fresh, and therefore $\{ a' \} = \mathcal{E}_n \setminus \mathcal{E}_{n-1}$, and $F_n = \mathit{ProtoNonce}(a')$, Condition 1 holds. Condition 2, Condition 3, Condition 5, Condition 6, Condition 7 and Condition 8 hold trivially.

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1}, \sigma_{n-1}, \mathcal{L}_{n-1}) \overset{K(t)}{\longrightarrow} (\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1}, \sigma_{n-1}, \mathcal{L}_{n-1})$. This step requires that $\nu\mathcal{E}_{n-1}.\sigma_{n-1} \vdash t$. From Lemma 8 follows that there is an execution $\emptyset \xrightarrow{F_1} S_1 \xrightarrow{F_2} \ldots \xrightarrow{F_{n'}} S_{n'} \rightarrow^* S \in \mathit{exec}^{msr}_{E}([\![P]\!])$ such that $!\mathsf{K}(t) \in_E S$ and $S_{n'} \rightarrow^*_R S$ for $R = \{ \textsc{MDOut}, \textsc{MDPub}, \textsc{MDFresh}, \textsc{MDAppl} \}$.

From S, we can go one further step using $\textsc{MDIn}$, since $!\mathsf{K}(t) \in S$:

$$\emptyset \xrightarrow{F_1}_{[\![P]\!]} S_1 \xrightarrow{F_2}_{[\![P]\!]} \ldots \xrightarrow{F_{n'}}_{[\![P]\!]} S_{n'} \rightarrow^*_{R \subset [\![P]\!]} S = S_{n'+s-1} \xrightarrow{K(t)}_{[\![P]\!]} S_{n'+s} \in \mathit{exec}^{msr}([\![P]\!])$$

where $S_{n'+s} = S \cup \{\mathsf{In}(t)\}$.

From the fact that $S_{f(n-1)} \rightarrow^*_R S_{f(n)} = S$, and the induction hypothesis, we can conclude that Condition 8 holds. Condition 4 holds since $\mathcal{P}_n = \mathcal{P}_{n-1}$ and no `state`-facts where neither removed nor added. Condition 1, Condition 2, Condition 3, Condition 5, Condition 6 and Condition 7 hold trivially.

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{ \mathrm{out}(t, t'); Q \}, \sigma_{n-1}, \mathcal{L}_{n-1}) \xrightarrow{K(t)} (\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}' \cup^{\#} \{ Q \}, \sigma_{n-1} \cup \{^{t'}/_{x}\}, \mathcal{L}_{n-1})$. This step requires that x is fresh and $\nu\mathcal{E}_{n-1}.\sigma \vdash t$. Using Lemma 8, we have that there is an execution $\emptyset \xrightarrow{F_1} S_1 \xrightarrow{F_2} \ldots \xrightarrow{F_{f(n)}} S_{f(n-1)} \rightarrow^* S \in \mathit{exec}^{msr}_{E}([\![P]\!])$ such that $!\mathsf{K}(t) \in_E S$ and $S_{f(n-1)} \rightarrow^*_R S$ for $R = \{ \textsc{MDOut}, \textsc{MDPub}, \textsc{MDFresh}, \textsc{MDAppl} \}$. Let p and $\tilde{t}$ such that $\{\mathrm{out}(t, t'); Q\} \leftrightarrow_P \mathsf{state}_p(\tilde{t})$. By Definition 20, there is a $\mathit{ri} \in \mathit{ginsts}([\![P]\!]_{=p})$ such that $\mathsf{state}_p(\tilde{t})$ is part of its premise. From the definition of $[\![P]\!]_{=p}$, we see that we can choose $\mathit{ri} = [\mathsf{state}_p(\tilde{t}), \mathsf{In}(t)] \;{-}[\mathrm{InEvent}(t)]{\rightarrow}\; [\mathsf{Out}(t'), \mathsf{state}_{p\cdot 1}(\tilde{t})]$. To apply this rule, we need the fact $\mathsf{In}(t)$. Since $\nu\mathcal{E}_{n-1}.\sigma \vdash t$, as mentioned before, we can apply Lemma 8. It follows that there is an execution $\emptyset \xrightarrow{F_1} S_1 \xrightarrow{F_2} \ldots \xrightarrow{F_{n'}} S_{n'} \rightarrow^* S \in \mathit{exec}^{msr}_{E}([\![P]\!])$ such that $!\mathsf{K}(t) \in_E S$ and $S_{n'} \rightarrow^*_R S$ for $R = \{ \textsc{MDOut}, \textsc{MDPub}, \textsc{MDFresh}, \textsc{MDAppl} \}$. From S, we can now go two steps further, using $\textsc{MDIn}$ and ri:

$$\emptyset \xrightarrow{F_1}_{[\![P]\!]} S_1 \ldots \xrightarrow{F_{n'}}_{[\![P]\!]} S_{n'} \rightarrow^*_{R \subset [\![P]\!]} S = S_{n'+s-2} \xrightarrow{K(t)}_{[\![P]\!]} S_{n'+s-1} \xrightarrow{\mathrm{InEvent}(t)}_{[\![P]\!]} S_{n'+s} \in \mathit{exec}^{msr}([\![P]\!])$$

where $S_{n'+s-1} = S \cup^{\#} \{ \mathsf{In}(t) \}^{\#}$ and $S_{f(n)} = S \setminus^{\#} \{ \mathsf{state}_p(\tilde{t}) \} \cup^{\#} \{ \mathsf{Out}(t'), \mathsf{state}_{p\cdot 1}(\tilde{t}) \}$.

Taking $k = n' + s - 1$ we immediately obtain that Condition 8 holds. Note first that, since $S_{n'} \rightarrow_R S$, $\mathit{set}(S_{n'}) \setminus \{ \mathsf{Fr}(t), \mathsf{Out}(t) | t \in \mathcal{M} \} \subset \mathit{set}(S)$ and $\mathit{set}(S) \setminus \{ !\mathsf{K}(t) | t \in \mathcal{M} \} \subset \mathit{set}(S_{n'})$. Since $\mathcal{P}_n = \mathcal{P}_{n-1} \setminus \{\mathrm{out}(t, t'); Q\} \cup \{Q\}$ and $\{Q\} \leftrightarrow_P \{\mathsf{state}_{p\cdot 1}(\tilde{t})\}$ (by definition of $[\![P]\!]_{=p}$), we have that $\mathcal{P}_n \leftrightarrow_P S_{f(n)}$, i.e., Condition 4 holds. Condition 5 holds since t' was added to σ_{n-1} and $\mathsf{Out}(t)$ added to $S_{f(n-1)}$. Condition 7 holds since $\mathrm{K}(t)$ appears right before $\mathrm{InEvent}(t)$. Condition 1, Condition 2, Condition 3 and Condition 6 hold trivially.

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{\mathrm{in}(t, N); Q\}, \sigma_{n-1}, \mathcal{L}_{n-1}) \rightarrow (\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}' \cup^{\#} \{ Q\theta \}, \sigma_{n-1}, \mathcal{L}_{n-1})$. This step requires that θ is grounding for N and that $\nu\mathcal{E}_{n-1}.\sigma_{n-1} \vdash \langle t, N\theta \rangle$. Using Lemma 8, we have that there is an execution $\emptyset \xrightarrow{F_1} S_1 \xrightarrow{F_2} \ldots \xrightarrow{F_{f(n-1)}} S_{f(n-1)} \rightarrow^* S \in \mathit{exec}^{msr}_{E}([\![P]\!])$ such that $!\mathsf{K}(t) \in_E S$ and $S_{f(n-1)} \rightarrow^*_R S$ for $R = \{ \textsc{MDOut}, \textsc{MDPub}, \textsc{MDFresh}, \textsc{MDAppl} \}$. The same holds for $N\theta$. We can combine those executions, by removing duplicate instantiations of $\textsc{Fresh}$, $\textsc{MDFresh}$ and $\textsc{MDOut}$. (This is possible since $!\mathsf{K}$ is persistent.) Let $\emptyset \xrightarrow{F_1} S_1 \xrightarrow{F_2} \ldots \xrightarrow{F_{f(n-1)}} S_{f(n-1)} \rightarrow^*_R \overline{S} \in \mathit{exec}^{msr}_{E}([\![P]\!])$ this combined execution, and $!\mathsf{K}(t), !\mathsf{K}(N\theta) \in_E \overline{S}$.

Let p and $\tilde{t}$ be such that, $\mathrm{in}(t, N); Q \leftrightarrow_P \mathsf{state}_p(\tilde{t})$. By Definition 20 there is a $\mathit{ri} \in \mathit{ginsts}([\![P]\!]_{=p})$ such that $\mathsf{state}_p(\tilde{t})$ is part of its premise. From the definition of $[\![P]\!]_{=p}$ and the fact that θ is grounding for $N\theta$, we have $\mathsf{state}_p(\tilde{t})$ in their premise, namely,

$$ri = [\mathsf{state}_p(\tilde{t}), \mathsf{In}(\langle t, N\theta\rangle)] \ -[\mathrm{InEvent}(\langle t, N\theta\rangle)]\!\!\rightarrow \ [\mathsf{state}_{p\cdot 1}(\tilde{t} \cup (\mathit{vars}(N)\theta)].$$

From $S_{n'}$, we can first apply the above transition $S_{n'} \rightarrow^*_R \overline{S}$, and then, (since $!\mathsf{K}(t), !\mathsf{K}(N\theta)$, $\mathsf{state}_p(\tilde{x}) \in \overline{S}$), MDAppl for the pair constructer, MDIn and ri:

$$\begin{aligned}\emptyset \xrightarrow{F_1}_{\llbracket P\rrbracket} S_1 \ldots \xrightarrow{F_{n'}}_{\llbracket P\rrbracket} S_{n'} \rightarrow^*_{R\subset\llbracket P\rrbracket} \overline{S} = S_{n'+s-3} \\ \xrightarrow{(\textsc{MdAppl})}_{\llbracket P\rrbracket} S_{n'+s-2} \xrightarrow{K(\langle t,N\theta\rangle)}_{\llbracket P\rrbracket} S_{n'+s-1} \xrightarrow{\mathrm{InEvent}(\langle t,N\theta\rangle)}_{\llbracket P\rrbracket} S_{n'+s} \in \mathit{exec}^{msr}(\llbracket P\rrbracket), \text{ where}\end{aligned}$$

- since $S_{n'} \rightarrow_R S$, S is such that $\mathit{set}(S_{n'}) \setminus \{\, \mathsf{Fr}(t), \mathsf{Out}(t) | t \in \mathcal{M}\,\} \subseteq \mathit{set}(S)$, $\mathit{set}(S) \setminus \{\, !\mathsf{K}(t) | t \in \mathcal{M}\,\} \subseteq \mathit{set}(S_{n'})$, and $!\mathsf{K}(t), !\mathsf{K}(N\theta) \in S$
- $S_{n'+s-2} = S \cup^{\#} \{\, !\mathsf{K}(\langle t, N\theta\rangle)\,\}^{\#}$,
- $S_{n'+s-1} = S \cup^{\#} \{\, \mathsf{In}(\langle t, N\theta\rangle)\,\}^{\#}$,
- $S_{n'+s} = S \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t})\,\} \cup^{\#} \{\, \mathsf{state}_{p\cdot 1}(\tilde{t} \cup (\mathit{vars}(N)\theta))\,\}$.

Letting $k = n' + s - 1$ we immediately have that Condition 8 holds.

We now show that Condition 4 holds. Since by induction hypothesis, $\mathrm{in}(t, N); Q \leftrightarrow_P \mathsf{state}_p(\tilde{t})$, we have that $P|_p\tau = \mathrm{in}(t, N); Q\rho$ for some τ and ρ. Therefore we also have that $P|_{p\cdot 1}\tau \cup (\theta\rho) = Q\rho(\theta\rho)$ and it is easy to see from definition of $\llbracket P\rrbracket_{=p}$ that $\{Q\theta\} \leftrightarrow_P \{\mathsf{state}_{p\cdot 1}(\tilde{t}, (\mathit{vars}(N)\theta))\}$. Since $\mathcal{P}_n = \mathcal{P}_{n-1} \setminus^{\#} \{\mathrm{in}(t, N); Q\} \cup^{\#} \{Q\}$, we have that $\mathcal{P}_n \leftrightarrow_P S_{f(n)}$, i. e., Condition 4 holds. Condition 7 holds since $\mathrm{K}\langle t, N\theta\rangle)$ appears right before $\mathrm{InEvent}\langle t, N\theta\rangle)$. Condition 1, Condition 2, Condition 3, Condition 5 and Condition 7 hold trivially.

Case: $(\mathcal{E}, \mathcal{S}, \mathcal{S}^{\mathrm{MS}}, \mathcal{P} \cup \{\mathrm{out}(c, m); Q\} \cup \{\mathrm{in}(c', N); R\}, \sigma, \mathcal{L}) \ \rightarrow (\mathcal{E}, \mathcal{S}, \mathcal{S}^{\mathrm{MS}}, \mathcal{P} \cup \{\, Q, R\theta\,\}, \sigma, \mathcal{L})$**.** This step requires that θ grounding for N, $t =_E N\theta$ and $c =_E c'$. Let p, p' and $\tilde{t}, \tilde{N}$ such that $\{\mathrm{out}(c, m); P\} \leftrightarrow_P \mathsf{state}_p(\tilde{t})$, $\{\mathrm{in}(c', N); Q\} \leftrightarrow_P \mathsf{state}_{p'}(\tilde{t}')$, and there are $ri \in \mathit{ginsts}(\llbracket P\rrbracket_{=p})$ and $ri' \in \mathit{ginsts}(\llbracket P\rrbracket_{=p'})$ such that $\mathsf{state}_p(\tilde{t})$ and $\mathsf{state}_{p'}(\tilde{t}')$ are part of their respective premise. From the definition of $\llbracket P\rrbracket_{=p}$ and the fact that θ is grounding for N, we have:

$$\begin{aligned} ri_1 &= & [\mathsf{state}_p(\tilde{t})] &\rightarrow [\mathsf{Msg}(t, N\theta), \mathsf{state}^{\mathsf{semi}}_{p\cdot 1}(\tilde{t})] \\ ri_2 &= & [\mathsf{state}_{p'}(\tilde{t}'), \mathsf{Msg}(t, N\theta)] &\rightarrow [\mathsf{state}_{p'\cdot 1}(\tilde{t}' \cup (\mathit{vars}(N)\theta)), \mathsf{Ack}(t, N\theta)] \\ ri_3 &= & [\mathsf{state}^{\mathsf{semi}}_{p}(\tilde{t}), \mathsf{Ack}(t, N\theta)] &\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{t})]. \end{aligned}$$

This allows to extend the previous execution by 3 steps:

$$\emptyset \xrightarrow{F_1}_{\llbracket P\rrbracket} S_1 \ldots \xrightarrow{F_{n'}}_{\llbracket P\rrbracket} S_{n'} \overset{(ri_1)}{\rightarrow}_{\llbracket P\rrbracket} S_{n'+s-2} \overset{(ri_2)}{\rightarrow}_{\llbracket P\rrbracket} S_{n'+s-1} \overset{(ri_2)}{\rightarrow}_{\llbracket P\rrbracket} S_{n'+s} \in \mathit{exec}^{msr}(\llbracket P\rrbracket)$$

where:

- $S_{n'+s-2} = S_{n'} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t})\,\} \cup^{\#} \{\, \mathsf{Msg}(t, N\theta), \mathsf{state}^{\mathsf{semi}}_{p\cdot 1}(\tilde{t})\,\}^{\#}$,
- $S_{n'+s-1} = S_{n'} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t}), \mathsf{state}_{p'}(\tilde{t}')\,\} \cup^{\#} \{\, \mathsf{state}^{\mathsf{semi}}_{p\cdot 1}(\tilde{t}), \mathsf{state}_{p'\cdot 1}(\tilde{t}' \cup (\mathit{vars}(N)\theta)), \mathsf{Ack}(t, N\theta)\,\}^{\#}$,
- $S_{n'+s} = S_{n'} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t}), \mathsf{state}_{p'}(\tilde{t}')\,\} \cup^{\#} \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}), \mathsf{state}_{p'\cdot 1}(\tilde{t}' \cup (\mathit{vars}(N)\theta))\,\}$.

We have that $\mathcal{P}_n = \mathcal{P}_{n-1} \setminus^{\#} \{\, \mathrm{out}(c, m); Q,\ \mathrm{in}(c', t'); R\,\} \cup^{\#} \{\, Q, R\theta\,\}^{\#}$. Exactly as in the two previous cases we have that $Q \leftrightarrow \mathsf{state}_{p\cdot 1}(\tilde{t})$, as well as $R\theta \leftrightarrow \mathsf{state}_{p'\cdot 1}(\tilde{t}')$. Hence we have that, Condition 4 holds. Condition 1, Condition 2, Condition 3, Condition 5, Condition 6, Condition 8 and Condition 7 hold trivially.

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{\, \textsf{if } t = t' \textsf{ then } Q \textsf{ else } Q' \,\}, \sigma_{n-1}, \mathcal{L}_{n-1}) \ \rightarrow (\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}' \cup \{\, Q \,\}, \sigma_{n-1}, \mathcal{L}_{n-1})$. This step requires that $t =_E t'$. By induction hypothesis $\mathcal{P}_{n-1} \leftrightarrow_P S_{n'}$. Let p and $\tilde{t}$ be such that if $t = t'$ then Q else $Q' \leftrightarrow_P \mathsf{state}_p(\tilde{t})$. By Definition 20, there is a $ri \in ginsts(\llbracket P \rrbracket_{=p})$ such that $\mathsf{state}_p(\tilde{t})$ is part of its premise. By definition of $\llbracket P \rrbracket_{=p}$, we can choose $ri = [\mathsf{state}_p(\tilde{t})] \ {-}[\mathrm{Eq}(t,t')]{\rightarrow}\ [\mathsf{state}_{p\cdot 1}(\tilde{t})]$. We can extend the previous execution by one step using ri, therefore:

$$\emptyset \xrightarrow{F_1}_{\llbracket P \rrbracket} S_1 \xrightarrow{F_2}_{\llbracket P \rrbracket} \dots \xrightarrow{F_{n'}}_{\llbracket P \rrbracket} S_{n'} \xrightarrow{Eq(t,t')}_{\llbracket P \rrbracket} S_{n'+1} \in exec^{msr}(\llbracket P \rrbracket)$$

with $S_{n'+1} = \{\, S_{n'} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t}) \,\}^{\#} \cup^{\#} \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}) \,\}^{\#} \,\}$. It is left to show that Conditions 1 to 8 hold for n. The last step is labelled $F_{f(n)} = \{\, Eq(t,t') \,\}^{\#}$. As $t =_E t'$, Condition 7 holds, in particular, α_{eq} is not violated. Since Eq is reserved, Condition 8 holds as well.

As before, since $\mathcal{P}_n = \mathcal{P}_{n-1} \setminus^{\#} \{\text{if } t = t' \text{ then } Q \text{ else } Q'\,\} \cup^{\#} \{Q\}$ and $\{Q\} \leftrightarrow \{\mathsf{state}_{p\cdot 1}(\tilde{t}, a)\}$ (by definition of the translation), we have that $\mathcal{P}_n \leftrightarrow_P S_{f(n)}$, and therefore Condition 4 holds. Condition 1, Condition 2, Condition 3, Condition 5 and Condition 6 hold trivially.

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{\, \textsf{if } t = t' \textsf{ then } Q' \textsf{ else } Q \,\}, \sigma_{n-1}, \mathcal{L}_{n-1}) \ \rightarrow (\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}' \cup \{\, Q' \,\}, \sigma_{n-1}, \mathcal{L}_{n-1})$. This step requires that $t \neq_E t'$. This proof step is similar to the previous case, except ri is chosen to be

$$[\mathsf{state}_p(\tilde{t})] \ {-}[\mathrm{NotEq}(t,t')]{\rightarrow}\ [\mathsf{state}_{p\cdot 2}(\tilde{t})].$$

The condition in α_{noteq} holds since $t \neq_E t'$.

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{\, \textsf{event}(F)\textsf{;}Q \,\}, \sigma_{n-1}, \mathcal{L}_{n-1}) \xrightarrow{F} (\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}' \cup \{\, Q \,\}, \sigma_{n-1}, \mathcal{L}_{n-1})$. By induction hypothesis $\mathcal{P}_{n-1} \leftrightarrow_P S_{n'}$. Let p and $\tilde{t}$ be such that $\mathsf{event}(F);Q \leftrightarrow_P \mathsf{state}_p(\tilde{t})$. By Definition 20, there is a $ri \in ginsts(\llbracket P \rrbracket_{=p})$ such that $\mathsf{state}_p(\tilde{t})$ is part of its premise. By definition of $\llbracket P \rrbracket_{=p}$, we can choose $ri = [\mathsf{state}_p(\tilde{t})] \ {-}[F, \mathrm{Event}()]{\rightarrow}\ [\mathsf{state}_{p\cdot 1}(\tilde{t})]$. We can extend the previous execution by one step using ri, therefore:

$$\emptyset \xrightarrow{F_1}_{\llbracket P \rrbracket} S_1 \xrightarrow{F_2}_{\llbracket P \rrbracket} \dots \xrightarrow{F_{n'}}_{\llbracket P \rrbracket} S_{n'} \xrightarrow{F,\mathrm{Event}()}_{\llbracket P \rrbracket} S_{n'+1} \in exec^{msr}(\llbracket P \rrbracket)$$

with $S_{n'+1} = S_{n')} \setminus^{\#} \{\mathsf{state}_p(\tilde{t})\} \cup^{\#} \{\mathsf{state}_{p\cdot 1}(\tilde{t})\}$. It is left to show that Conditions 1 to 8 hold for n. Condition 4 holds because $\mathcal{P}_n = \mathcal{P}_{n-1} \setminus^{\#} \{\, \mathrm{event}(F);Q \,\} \cup^{\#} \{\, Q \,\}$ and $\{Q\} \leftrightarrow \{\mathsf{state}_{p\cdot 1}(\tilde{t})\}$ (by definition of $\llbracket P \rrbracket_{=p}$). Taking $k = f(n)$ Condition 8 holds. Condition 1, Condition 2, Condition 3, Condition 5, Condition 6 and Condition 7 hold trivially.

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{\, \textsf{insert } t, t'\textsf{;}\ Q \,\}, \sigma_{n-1}, \mathcal{L}_{n-1}) \rightarrow (\mathcal{E}_{n-1}, \mathcal{S}_n = \mathcal{S}_{n-1}[t \mapsto t'], \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}' \cup \{\, Q \,\}, \sigma_{n-1}, \mathcal{L}_{n-1})$. By induction hypothesis $\mathcal{P}_{n-1} \leftrightarrow_P S_{n'}$. Let p and $\tilde{t}$ be such that insert t, t'; $Q \leftrightarrow_P \mathsf{state}_p(\tilde{t})$. By Definition 20, there is a $ri \in ginsts(\llbracket P \rrbracket_{=p})$ such that $\mathsf{state}_p(\tilde{t})$ is part of its premise. By definition of $\llbracket P \rrbracket_{=p}$, we can choose $ri = [\mathsf{state}_p(\tilde{t})] \ {-}[\mathrm{Insert}(t,t')]{\rightarrow}\ [\mathsf{state}_{p\cdot 1}(\tilde{t})]$. We can extend the previous execution by one step using ri, therefore:

$$\emptyset \xrightarrow{F_1}_{\llbracket P \rrbracket} S_1 \xrightarrow{F_2}_{\llbracket P \rrbracket} \dots \xrightarrow{F_{n'}}_{\llbracket P \rrbracket} S_{n'} \xrightarrow{\mathrm{Insert}(t,t')}_{\llbracket P \rrbracket} S_{n'+1} \in exec^{msr}(\llbracket P \rrbracket)$$

with $S_{n'+1} = S_{f(n-1)} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t}) \,\}^{\#} \cup^{\#} \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}) \,\}^{\#}$. It is left to show that Conditions 1 to 8 hold for n.

This step is labelled $F_{f(n)} = \mathrm{Insert}(t,t')$, hence Condition 8 holds. To see that Condition 2 holds we let $j = f(n)$ for which both conjuncts trivially hold. Since, by induction hypothesis, Condition 7 holds, i.e., $[F_1, \dots F_{n'}] \vDash \alpha$, it holds for this step too. In particular, if $[F_1, \dots F_{n'}] \vDash \alpha_{in}$ and $[F_1, \dots F_{n'}] \vDash \alpha_{notin}$, we also have that $[F_1, \dots F_{n'}, F_{f(n)}] \vDash \alpha_{in}$ and $[F_1, \dots F_{n'}, F_{f(n)}] \vDash \alpha_{notin}$: as the Insert-action was added at the last position of the trace, it appears after any InIn or IsNotSet-action and by the semantics of the logic the formula holds.

Since $\mathcal{P}_n = \mathcal{P}_{n-1} \setminus^{\#} \{\, \mathrm{insert}\ t, t';\ Q \,\} \cup^{\#} \{\, Q \,\}$ and $\{Q\} \leftrightarrow \{\mathsf{state}_{p\cdot 1}(\tilde{t})\}$ (by definition of $\llbracket P \rrbracket_{=p}$), we have that Condition 4 holds. Condition 1, Condition 3, Condition 5 and Condition 6 hold trivially.

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{\, \mathsf{delete}\ t;\ Q \,\}, \sigma_{n-1}, \mathcal{L}_{n-1}) \rightarrow (\mathcal{E}_{n-1}, \mathcal{S}_n = \mathcal{S}_{n-1}[t \mapsto \bot],$ $\mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}' \cup \{\, Q \,\}, \sigma_{n-1}, \mathcal{L}_{n-1})$**.** By induction hypothesis $\mathcal{P}_{n-1} \leftrightarrow_P S_{n'}$. Let p and $\tilde{t}$ be such that $\mathsf{delete}\ t;\ Q \leftrightarrow_P \mathsf{state}_p(\tilde{t})$. By Definition 20, there is a $ri \in ginsts([\![P]\!]_{=p})$ such that $\mathsf{state}_p(\tilde{t})$ is part of its premise. By definition of $[\![P]\!]_{=p}$, we can choose $ri = [\mathsf{state}_p(\tilde{t})] -\![\mathrm{Delete}(t)]\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{t})]$. We can extend the previous execution by one step using ri, therefore:

$$\emptyset \xrightarrow{F_1}_{[\![P]\!]} S_1 \xrightarrow{F_2}_{[\![P]\!]} \dots \xrightarrow{F_{n'}}_{[\![P]\!]} S_{n'} \xrightarrow{\mathrm{Delete}(t)}_{[\![P]\!]} S_{n'+1} \in exec^{msr}([\![P]\!])$$

with $S_{n'+1} = S_{f(n-1)} \setminus^{\#} \{\mathsf{state}_p(\tilde{t})\} \cup^{\#} \{\mathsf{state}_{p\cdot 1}(\tilde{t})\}$. It is left to show that Conditions 1 to 8 hold for n.

This step is labelled $F_{f(n)} = \mathrm{Delete}(t)$, hence Condition 8 holds. Since, by induction hypothesis, Condition 7 holds, i.e., $[F_1, \dots F_{n'}] \vDash \alpha$, it holds for this step too. In particular, if $[F_1, \dots F_{n'}] \vDash \alpha_{in}$ and $[F_1, \dots F_{n'}] \vDash \alpha_{notin}$, we also have that $[F_1, \dots F_{n'}, F_{f(n)}] \vDash \alpha_{in}$ and $[F_1, \dots F_{n'}, F_{f(n)}] \vDash \alpha_{notin}$: as the Insert-action was added at the last position of the trace, it appears after any InIn or IsNotSet-actions and by the semantics of the logic the formula holds.

We now show that Condition 2 holds. We have that $\mathcal{S}_n = \mathcal{S}_{n-1}[t \mapsto \bot]$ and therefore, for all $t' \neq_E Tt$, $\mathcal{S}_n(x) = \mathcal{S}_{n-1}(x)$. Hence for all such t' we have by induction hypothesis that for some u,

$$\exists j \leq n'. \mathrm{Insert}(t', u) \in F_j \wedge \forall j', u'. j < j' \leq n' \rightarrow \mathrm{Insert}(t', u') \notin_E F_{j'} \wedge \mathrm{Delete}(t') \notin_E F_{j'}$$

As, $F_{n'+1} \neq_E \mathrm{Delete}(x, u)$ and, for all $u' \in \mathcal{M}$, $F_{n'+1} \neq_E \mathrm{Insert}(x, u')$ we also have that

$$\exists j \leq n'+1. \mathrm{Insert}(t', u) \in F_j \wedge \forall j', u'. j < j' \leq n'+1 \rightarrow \mathrm{Insert}(t', u') \notin_E F_{j'} \wedge \mathrm{Delete}(t') \notin_E F_{j'}.$$

For $t' =_E t$, the above condition can never be true, because $F_{n'+1} = \mathrm{Delete}(t)$ which allows us to conclude that Condition 2 holds.

Since $\mathcal{P}_n = \mathcal{P}_{n-1} \setminus^{\#} \{\, \mathrm{delete}\ t;\ Q \,\} \cup^{\#} \{\, Q \,\}$ and $\{P\} \leftrightarrow \{\mathsf{state}_{p\cdot 1}(\tilde{t})\}$ (by definition of $[\![P]\!]_{=p}$), we have that Condition 4 holds. Condition 1, Condition 3, Condition 5 and Condition 6 hold trivially.

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{\, \mathsf{lookup}\ t\ \mathsf{as}\ x\ \mathsf{in}\ Q\ \mathsf{else}\ Q' \,\}, \sigma_{n-1}, \mathcal{L}_{n-1}) \rightarrow (\mathcal{E}_{n-1}, \mathcal{S}_{n-1},$ $\mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}' \cup \{\, Q\{v/x\} \,\}, \sigma_{n-1}, \mathcal{L}_{n-1})$**.** This step requires that $\mathcal{S}_{n-1}(t') =_E v$ for some $t' =_E t$. By induction hypothesis $\mathcal{P}_{n-1} \leftrightarrow_P S_{n'}$. Let p and $\tilde{t}$ be such that $\mathsf{lookup}\ t\ \mathsf{as}\ v\ \mathsf{in}\ Q\ \mathsf{else}\ Q' \leftrightarrow_P \mathsf{state}_p(\tilde{t})$. By Definition 20, there is a $ri \in ginsts([\![P]\!]_{=p})$ such that $\mathsf{state}_p(\tilde{t})$ is part of its premise. By definition of $[\![P]\!]_{=p}$, we can choose $ri = [\mathsf{state}_p(\tilde{t})] -\![\mathrm{IsIn}(t, v)]\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{t}, v)]$. We can extend the previous execution by one step using ri, therefore:

$$\emptyset \xrightarrow{F_1}_{[\![P]\!]} S_1 \xrightarrow{F_2}_{[\![P]\!]} \dots \xrightarrow{F_{n'}}_{[\![P]\!]} S_{n'} \xrightarrow{\mathrm{IsIn}(t,v)}_{[\![P]\!]} S_{n'+1} \in exec^{msr}([\![P]\!])$$

with $S_{n'+1} = S_{f(n-1)} \setminus^{\#} \{\mathsf{state}_p(\tilde{t})\} \cup^{\#} \{\mathsf{state}_{p\cdot 1}(\tilde{t})\}$. It is left to show that Conditions 1 to 8 hold for n.

This step is labelled $F_{f(n)} = \mathrm{IsIn}(t, v)$, hence Condition 8 holds.

From the induction hypothesis, Condition 2, we have that there is a j such that $\mathrm{Insert}(t, t') \in_E F_j$, $j \leq n'$ and

$$\forall j', u'.\ j < j' \leq n' \rightarrow \mathrm{Insert}(t, u') \notin_E F_{j'} \wedge \mathrm{Delete}(t) \notin_E F_{j'}$$

This can be strengthened, since $F_{f(n)} = \{\, \mathrm{IsIn}(t, v) \,\}$:

$$\forall j', u'.\ j < j' \leq f(n) \rightarrow \mathrm{Insert}(t, u') \notin_E F_{j'} \wedge \mathrm{Delete}(t) \notin_E F_{j'}$$

This allows to conclude that Condition 2 holds. From Condition 2 it also follows that Condition 7, in particular α_{in}, holds.

We now show that Condition 4 holds. By induction hypothesis we have that $\mathsf{lookup}\ t\ \mathsf{as}\ x\ \mathsf{in}\ Q\ \mathsf{else}\ Q' \leftrightarrow_P \mathsf{state}_p(\tilde{t})$, and hence $P|_p\tau = (\mathsf{lookup}\ t\ \mathsf{as}\ x\ \mathsf{in}\ Q\ \mathsf{else}\ Q')\rho$ for some τ and ρ. Therefore we also have that $P|_{p\cdot 1}\tau \cup (\{^{v\rho}/_x\}) = Q\rho\{^{v\rho}/_x\})$ and it is easy to see from definition of $[\![P]\!]_{=p}$ that $\{Q\{^v/_x\}\} \leftrightarrow_P \{\mathsf{state}_{p\cdot 1}(\tilde{t}, v)\}$. Since $\mathcal{P}_n = \mathcal{P}_{n-1} \setminus^{\#} \{\, \mathsf{lookup}\ t\ \mathsf{as}\ x\ \mathsf{in}\ Q\ \mathsf{else}\ Q' \,\} \cup^{\#} \{\, Q\{^v/_x\} \,\}$ we have that $\mathcal{P}_n \leftrightarrow_P S_{f(n)}$, i. e., Condition 4 holds.

Condition 1, Condition 3, Condition 5 and Condition 6 hold trivially.

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{\, \textsf{lookup}\ t\ \textsf{as}\ x\ \textsf{in}\ Q\ \textsf{else}\ Q' \,\}, \sigma_{n-1}, \mathcal{L}_{n-1}) \rightarrow (\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}' \cup \{\, Q' \,\}, \sigma_{n-1}, \mathcal{L}_{n-1})$**.** This step requires that $S(t')$ is undefined for all $t' =_E t$. By induction hypothesis $\mathcal{P}_{n-1} \leftrightarrow_P S_{n'}$. Let p and $\tilde{t}$ be such that $\textsf{lookup}\ t\ \textsf{as}\ x\ \textsf{in}\ Q\ \textsf{else}\ Q' \leftrightarrow_P \textsf{state}_p(\tilde{t})$. By Definition 20, there is a $ri \in \mathit{ginsts}([\![P]\!]_{=p})$ such that $\textsf{state}_p(\tilde{t})$ is part of its premise. By definition of $[\![P]\!]_{=p}$, we can choose $ri = [\textsf{state}_p(\tilde{t})] \;{-}[\mathrm{IsNotSet}(t)]{\rightarrow}\; [\textsf{state}_{p\cdot 1}(\tilde{t})]$. We can extend the previous execution by one step using ri, therefore:

$$\emptyset \xrightarrow{F_1}_{[\![P]\!]} S_1 \xrightarrow{F_2}_{[\![P]\!]} \dots \xrightarrow{F_{n'}}_{[\![P]\!]} S_{n'} \xrightarrow{\mathrm{IsNotSet}(t)}_{[\![P]\!]} S_{n'+1} \in \mathit{exec}^{msr}([\![P]\!])$$

with $S_{n'+1} = S_{f(n-1)} \setminus^{\#} \textsf{state}_p(\tilde{t}) \cup^{\#} \textsf{state}_{p\cdot 1}(\tilde{t})$. It is left to show that Conditions 1 to 8 hold for n.

This step is labelled $F_{f(n)} = \mathrm{IsNotSet}(t)$, hence Condition 8 holds. Condition 2 also holds trivially and will be used to show Condition 7. Since this step requires that $S(t')$ is undefined for all $t' =_E t$, we have by Condition 2 that

$$\begin{aligned}&\forall j \leq f(n), u.\ \mathrm{Insert}(t,u) \in_E F_j \\ &\qquad \rightarrow \exists j', u'. j < j' \leq f(n) \wedge (\mathrm{Insert}(t,u') \in_E F_{j'} \vee \mathrm{Delete}(t) \in_E F_{j'})\end{aligned}$$

Now suppose that

$$\exists i \leq f(n), y.\mathrm{Insert}(t,y) \in_E F_i)$$

As there exists an insert, there is a last insert and hence we also have

$$\exists i \leq f(n), y.\mathrm{Insert}(t,y) \in_E F_i \quad \wedge \quad \forall i', y'.i < i' \leq f(n) \rightarrow \mathrm{Insert}(t,y') \notin_E F_{i'}$$

Applying Condition 2 (cf above) we obtain that

$$\begin{aligned}&\exists i \leq f(n), y.\mathrm{Insert}(t,y) \in_E F_i \quad \wedge \quad \forall i', y'.\ i < i' \leq f(n) \rightarrow \mathrm{Insert}(t,y') \notin_E F_{i'} \\ &\qquad \wedge \quad \exists j', u'.\ i < j' \leq f(n) \wedge (\mathrm{Insert}(t,u') \in_E F_{j'} \vee \mathrm{Delete}(t) \in_E F_{j'})\end{aligned}$$

which simplifies to

$$\begin{aligned}&\exists i \leq f(n), y.\mathrm{Insert}(t,y) \in_E F_i \quad \wedge \quad \forall i', y'.\ i < i' \leq f(n) \rightarrow \mathrm{Insert}(t',y') \notin F_{i'} \\ &\qquad \wedge \quad \exists j'.\ i < j' \leq f(n) \wedge \mathrm{Delete}(t) \in_E F_{j'}\end{aligned}$$

Now we weaken the statement by dropping the first conjunct and restricting the quantification $\forall i'.i < i' \leq f(n)$ to $\forall i'.j' < i' \leq f(n)$, since $i < j'$.

$$\exists i \leq f(n).\ \exists j'.\ i < j' \leq f(n) \wedge \forall i'.\ j' < i' \leq f(n) \rightarrow \mathrm{Insert}(t',y') \notin F_{i'} \wedge \mathrm{Delete}(t) \in_E F_{j'}$$

We further weaken the statement by weakening the scope of the existential quantification $\exists j'.\ i < j' \leq f(n)$ to $\exists j'.\ j' \leq f(n)$. Afterwards, i is not needed anymore.

$$\exists j'.\ j' \leq f(n) \wedge \forall i'.\ j' < i' \leq f(n) \rightarrow \mathrm{Insert}(t',y') \notin F_{i'} \wedge \mathrm{Delete}(t) \in_E F_{j'}$$

This statement was obtained under the hypothesis that $\exists i \leq f(n), y.\mathrm{Insert}(t,y) \in_E F_i)$. Hence we have that

$$\begin{aligned}&\forall i \leq f(n), y.\mathrm{Insert}(t,y) \notin_E F_i \\ &\vee \exists j' \leq f(n).\ \mathrm{Delete}(t) \in_E F_{j'} \wedge \forall i'.\ j' < i' \leq f(n) \rightarrow \mathrm{Insert}(t',y') \notin F_{i'}\end{aligned}$$

This shows that Condition 7, in particular α_{notin}, holds.

Since $\mathcal{P}_n = \mathcal{P}_{n-1} \setminus^{\#} \{\, \textsf{lookup}\ t\ \textsf{as}\ x\ \textsf{in}\ Q\ \textsf{else}\ Q' \,\} \cup^{\#} \{\, Q' \,\}$ and $\{Q'\} \leftrightarrow \{\textsf{state}_{p\cdot 1}(\tilde{t})\}$ (by definition of $[\![P]\!]_{=p}$), we have that Condition 4 holds. Condition 1, Condition 3, Condition 5 and Condition 6 hold trivially.

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{\, \mathsf{lock}\ t;\ Q \,\}, \sigma_{n-1}, \mathcal{L}_{n-1}) \rightarrow (\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}' \cup^{\#} \{\, Q' \,\},$ $\sigma_{n-1}, \mathcal{L}_{n-1} \cup \{\, t \,\})$**.** This step requires that for all $t' =_E t$, $t' \notin \mathcal{L}_{n-1}$. Let p and $\tilde{t}$ such that $\mathsf{lock}\ t;\ Q \leftrightarrow_P \mathsf{state}_p(\tilde{t})$. By Definition 20, there is a $ri \in \mathit{ginsts}([\![P]\!]_{=p})$ such that $\mathsf{state}_p(\tilde{t})$ is part of its premise. By definition of $[\![P]\!]_{=p}$, we can choose $ri = [\mathsf{Fr}(l), \mathsf{state}_p(\tilde{t})] \,\text{--}[\mathrm{Lock}(l,t)]\!\!\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{t}, l)]$ for a fresh name l, that never appeared in a Fr-fact in $\cup_{j \leq f(n-1)} S_j$. We can extend the previous execution by $s = 2$ steps using an instance of Fresh for l and ri:

$$\emptyset \xrightarrow{F_1}_{[\![P]\!]} S_1 \xrightarrow{F_2}_{[\![P]\!]} \dots \xrightarrow{F_{n'}}_{[\![P]\!]} S_{n'} \rightarrow_{\{\,\text{FRESH}\,\}} S_{n'+s-1} \xrightarrow{\mathrm{Lock}(l,t)}_{[\![P]\!]} S_{n'+s} \in \mathit{exec}^{msr}([\![P]\!])$$

with $S_{n'+s-1} = S_{f(n-1)} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t}) \,\}^{\#} \cup^{\#} \{\, \mathsf{Fr}(l) \,\}$ and $S_{n'+s} = S_{f(n-1)} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t}) \,\}^{\#} \cup^{\#} \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}) \,\}^{\#}$. It is left to show that Conditions 1 to 8 hold for n.

The step from $S_{f(n)-1}$ to $S_{f(n)}$ is labelled $F_{f(n)} = \mathrm{Lock}(l,t)$, hence Condition 8 and Condition 2 hold.

$F_{f(n)}$ also preserves Condition 6 for the new set of active locks $\mathcal{L}_{f(n)} = \mathcal{L}_{f(n-1)} \cup \{\, t \,\}$.

In the following we show by contradiction that α_{lock}, and therefore Condition 7 holds. α_{lock} held in the previous step, and $F_{f(n-1)+1}$ is empty, so we assume (by contradiction), that $F_{f(n)} = \mathrm{Lock}(l,t)$ violates α_{lock}. If this was the case, then:

$$\begin{aligned} &\exists i < f(n), l_1.\ \mathrm{Lock}(l_1,t) \in_E F_i \wedge \\ &\qquad \wedge \forall j.\ i < j < f(n) \rightarrow \mathrm{Unlock}(l_1,t) \notin_E F_j \\ &\qquad\quad \vee \exists l_2, k.\ i < k < j \wedge (\mathrm{Lock}(\mathrm{l_2},\mathrm{t}) \in_E F_k \vee \mathrm{Unlock}(\mathrm{l_2},\mathrm{t}) \in_E F_k) \end{aligned} \tag{2}$$

Since the semantics of the calculus requires that for all $t' =_E t$, $t' \notin \mathcal{L}_{n-1}$, by induction hypothesis, Condition 6, we have that

$$\begin{aligned} &\forall i < f(n-1), l_1.\ \mathrm{Lock}(l_1,t) \in_E F_i \rightarrow \\ &\qquad \exists j.\ i < j < f(n-1) \wedge \mathrm{Unlock}(l_1,t) \in_E F_j \end{aligned}$$

Since $F_{f(n-1)+1} = \emptyset$ and $f(n) = f(n-1) + 2$, we have:

$$\begin{aligned} &\forall i < f(n), l_1.\mathrm{Lock}(l_1,t) \in_E F_i \rightarrow \\ &\qquad \exists j.\ i < j < f(n) \wedge \mathrm{Unlock}(l_1,t) \in_E F_j \end{aligned}$$

We apply Proposition 4 for the total order $>$ on the integer interval $i+1..f(n)-1$:

$$\begin{aligned} &\forall i < f(n), l_1.\ \mathrm{Lock}(l_1,t) \in_E F_i \rightarrow \\ &\qquad \exists j.\ i < j < f(n) \wedge \mathrm{Unlock}(l_1,t) \in_E F_j \\ &\qquad \wedge \quad \forall k.\ i < k < j \rightarrow \mathrm{Unlock}(l_1,t) \notin_E F_k \end{aligned}$$

Combining this with (2) we obtain that

$$\begin{aligned} &\exists i < f(n), l_1.\ \mathrm{Lock}(l_1,t) \in_E F_i \wedge \\ &\qquad \exists j.\ i < j < f(n) \wedge \mathrm{Unlock}(l_1,t) \in_E F_j \\ &\qquad \wedge \exists l_2, k.\ i < k < j \wedge (\mathrm{Lock}(\mathrm{l_2},\mathrm{t}) \in_E F_k \vee (\mathrm{Unlock}(\mathrm{l_2},\mathrm{t}) \in_E F_k \wedge l_2 \neq_E l_1)) \end{aligned}$$

Fix $i < f(n)$, j such that $i < j < f(n)$, and l_1 such that $\mathrm{Lock}(l_1,t) \in_E F_i$ and $\mathrm{Unlock}(l_1,t) \in_E F_j$. Then, there are l_2 and k such that $i < k < j$ and either $\mathrm{Lock}(\mathrm{l_2},\mathrm{t}) \in_E F_k$ or $\mathrm{Unlock}(\mathrm{l_2},\mathrm{t}) \in_E F_k$, but $l_2 \neq_E l_1$. We proceed by case distinction.

Case 1: there is no unlock in between i and j, i. e., for all m, $i < m < j$, $\mathrm{Unlock}(l',t) \notin F_m$. Then there is a k and l_2 such that $\mathrm{Lock}(\mathrm{l_2},\mathrm{t}) \in_E F_k$. In this case, α_{lock} is already invalid at the trace produced by the k-prefix of the execution, contradicting the induction hypothesis.

Case 2: there are l' and m, $i < m < j$ such that $\mathrm{Unlock}(l',t) \in F_m$ (see Figure 17).

We first observe that for any l, u, i_1, i_2 , if $\mathrm{Unlock}(\mathrm{l},\mathrm{u}) \in_E F_{i_1}$ and $\mathrm{Unlock}(\mathrm{l},\mathrm{u}) \in_E F_{i_2}$, then $i_1 = i_2$. We proceed by contradiction. By definition of $[\![P]\!]$ and well-formedness of P, the steps from $i_1 - 1$ to i_1

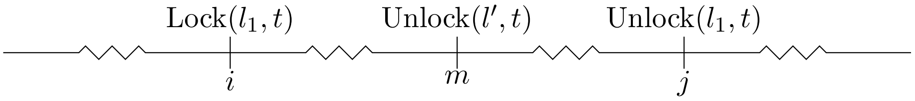

Figure 13: Visualisation of Case 2.

and from $i_2 - 1$ to i_2 must be ground instances of rules $[\![P]\!]_{=q}$ and $[\![P]\!]_{=q'}$ such that $P|_q$ and $P|_{q'}$ start with unlock commands that are labelled the same and have the same parameter, since every variable $lock_l$ in $[\![P]\!]$ appears in a Fr-fact in the translation for the corresponding lock command. By definition of $\overline{P}$, this means q and q' have a common prefix q_l that starts with a lock with this label.

Let $q_l \leq q$ denote that q_l is a prefix of q. Since $\overline{P}$ gives $\bot$ if there is a replication or a parallel between q_l and q or q', and since P is well-formed (does not contain $\bot$), we have that every state fact state_r for $q_l \leq r \leq q$ or $q_l \leq r \leq q'$ appearing in $[\![P]\!]$ is a linear fact, since no replication is allowed between q_l and q or q'. This implies that $q' \neq q$. Furthermore, every rule in $\cup_{q_l \leq r \leq q \vee q_l \leq r \leq q'} [\![P]\!]_{=r}$ adds at most one fact state_r and if it adds one fact, it either removes a fact $\mathsf{state}_{r'}$ where $r = r' \cdot 1$ or $r' \cdot 2$, or removes a fact $\mathsf{state}^{\mathsf{semi}}_{r'}$ where $r = r' \cdot 1$, which in turn requires removing $\mathsf{state}_{r'}$ (see translation of out). Therefore, either $q \leq q'$ or $q' \leq q$. But this implies that both have different labels, and since $[\![P]\!]_{=q_l}$ requires $\mathsf{Fr}(l)$, and E distinguishes fresh names, we have a contradiction. (A similiar observation is possible for locks: For any l, u, i_1, i_2 , if $\mathrm{Lock(l,u)} \in_E F_{i_1}$ and $\mathrm{Lock(l,u)} \in_E F_{i_2}$, then $i_1 = i_2$, since by definition of the translation, the transition from $i_1 - 1$ to i_1 or $i - 2 - 1$ to i_2 removes fact $\mathsf{Fr}(l)$.)

From the first observation we learn that , $l' \neq_E l_1$ for any l' and m, $i < m < j$ such that $\mathrm{Unlock}(l', t) \in F_m$. We now choose the smallest such m. By definition of $[\![P]\!]$, the step from S_{m-1} to S_m must be ground instance of a rule from $[\![P]\!]_{=q}$ for $P|_q$ starting with unlock. Since P is well-formed, there is a q_l such that $P|_{q_l}$ starts with lock, with the same label and parameter as the unlock. As before, since P is well-formed, and therefore there are no replications and parallels between q_l and q, there must be n such that $\mathrm{Lock}(l', t) \in F_n$ and $n < m$. We proceed again by case distinction.
Case 2a: $n < i$ (see Figure 14). By the fact that $m > i$ we have that there is no o such that $n < o < i$ and $\mathrm{Unlock}(l', t) \in_E F_o$ (see first observation). Therefore, the trace produced by the i-prefix of this execution does already not satisfy α_{lock}, i. e., $[F_1, \ldots, F_i] \not\vDash \alpha_{lock}$.

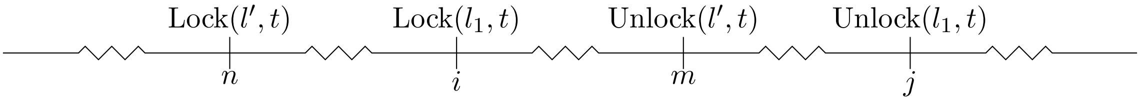

Figure 14: Visualisation of Case 2a.

Case 2b: $i < n$ (see Figure 15). Again, α_{lock} is not satisfied, i.e., $[F_1, \ldots, F_n] \not\vDash \alpha_{lock}$, since there is no o such that $i < o < n$ and $\mathrm{Unlock}(l_1, t) \in_E F_o$.

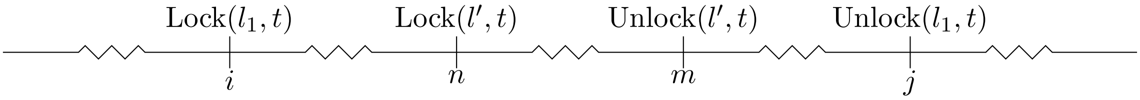

Figure 15: Visualisation of Case 2b.

Since we could, under the assumption that Condition 1 to Condition 8 hold for $i \leq n'$, reduce every case in which $[F_1, \ldots, F_{n'+1}] \not\vDash \alpha_{lock}$ to a contradiction, we can conclude that Condition 7 holds for $n' + 1$.

Since $\mathcal{P}_n = \mathcal{P}_{n-1} \setminus^\# \{\, \mathrm{lock}\ t;\ Q \,\} \cup^\# \{\, Q \,\}$ and $\{Q\} \leftrightarrow \{\mathsf{state}_{p \cdot 1}(\tilde{t})\}$ (by definition of the translation), we have that Condition 4 holds. Condition 1, Condition 3 and Condition 5 hold trivially.

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{\, \mathsf{unlock}\ t;\ Q \,\}, \sigma_{n-1}, \mathcal{L}_{n-1}) \rightarrow (\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1},$ $\mathcal{P}' \cup^\# \{\, Q' \,\}, \sigma_{n-1}, \mathcal{L}_{n-1} \setminus \{\, t' \colon t' =_E t \,\})$. By induction hypothesis $\mathcal{P}_{n-1} \leftrightarrow_P S_{n'}$. Let p and $\tilde{t}$ be such that $\mathsf{unlock}\ t;\ Q \leftrightarrow_P \mathsf{state}_p(\tilde{t})$. By Definition 20, there is a $ri \in ginsts([\![P]\!]_{=p})$ such that $\mathsf{state}_p(\tilde{t})$ is part of its premise. By definition of $[\![P]\!]_{=p}$, we can choose $ri = [\mathsf{state}_p(\tilde{t})] -[\mathrm{Unlock}(l,t)]\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{t})]$. We can extend the previous execution by one step using ri, therefore:

$$\emptyset \xrightarrow{F_1}_{[\![P]\!]} S_1 \xrightarrow{F_2}_{[\![P]\!]} \dots \xrightarrow{F_{n'}}_{[\![P]\!]} S_{n'} \xrightarrow{\mathrm{Unlock}(l,t)}_{[\![P]\!]} S_{n'+1} \in \mathit{exec}^{\mathit{msr}}([\![P]\!])$$

with $S_{n'+1} = S_{f(n-1)} \setminus^{\#} \{\mathsf{state}_p(\tilde{t})\} \cup^{\#} \{\mathsf{state}_{p\cdot 1}(\tilde{t})\}$. It is left to show that Conditions 1 to 8 hold for n.

The step from $S_{f(n-1)}$ to $S_{f(n)}$ is labelled $F_{f(n)} = \mathrm{Unlock}(l,t)$, hence Condition 8 and Condition 2 hold.

In order to show that Condition 6 holds, we perform a case distinction. Assume $t \notin_E \in \mathcal{L}_{n-1}$. Then, $\mathcal{L}_{f(n-1)} = \mathcal{L}_{f(n)}$. In this case, Condition 6 holds by induction hypothesis. In the following, we assume $t \in_E \mathcal{L}_{n-1}$. Thus, there is $j \in n', l'$ such that $\mathrm{Lock}(l',t) \in_E F_j$ and for all k such that $j < k \leq n'$, $\mathrm{Unlock}(l',t) \notin_E F_k$.

Since $P|_p$ is an unlock node and P is well-formed, there is a prefix q of p, such that $P|_q$ is a lock with the same parameter and annotation. By definition of $\overline{P}$, there is no parallel and no replication between q and p. Note that any rule in $[\![P]\!]$ that produces a state named state_p for a non-empty p is such that it requires a fact with name $\mathsf{state}_{p'}$ for $p = p' \cdot 1$ or $p = p' \cdot 2$ (in case of the translation of out, it might require $\mathsf{state}^{\mathsf{semi}}_{p'}$, which in turn requires $\mathsf{state}_{p'}$). This means that, since $\mathsf{state}_p(\tilde{t}) \in S_{n'}$, there is an i such that $\mathsf{state}_q(\tilde{t'}) \in S_i$ and $\mathsf{state}_q(\tilde{t'}) \notin S_{i-1}$ for $\tilde{t'}$ a prefix to t. This rule is an instance of $[\![P]\!]_{=q}$ and thus labelled $F_i = \mathrm{Lock}(l,t)$. We proceed by case distinction.

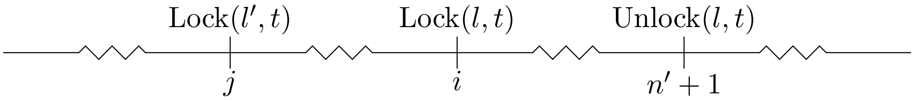

Figure 16: Visualisation of Case 1.

Case 1: $j < i$ (see Figure 16). By induction hypothesis, Condition 7 holds for the trace up to n'. But, $[F_1,\dots,F_i] \not\vDash \alpha_{lock}$, since we assumed that for all k such that $j < k \leq n'$, $\mathrm{Unlock}(l',t) \notin_E F_k$.

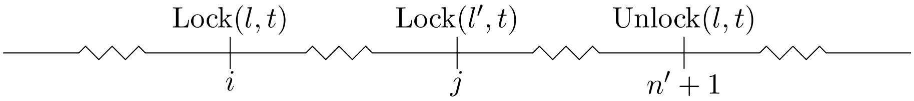

Figure 17: Visualisation of Case 2.

Case 2: $i < j$ (see Figure 17). As shown in the lock case, any k such that $\mathrm{Unlock}(l,t) \in_E F_k$ is $k = n'+1$. This contradicts Condition 7 for the trace up to j, since $[F_1,\dots,F_j] \not\vDash \alpha_{lock}$, because there is not k such that $i < k < j$ such that $\mathrm{Unlock}(l,t) \in_E F_k$. This concludes the proof that Condition 6 holds for $n+1$.

Condition 7 holds, since none of the axioms, in particular not α_{lock}, become unsatisfied if they were satisfied for the trace up to $f(n-1)$ and an Unlock is added.

Since $\mathcal{P}_n = \mathcal{P}_{n-1} \setminus^{\#} \{\, \text{unlock } t;\, Q \,\} \cup^{\#} \{\, Q \,\}$ and $\{Q\} \leftrightarrow \{\mathsf{state}_{p\cdot 1}(\tilde{t})\}$ (by definition of the translation), we have that Condition 4 holds. Condition 1, Condition 3 and Condition 5 hold trivially.

Case: $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathcal{P}_{n-1} = \mathcal{P}' \cup \{\, l -[a]\rightarrow r;\ Q \,\}, \sigma_{n-1}, \mathcal{L}_{n-1}) \xrightarrow{a}$ $(\mathcal{E}_{n-1}, \mathcal{S}_{n-1}, \mathcal{S}^{\mathrm{MS}}_{n-1} \setminus \mathit{lfacts}(l') \cup^{\#} \mathit{mset}(r), \mathcal{P}' \cup^{\#} \{\, Q \,\}, \sigma_{n-1}, \mathcal{L}_{n-1})$. This step requires that $l' -[a']\rightarrow r' \in_E \mathit{ginsts}(l -[a]\rightarrow r)$ and $\mathit{lfacts}(l') \subset^{\#} \mathcal{S}^{\mathrm{MS}}_{n-1}, \mathit{pfacts}(l') \subset \mathit{mset}(\mathcal{S}^{\mathrm{MS}}_{n-1})$. Let θ be a substitution such that $(l -[a]\rightarrow r)\theta = (l' -[a']\rightarrow r')$. Since, by induction hypothesis, $\mathcal{S}^{\mathrm{MS}}_{n-1} = S_{n'} \setminus^{\#} \mathcal{F}_{res}$, we therefore have $\mathit{lfacts}(l') \subset^{\#} S_{n'}, \mathit{pfacts}(l') \subset \mathit{mset}(S_{n'})$. By induction hypothesis $\mathcal{P}_{n-1} \leftrightarrow_P S_{n'}$. Let p and $\tilde{t}$ be such that $l -[a]\rightarrow r;\ \mathsf{Q} \leftrightarrow_P \mathsf{state}_p(\tilde{t})$. By Definition 20, there is a $ri \in \mathit{ginsts}([\![P]\!]_{=p})$ such that $\mathsf{state}_p(\tilde{t})$ is part of its premise. By definition of $[\![P]\!]_{=p}$, we can choose $ri = [\mathsf{state}_p(\tilde{t}), l'] -[a', \mathrm{Event}()]\rightarrow [r', \mathsf{state}_{p\cdot 1}(\tilde{t} \cup (\mathit{vars}(l)\theta))]$. We can extend the previous execution by one step using ri, therefore:

$$\emptyset \xrightarrow{F_1}_{[\![P]\!]} S_1 \xrightarrow{F_2}_{[\![P]\!]} \dots \xrightarrow{F_{n'}}_{[\![P]\!]} S_{n'} \xrightarrow{a',\mathrm{Event}()}_{[\![P]\!]} S_{n'+1} \in \mathit{exec}^{\mathit{msr}}([\![P]\!])$$

with $S_{n'+1} = S_{f(n-1)} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t}) \,\}^{\#} \setminus^{\#} \mathit{lfacts}(l') \cup^{\#} \{\, \mathsf{state}_{p\cdot 1}(\tilde{t} \cup (\mathit{vars}(l)\theta)) \,\}^{\#} \cup^{\#} \mathit{mset}(r')$. It is left to show that Conditions 1 to 8 hold for n.

Condition 3 holds since

$$\begin{aligned}
\mathcal{S}_n^{\mathrm{MS}} &= \mathcal{S}_{n-1}^{\mathrm{MS}} \setminus^{\#} \mathit{lfacts}(l') \cup^{\#} \mathit{mset}(r) \\
&= (S_{n'} \setminus^{\#} \mathcal{F}_{res}) \setminus \mathit{lfacts}(l') \cup^{\#} \mathit{mset}(r) && \text{(induction hypothesis)} \\
&= (S_{n'} \setminus^{\#} \mathit{lfacts}(l') \cup^{\#} \mathit{mset}(r) \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t}) \,\}^{\#} \cup^{\#} \{\, \mathsf{state}_{p\cdot 1}(\tilde{t} \cup (\mathit{vars}(l)\theta)) \,\}^{\#}) \setminus^{\#} \mathcal{F}_{res} \\
& && \text{(since } \mathsf{state}_p(\tilde{t}), \mathsf{state}_{p\cdot 1}(\tilde{t} \cup (\mathit{vars}(l)\theta)) \in \mathcal{F}_{res}\text{)} \\
&= S_{f(n)} \setminus^{\#} \mathcal{F}_{res}
\end{aligned}$$

The step from $S_{f(n-1)}$ to $S_{f(n)}$ is labelled $F_{f(n)} = a$, and does not contain actions in $\mathcal{F}_{res}$, since P is well-formed. Hence Condition 2, Condition 6, Condition 7 and Condition 8 hold.

Since $\mathcal{P}_n = \mathcal{P}_{n-1} \setminus^{\#} \{\, l \text{ -}[a]\!\rightarrow r;\, Q \,\} \cup^{\#} \{\, Q \,\}$ and $\{Q\} \leftrightarrow \{\mathsf{state}_{p\cdot 1}(\tilde{t} \cup (\mathit{vars}(l)\theta))\}$ (by definition of $[\![P]\!]_{=p}$), we have that Condition 4 holds. Condition 1, and Condition 5 hold trivially.

□

Definition 21 (normal msr execution)**.** *A msr execution* $\emptyset \xrightarrow{E_1}_{[\![P]\!]} \cdots \xrightarrow{E_n}_{[\![P]\!]} S_n \in \mathit{exec}^{msr}([\![P]\!])$ *for the multiset rewrite system* $[\![P]\!]$ *defined by a ground process* P *is* normal *if:*

1. *The first transition is an instance of the* Init *rule, i. e.,* $S_1 = \mathsf{state}_{[]}()$ *and there is at least this transition.*

2. S_n *neither contains any fact with the symbol* $\mathsf{state}_p^{\mathsf{semi}}$ *for any* p*, nor any fact with symbol* Ack*.*

3. *if for some* i *and* $t_1, t_2 \in \mathcal{M}$*,* $\mathsf{Ack}(t_1, t_2) \in (S_{i-1} \setminus^{\#} S_i)$*, then there are* p *and* q *such that:*

$$S_{i-3} \rightarrow_{R_1} S_{i-2} \rightarrow_{R_2} S_{i-1} \rightarrow_{R_3} S_i \quad \textit{, where:}$$

   - $R_1 = [\mathsf{state}_p(\tilde{x})] \rightarrow [\mathsf{Msg}(t_1, t_2), \mathsf{state}_p^{\mathsf{semi}}(\tilde{x})]$
   - $R_2 = [\mathsf{state}_q(\tilde{y}), \mathsf{Msg}(t_1, t_2)] \rightarrow [\mathsf{state}_{q\cdot 1}(\tilde{y} \cup \tilde{y}'), \mathsf{Ack}(t_1, t_2)]$
   - $R_3 = [\mathsf{state}_p^{\mathsf{semi}}(\tilde{x}), \mathsf{Ack}(t_1, t_2)] \rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{x})]$.

4. $S_{n-1} \xrightarrow{E_n}_{[\![P,[],[]]\!],\mathrm{MDIn},\mathrm{Init}} S_n$

5. *if* $\mathsf{In}(t) \in (S_{i-1} \setminus^{\#} S_i)$ *for some* i *and* $t \in \mathcal{M}$*, then* $S_{i-2} \xrightarrow{K(t)}_{\mathrm{MDIn}} S_{i-1}$

6. *if* $n \geq 2$ *and no* Ack*-fact in* $(S_{i-1} \setminus^{\#} S_i)$*, then there exists* $m < n$ *such that* $S_m \rightarrow_R^* S_{n-1}$ *for* $R = \{\, \mathrm{MDOut}, \mathrm{MDPub}, \mathrm{MDFresh}, \mathrm{MDAppl} \,\} \cup \mathrm{Fresh}$ *and* $\emptyset \xrightarrow{E_1}_{[\![P]\!]} \cdots \xrightarrow{E_m}_{[\![P]\!]} S_m \in \mathit{exec}^{msr}([\![P]\!])$ *is normal.*

7. *if for some* $t_1, t_2 \in \mathcal{M}$*,* $\mathsf{Ack}(t_1, t_2) \in (S_{n-1} \setminus^{\#} S_n)$*, then there exists* $m \leq n - 3$ *such that* $S_m \rightarrow_R^* S_{n-3}$ *for* $R = \{\, \mathrm{MDOut}, \mathrm{MDPub}, \mathrm{MDFresh}, \mathrm{MDAppl} \,\} \cup \mathrm{Fresh}$ *and* $\emptyset \xrightarrow{E_1}_{[\![P]\!]} \cdots \xrightarrow{E_m}_{[\![P]\!]} S_m \in \mathit{exec}^{msr}([\![P]\!])$ *is normal.*

Lemma 11 (Normalisation)**.** *Le* P *be a well-formed ground process. If*

$$S_0 = \emptyset \xrightarrow{E_1}_{[\![P]\!]} S_1 \xrightarrow{E_2}_{[\![P]\!]} \ldots \xrightarrow{E_n}_{[\![P]\!]} S_n \in \mathit{exec}^{msr}([\![P]\!])$$

and $[E_1, \ldots, E_n] \vDash \alpha$*, then there exists a normal msr execution*

$$T_0 = \emptyset \xrightarrow{F_1}_{[\![P]\!]} T_1 \xrightarrow{F_2}_{[\![P]\!]} \ldots \xrightarrow{F_{n'}}_{[\![P]\!]} T_{n'} \in \mathit{exec}^{msr}([\![P]\!])$$

such that $\mathit{hide}([E_1, \ldots, E_n]) = \mathit{hide}(F_1, \ldots, F_{n'})$ *and* $[F_1, \ldots, F_{n'}] \vDash \alpha$*.*

Proof. We will modify $S_0 \xrightarrow{E_1}_{[\![P]\!]} \ldots \xrightarrow{E_n}_{[\![P]\!]} S_n$ by applying one transformation after the other, each resulting in an msr execution that still fulfills the conditions on its trace.

1. If an application of the Init rule appears in $S_0 \xrightarrow{E_1}_{[\![P]\!]} \dots \xrightarrow{E_n}_{[\![P]\!]} S_n$, we move it to the front. Therefore, $S_1 = \mathsf{state}_{[]}()$. This is possible since the left-hand side of the Init rule is empty. If the rule is never instantiated, we prepend it to the trace. Since Init() $\in \mathcal{F}_{res}$, the resulting msr execution
$$S_0^{(1)} \xrightarrow{E_1^{(1)}}_{[\![P]\!]} \dots \xrightarrow{E_n^{(1)}}_{[\![P]\!]} S_{n^{(1)}}^{(1)}$$
is such that $\mathit{hide}([E_1,\dots,E_n]) = \mathit{hide}([E_1^{(1)},\dots,E_{n^{(1)}}^{(1)}])$. Since Init() is only added if it was not present before, $[E_1^{(1)},\dots,E_{n^{(1)}}^{(1)}] \vDash \alpha$, especially α_{init}.

2. For each fact $\mathsf{Ack}(t_1,t_2)$ contained in $S_{n^{(1}}^{(1)}$, it also contains a fact $\mathsf{state}_p^{\mathsf{semi}}(\tilde{t})$ for some p and $\tilde{t}$ such that there exists a rule of type R_3 that consumes both of them, since $\mathsf{Ack}(t_1,t_2)$ can only be produced by a rule of type R_2 which consumes $\mathsf{Msg}(t_1,t_2)$ which in turn can only be produced along with a fact $\mathsf{state}_p^{\mathsf{semi}}(\tilde{t})$, and by definition of $[\![P]\!]$, there exists a rule in $[\![P]\!]_{=p}$ of form R_3 that consumes $\mathsf{Ack}(t_1,t_2)$ and $\mathsf{state}_p^{\mathsf{semi}}(\tilde{t})$. We append as many applications of rules of type R_3 as there are facts $\mathsf{Ack}(t_1,t_2) \in S_{n^{(1)}}^{(1)}$, and repeat this for all t_1,t_2 such that $\mathsf{Ack}(t_1,t_2) \in S_{n^{(1)}}^{(1)}$. Then, $S_{n^{(1)}}^{(1)} \rightarrow_{[\![P]\!]} S_{n'}^{(1)}$ and $S_{n'}^{(1)}$ does not contain Ack-facts anymore.

   If $S_{n'}^{(1)}$ contains a fact $\mathsf{state}_p^{\mathsf{semi}}(\tilde{t})$, we remove the last transition that produced this fact, i. e., for i such that $S_i = S_{i-1} \setminus^\# \{\, \mathsf{state}_p(\tilde{t}) \,\}^\# \cup^\# \{\, \mathsf{Msg}(t_1,t_2), \mathsf{state}_p^{\mathsf{semi}}(\tilde{t}) \,\}^\#$, we define
$$S_j^{(1)'} := \begin{cases} S_j^{(1)} & \text{if } j \le i-1 \\ S_{j+1}^{(1)} \setminus^\# \{\, \mathsf{Msg}(t_1,t_2), \mathsf{state}_p^{\mathsf{semi}}(\tilde{t}) \,\}^\# \cup^\# \{\, \mathsf{state}_p(\tilde{t}) \,\}^\# & \text{if } i-1 < j < n' \end{cases}$$
   The resulting execution is valid, since $\mathsf{state}_p^{\mathsf{semi}}(\tilde{t}) \in S_{n'}^{(1)}$ and since $\mathsf{Msg}(t_1,t_2) \in S_{n'}^{(1)}$. The latter is the case because if $\mathsf{Msg}(t_1,t_2)$ would be consumned at a later point, say j, $j+1$ would contain $\mathsf{Ack}(t_1,t_2)$, but since $S_{n'-1}^{(1)'}$ does not contain Ack-facts, they can only be consumned by a rule of type R_3, which would have consumned $\mathsf{state}_p^{\mathsf{semi}}(\tilde{t})$. We repeat this procedure for every remaining $\mathsf{state}_p^{\mathsf{semi}}(\tilde{t}) \in S_{n'}^{(1)}$, and call the resulting trace
$$S_0^{(2)} \xrightarrow{E_1^{(2)}}_{[\![P]\!]} \dots \xrightarrow{E_n^{(2)}}_{[\![P]\!]} S_{n^{(2)}}^{(2)}$$
   Since no rule added or removed or removed has an action, $\mathit{hide}([E_1,\dots,E_n]) = \mathit{hide}([E_1^{(2)}, \dots, E_{n^{(2)}}^{(2)}])$ and $[E_1^{(2)},\dots,E_{n^{(2)}}^{(2)}] \vDash \alpha$.

3. We transform $S_0^{(1)} \xrightarrow{E_1^{(1)}}_{[\![P]\!]} \dots \xrightarrow{E_n^{(1)}}_{[\![P]\!]} S_{n^{(1)}}^{(1)}$ as follows (all equalities are modulo E): Let us call instances of R_1, R_2 or R_3 that appear outside a chain
$$S_{i-3} \rightarrow_{R_1} S_{i-2} \rightarrow_{R_2} S_{i-1} \rightarrow_{R_3} S_i$$
   for some i $t_1,t_2 \in \mathcal{M}$ "unmarked". Do the following for the smallest i that is an unmarked instance of R_3 (we will call the instance of R_3 ri_3 and suppose it is applied from S_{i-1} to S_i): Apply ri_3 after $j < i$ such that S_{j-1} to S_j is the first unmarked instance of R_2, for some q and $\tilde{y}$,i. e., this instance produces a fact $\mathsf{state}_{q\cdot 1}(\tilde{y},\tilde{y}')$ and a fact $\mathsf{Ack}(t_1,t_2)$. Since there is no rule between j and i that might consume $\mathsf{Ack}(t_1,t_2)$ (only rules of form R_3 do, and ri_3 is the first unmarked instance of such a rule) and since ri_3 does not consume $\mathsf{state}_{q\cdot 1}(\tilde{y},\tilde{y}')$, we can move ri_3 between j and $j+1$, adding the conclusions of ri_3 and removing the premises of ri_3 from every $S_{j+1},\dots,S_i$. Note that unmarked instances of R_2 and R_3 are guaranteed to be preceeded by a marked R_1, and therefore only remove facts of form $\mathsf{Ack}(\dots)$ or $\mathsf{Msg}(\dots)$ that have been added in that preceeding step. Since the transition at step j requires a fact $\mathsf{Msg}(t_1,t_2)$, there is an instance of R_1 prior to j, say at $k < j$, since only rules of form R_1 produces facts labelled $\mathsf{Msg}(t_1,t_2)$. Since ri_3 is now applied from Sj to S_{j+1}, we have that an instance ri_1 of a rule of

form R_1 that produces $\mathsf{state}_p^{\mathsf{semi}}(\tilde{t})$ must appear before j, i. e., $ri_1 \in \mathit{ginsts}([\![P]\!]_{=p})$. Therefore, it produces a fact $\mathsf{Msg}(t_1, t_2)$ indeed. We choose the largest k that has an unmarked R_1 that produces $\mathsf{Msg}(t_1, t_2)$ and $\mathsf{state}_p^{\mathsf{semi}}(\tilde{t})$ and move it right before j, resulting in the following msr execution:

$$S_t^{(1)'} := \begin{cases} S_t^{(1)} & \text{if } t < k \\ S_{t+1}^{(1)} \cup^{\#} \{\, \mathsf{Msg}(t_1, t_2), \mathsf{state}_p^{\mathsf{semi}}(\tilde{t}) \,\}^{\#} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t}) \,\}^{\#} & \text{if } k \leq t < j-1 \\ S_{(t)}^{(1)} & \text{if } j-1 \leq t < j+1 \\ S_{(t-1)}^{(1)} \setminus^{\#} \{\, \mathsf{state}_p^{\mathsf{semi}}(\tilde{t}), \mathsf{Ack}(t_1, t_2) \,\}^{\#} \cup^{\#} \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}) \,\}^{\#} & \text{if } j+1 \leq t < i+1 \\ S_t^{(1)} & \text{if } i+1 \leq t \end{cases}$$

We apply this procedure until it reaches a fixpoint and call the resulting trace

$$S_0^{(3)} \xrightarrow{E_1^{(3)}}_{[\![P]\!]} \cdots \xrightarrow{E_n^{(3)}}_{[\![P]\!]} S_{n^{(3)}}^{(3)}$$

Since no rule moved during the procedure has an action, $\mathit{hide}([E_1, \ldots, E_n]) = \mathit{hide}([E_1^{(3)}, \ldots, E_{n^{(3)}}^{(3)}])$ and $[E_1^{(3)}, \ldots, E_{n^{(3)}}^{(3)}] \vDash \alpha$.

4. If the last transition is in $\{\, \text{MDOut}, \text{MDPub}, \text{MDFresh}, \text{MDAppl}, \text{Fresh} \,\}$, we remove it. Repeat until fixpoint is reached and call the resulting trace

$$S_0^{(4)} \xrightarrow{E_1^{(4)}}_{[\![P]\!]} \cdots \xrightarrow{E_n^{(4)}}_{[\![P]\!]} S_{n^{(4)}}^{(4)}$$

Since no rule removed during the procedure has an action, $\mathit{hide}([E_1, \ldots, E_n]) = \mathit{hide}([E_1^{(4)}, \ldots, E_{n^{(4)}}^{(4)}])$ and $[E_1^{(4)}, \ldots, E_{n^{(4)}}^{(4)}] \vDash \alpha$.

5. If there is $\mathsf{In}(t) \in S_{n^{(4)}-1}^{(4)}$, then there is a transition where $\mathsf{In}(t)$ is produced and never consumned until $n^{(4)} - 1$. The only rule producing $\mathsf{In}(t)$ is MDIn. We can move this transition to just before $n^{(4)} - 1$ and call the resulting trace

$$S_0^{(5)} \xrightarrow{E_1^{(5)}}_{[\![P]\!]} \cdots \xrightarrow{E_n^{(5)}}_{[\![P]\!]} S_{n^{(5)}}^{(5)}$$

Since $[E_1^{(4)}, \ldots, E_{n^{(4)}}^{(4)}] \vDash \alpha$, especially α_{inev}, there is no action that is not in $\mathcal{F}_{\mathit{res}}$ between the abovementioned instance of MDIn, therefore, $\mathit{hide}([E_1, \ldots, E_n]) = \mathit{hide}([E_1^{(5)}, \ldots, E_{n^{(5)}}^{(5)}])$ holds. Since α_{inev} is the only part of α that mentions K, and since the tranformation preserved α_{inev}, we have that $[E_1^{(5)}, \ldots, E_{n^{(5)}}^{(5)}] \vDash \alpha$.

6. We will show that 6 and 7 hold for

$$S_0^{(5)} \xrightarrow{E_1^{(5)}}_{[\![P]\!]} \cdots \xrightarrow{E_n^{(5)}}_{[\![P]\!]} S_{n^{(5)}}^{(5)}$$

in one step.

If $n^{(5)} \geq 2$ and there is no Ack-fact in $S_{n^{(5)}-1}^{(5)} \setminus S_{n^{(5)}}^{(5)}$, then we chose the largest $m < n$ such that $S_{m-1}^{(5)} \xrightarrow{E_m^{(5)}}_{[\![P,[],[]]\!],\text{Init},\text{MDIn}} S_m^{(5)}$, or, if there is an Ack-fact in $S_{n^{(5)}-1}^{(5)} \setminus S_{n^{(5)}}^{(5)}$, we will chose the largest $m' < n-2$ such that $S_{m'-1}^{(5)} \xrightarrow{E_{m'}^{(5)}}_{[\![P,[],[]]\!],\text{Init},\text{MDIn}} S_{m'}^{(5)}$.

This trivially fulfills 4. $S_m^{(5)} \rightarrow_R^* S_{n^{(5)}}^{(5)}$ and $S_{m'}^{(5)} \rightarrow_R^* S_{n^{(5)}-3}^{(5)}$, since otherwise there would be a larger m or m'. This also implies 2, as none of the rules in $R = \{\, \text{MDOut}, \text{MDPub}, \text{MDFresh}, \text{MDAppl}, \text{Fresh} \,\}$ remove Ack- or $\mathsf{state}^{\mathsf{semi}}$-facts, and the chain of rules R_1, R_2, R_3 consumes as many as it produces. Thus, if they where in $S_m^{(5)}$, they would be in $S_{n^{(5)}}^{(5)}$, too. Since $n > 2$,

$m > 1$, and therefore 1. 3 holds for all parts of the trace, and therefore also for the m prefix. Similar for 5.

Since we can literally apply the same argument for the largest $\tilde{m} < m$ such that

$$S^{(5)}_{m-1} \xrightarrow{E^{(5)}_m}_{[\![P,[],[]]\!],\textsc{Init},\text{MDIn}} S^{(5)}_m$$

or, in case that there is an Ack-fact in $S^{(5)}_{m-1} \setminus S^{(5)}_m$, for the largest $\tilde{m} < m - 2$, can show that 6 and 7 hold for the trace up to m or m', concluding it is normal.

□

Definition 22. *Let P be a ground process, $\mathcal{P}$ be a multiset of processes and S a multiset of multiset rewrite rules. We write $\mathcal{P} \leftrightsquigarrow_P S$ if there exists a bijection between $\mathcal{P}$ and the multiset $\{\mathsf{state}_p(\tilde{t}) \mid \exists p, \tilde{t}.\ \mathsf{state}_p(\tilde{t}) \in^\# S\}^\#$ such that whenever $Q \in^\# \mathcal{P}$ is mapped to $\mathsf{state}_p(\tilde{t}) \in^\# S$, then:*

1. *$\mathsf{state}_p(\tilde{t}) \in_E prems(R)$ for $R \in ginsts([\![P]\!]_{=p})$.*
2. *Let θ be a grounding substitution for $state(\tilde{x}) \in prems([\![P]\!]_{=p})$ such that $\tilde{t} = \tilde{x}\theta$. Then*

   $$(P|_p\tau)\rho =_E Q$$

   for a substitution τ, and a bijective renaming ρ of fresh, but not bound names in Q, defined as follows:

   $$\begin{aligned} \tau(x) :=& \theta(x) && \text{if } x \text{ not a reserved variable} \\ \rho(a) :=& a' && \text{if } \theta(n_a) = a' \end{aligned}$$

When $\mathcal{P} \leftrightsquigarrow_P S$, $Q \in^\# \mathcal{P}$ and $\mathsf{state}_p(\tilde{t}) \in^\# S$ we also write $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$ if this bijection maps Q to $\mathsf{state}_p(\tilde{t})$.

Remark 3. *Note that $\leftrightsquigarrow_P$ has the following properties (by the fact that it defines a bijection between multisets).*

- *If $\mathcal{P}_1 \leftrightsquigarrow_P S_1$ and $\mathcal{P}_2 \leftrightsquigarrow_P S_2$ then $\mathcal{P}_1 \cup^\# \mathcal{P}_2 \leftrightsquigarrow_P S_1 \cup^\# S_2$.*
- *If $\mathcal{P}_1 \leftrightsquigarrow_P S_1$ and $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$ for $Q \in \mathcal{P}_1$ and $\mathsf{state}_p(\tilde{t}) \in S_1$ (i.e. Q and $\mathsf{state}_p(\tilde{t})$ are related by the bijection defined by $\mathcal{P}_1 \leftrightsquigarrow_P S_1$) then $\mathcal{P}_1 \setminus^\# \{Q\} \leftrightsquigarrow_P S_1 \setminus^\# \{\mathsf{state}_p(\tilde{t})\}$.*

Lemma 12. *Le P be a well-formed ground process. If*

$$S_0 = \emptyset \xrightarrow{E_1}_{[\![P]\!]} S_1 \xrightarrow{E_2}_{[\![P]\!]} \dots \xrightarrow{E_n}_{[\![P]\!]} S_n \in exec^{msr}([\![P]\!])$$

is normal (see Definition 21) and $[E_1, \dots, E_n] \vDash \alpha$ (see Definition 14), then there are $(\mathcal{E}_0, \mathcal{S}_0, \mathcal{S}^{\mathrm{MS}}_0, \mathcal{P}_0, \sigma_0, \mathcal{L}_0), \dots, (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\mathrm{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'})$ and $F_1, \dots, F_{n'}$ such that:

$$(\mathcal{E}_0, \mathcal{S}_0, \mathcal{S}^{\mathrm{MS}}_0, \mathcal{P}_0, \sigma_0, \mathcal{L}_0) \xrightarrow{F_1} (\mathcal{E}_1, \mathcal{S}_1, \mathcal{S}^{\mathrm{MS}}_1, \mathcal{P}_1, \sigma_1, \mathcal{L}_1) \xrightarrow{F_2} \dots \xrightarrow{F_{n'}} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\mathrm{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'})$$

where $(\mathcal{E}_0, \mathcal{S}_0, \mathcal{S}^{\mathrm{MS}}_0, \mathcal{P}_0, \sigma_0, \mathcal{L}_0) = (\emptyset, \emptyset, \emptyset, \{\, P \,\}, \emptyset, \emptyset)$ and there exists a monotonically increasing, surjective function $f \colon \mathbb{N}_n \setminus \{\, 0 \,\} \to \mathbb{N}_{n'}$ such that $f(n) = n'$ and for all $i \in \mathbb{N}_n$

1. *$\mathcal{E}_{f(i)} = \{\, a \in FN \mid ProtoNonce(a) \in_E \bigcup_{1 \le j \le i} E_j \,\}$*
2. *$\forall\, t \in \mathcal{M}.\ \mathcal{S}_{f(i)}(t) = \begin{cases} u & \text{if } \exists j \le i. \text{Insert}(t,u) \in_E E_j \\ & \quad \wedge \forall j', u'. j < j' \le i \to \text{Insert}(t,u') \notin_E E_{j'} \wedge \text{Delete}(t) \notin_E E_{j'} \\ \perp & \text{otherwise} \end{cases}$*
3. *$\mathcal{S}^{\mathrm{MS}}_{f(i)} =_E S_i \setminus^\# \mathcal{F}_{res}$*

4. $\mathcal{P}_{f(i)} \leftrightsquigarrow_P S_i$

5. $\{\, x\sigma_{f(i)} \mid x \in \mathbf{D}(\sigma_{f(i)}) \,\}^{\#} =_E \{\, \mathsf{Out}(t) \in \cup_{k \leq i} S_k \,\}^{\#}$

6. $\mathcal{L}_{f(i)} =_E \{\, t \mid \exists j \leq i, u.\ \mathrm{Lock}(u,t) \in_E E_j \wedge \forall j < k \leq i.\mathrm{Unlock}(u,t) \notin_E E_k \,\}$.

Furthermore,

7. $\mathit{hide}([E_1, \ldots, E_n]) =_E [F_1, \ldots, F'_n]$.

Proof. We proceed by induction over the number of transitions n.

Base Case. A normal msr execution contains at least an application of the init rule, thereby the shortest normal msr execution is

$$\emptyset \rightarrow_{[\![P]\!]} S_1 = \{\, \mathsf{state}_{[]}() \,\}^{\#}$$

We chose $n' = 0$ and thus

$$(\mathcal{E}_0, \mathcal{S}_0, \mathcal{S}_0^{\mathrm{MS}}, \mathcal{P}_0, \sigma_0, \mathcal{L}_0) = (\emptyset, \emptyset, \emptyset, \{\, P \,\}^{\#}, \emptyset, \emptyset).$$

We define $f : \{\, 1 \,\} \rightarrow \{\, 0 \,\}$ such that $f(1) = 0$.

To show that Condition 4 holds, we have to show that $\mathcal{P}_0 \leftrightsquigarrow_P \{\, \mathsf{state}_{[]}(s : \mathit{fresh}) \,\}^{\#}$. Note that $\mathcal{P}_0 = \{\, P \,\}^{\#}$. We choose the bijection such that $P \leftrightsquigarrow_P \mathsf{state}_{[]}(s : \mathit{fresh})$.

By Definition 19, $[\![P]\!]_{=[]} = [\![P, [], []]\!]_{=[]}$. We see from Figure 12 that for every P we have that $\mathsf{state}_{[]}(s : \mathit{fresh}) \in \mathit{prems}(R\theta)$, for $R \in [\![P, [], []]\!]_{=[]}$ and $\theta = \emptyset$. This induces $\tau = \emptyset$ and $\rho = \emptyset$. Since $P|_{[]}\tau\rho = P$, we have $P \leftrightsquigarrow_P \mathsf{state}_{[]}()$, and therefore $\mathcal{P}_0 \leftrightsquigarrow_P S_1$.

Condition 1, Condition 2, Condition 3, Condition 5, Condition 6, and Condition 7 hold trivially.

Inductive step. Assume the invariant holds for $n - 1 \geq 1$. We have to show that the lemma holds for n transitions, i. e., we assume that

$$\emptyset \xrightarrow{E_1}_{[\![P]\!]} S_1 \xrightarrow{E_2}_{[\![P]\!]} \ldots \xrightarrow{E_n}_{[\![P]\!]} S_n \in \mathit{exec}^{\mathit{msr}}([\![P]\!])$$

is normal and $[E_1, \ldots, E_n] \vDash \alpha$. Then it is to show that there is

$$(\mathcal{E}_0, \mathcal{S}_0, \mathcal{S}_0^{\mathrm{MS}}, \mathcal{P}_0, \sigma_0, \mathcal{L}_0) \xrightarrow{F_1} (\mathcal{E}_1, \mathcal{S}_1, \mathcal{S}_1^{\mathrm{MS}}, \mathcal{P}_1, \sigma_1, \mathcal{L}_1) \xrightarrow{F_2} \ldots \xrightarrow{F'_n} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}_{n'}^{\mathrm{MS}}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'})$$

fulfilling Conditions 1 to 8.

Assume now for the following argument, that there is not fact with the symbol Ack in $S_{n-1} \setminus^{\#} S_n$. This is the case for all cases except for the case where rule instance applied from S_{n-1} to S_n has the form $ri = [\mathsf{state}_p^{\mathsf{semi}}(\tilde{s}), \mathsf{Ack}(t_1, t_2)] \;{-}[]{\rightarrow}\; [\mathsf{state}_{p\cdot 1}(\tilde{s})]$. This case will require a similiar, but different argument, which we will present when we come to this case.

Since $\emptyset \xrightarrow{E_1}_{[\![P]\!]} \cdots \xrightarrow{E_n}_{[\![P]\!]} S_n \in \mathit{exec}^{\mathit{msr}}([\![P]\!])$ is normal and $n \geq 2$, there exists $m < n$ such that $S_m \rightarrow^*_R S_n$ for $R = \{\, \textsc{MDOut}, \textsc{MDPub}, \textsc{MDFresh}, \textsc{MDAppl}, \textsc{Fresh} \,\}$ and $\emptyset \xrightarrow{E_1}_{[\![P]\!]} \cdots \xrightarrow{E_m}_{[\![P]\!]} S_m \in \mathit{exec}^{\mathit{msr}}([\![P]\!])$ is normal, too. This allows us to apply the induction hypothesis on $\emptyset \xrightarrow{E_1}_{[\![P]\!]} \cdots \xrightarrow{E_m}_{[\![P]\!]} S_m \in \mathit{exec}^{\mathit{msr}}([\![P]\!])$. Hence there is a monotonically increasing function from $\mathbb{N}_m \rightarrow \mathbb{N}_{n'}$ and an execution such that Conditions 1 to 8 hold. Let f_p be this function and note that $n' = f_p(m)$.

In the following case distinction, we will (unless stated otherwise) extend the previous execution by one step from $(\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}_{n'}^{\mathrm{MS}}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'})$ to $(\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}_{n'+1}^{\mathrm{MS}}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$, and prove that Conditions 1 to 7 hold for $n' + 1$. By induction hypothesis, they hold for all $i \leq n'$. We define a function $f \colon \mathbb{N}_n \rightarrow \mathbb{N}_{n'+1}$ as follows:

$$f(i) := \begin{cases} f_p(i) & \text{if } i \in \mathbb{N}_m \\ n' & \text{if } m < i < n \\ n' + 1 & \text{if } i = n \end{cases}$$

Since, $S_m \to_R^* S_n$ for $R = \{\, \textsc{MDOut}, \textsc{MDPub}, \textsc{MDFresh}, \textsc{MDAppl}, \textsc{Fresh} \,\}$, only $S_n \setminus^\# S_m$ contains only Fr-facts and $\mathsf{!K}$-facts, and $S_m \setminus^\# S_n$ contains only Fr-facts and Out-facts. Therefore, Condition 3,4 and 5 hold for all $i \leq n-1$. Since $E_{m+1}, \dots, E_{n-1} = \emptyset$, Condition 1, 2,6 and 7 hold for all $i \leq n-1$.

Fix a bijection such that $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$. We will abuse notation by writing $P \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$, if this bijection maps P to $\mathsf{state}_p(\tilde{t})$.

We now proceed by case distinction over the last type of transition from S_{n-1} to S_n. Let $l_{linear} =_E S_{n-1} \setminus S_n$ and $r =_E S_n \setminus S_{n-1}$. l_{linear} can only contain linear facts, while r can contain linear as well as persistent facts. The rule instance ri used to go from S_{n-1} to S_n has the following form:

$$[l_{linear}, l_{persistent}] \; -\![E_n]\!\!\rightarrow r$$

for some $l_{persistent} \subset_E^\# S_{n-1}$.

Note that l_{linear}, E_n and r uniquely identify which rule in $[\![P, [], []]\!]$ ri is an instance of – with exactly one exception: $[\![[] \; -\![a]\!\!\rightarrow []; \; P, p, \tilde{x}]\!] = [\![\mathsf{event}\ a; \; P, p, \tilde{x}]\!]$. Luckily, we can treat the last as a special case of the first.

If R is uniquely determined, we fix some $ri \in \mathit{ginsts}(R)$.

Case: $R =$ Init or $R \in$ MD $\setminus \{$ MDIn $\}$. In this case, $\emptyset \xrightarrow{E_1} \dots \xrightarrow{E_n} S_n$ is not a well-formed msr execution.

Case: $R =$ MDIn. Let $t \in \mathcal{M}$ such that $ri = R\tau = \mathsf{!K}(t) \; -\![K(t)]\!\!\rightarrow \mathsf{In}(t)$.

From the induction hypothesis, and since $E_{m+1}, \dots, E_n = \emptyset$, we have that

$$\mathcal{E}_{n'} = \{\, a \in \mathit{FN} \mid \mathit{ProtoNonce}(a) \in_E \bigcup_{1 \leq j \leq n} E_j \,\}.$$

From the induction hypothesis, and since no rule producing Out-facts is applied between step m and step n, we have that

$$\{\, x\sigma_{n'} \mid x \in \mathbf{D}(\sigma_{n'}) \,\}^\# =_E \{\, \mathsf{Out}(t) \in \cup_{k \leq n} S_k \,\}^\#.$$

Let $\tilde{r} = \{\, a \in \mathit{FN} \mid \mathit{RepNonce}(a) \in_E \bigcup_{1 \leq j \leq n} F_j \,\}$. Then, by Lemma 8 and Lemma 9, we have that $\nu\mathcal{E}_{n'}, \tilde{r}.\sigma_{n'} \vdash t$. Therefore, $\nu\mathcal{E}_{n'}.\sigma_{n'} \vdash t$. This allows us to chose the following transition:

$$\dots \xrightarrow{F_{n'}} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\mathrm{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \xrightarrow{K(t)} (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\mathrm{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $(\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\mathrm{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1}) = (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\mathrm{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'})$.

Conditions 1 to 8 hold trivially.

Case: $ri = [\mathsf{state}_p(\tilde{t})] \; -\![]\!\!\rightarrow []$ (for some p and $\tilde{t}$). By induction hypothesis, we have $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$, and thus, as previously established, $\mathcal{P}_{n'} \leftrightsquigarrow_P S_{n-1}$. Let $Q \in^\# \mathcal{P}_{n'}$ such that $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$. Let θ be a grounding substitution for $\mathsf{state}_p(\tilde{x}) \in \mathit{prems}([\![P]\!]_{=p})$ such that $\tilde{t} = \tilde{x}\theta$. Then θ induces a substitution τ and a bijective renaming ρ for fresh, but not bound names (in Q) such that $P|_p\tau\rho = Q$ (see Definition 22).

From the form of the rule R, and since $Q = P|_p\tau\rho$, we can deduce that $Q = \mathsf{0}$.

We therefore chose the following transition:

$$\dots \xrightarrow{F'_n} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\mathrm{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \xrightarrow{K(t)} (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\mathrm{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $\mathcal{E}_{n'+1} = \mathcal{E}_{n'}$, $\mathcal{S}_{n'+1} = \mathcal{S}_{n'}$, $\mathcal{S}^{\mathrm{MS}}_{n'+1} = \mathcal{S}^{\mathrm{MS}}_{n'}$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^\# \{\, \mathsf{0} \,\}^\#$, $\sigma_{n'+1} = \sigma_{n'}$ and $\mathcal{L}_{n'+1} = \mathcal{L}_{n'}$.

We define f as on page 51. Therefore, Conditions 1 to 8 hold for $i < n-1$. It is left to show that Conditions 1 to 8 hold for n.

Condition 4 holds since $Q \leftrightarrow \mathsf{state}_p(\tilde{t})$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^\# \{\, \mathsf{0} \,\}^\#$ and $S_n = S_{n-1} \setminus^\# \{\, \mathsf{state}_p(\tilde{t}) \,\}^\#$. Conditions 1, 2, 3, 5 and 7 hold trivially.

Case: $ri = [\mathsf{state}_p(\tilde{t})] -[]\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{t}), \mathsf{state}_{p\cdot 2}(\tilde{t})]$ **(for some p and $\tilde{t}$).** By induction hypothesis, we have $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$, and thus, as previously established, $\mathcal{P}_{n'} \leftrightsquigarrow_P S_{n-1}$. Let $Q \in^\# \mathcal{P}_{n'}$ such that $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$. Let θ be a grounding substitution for $\mathsf{state}_p(\tilde{x}) \in \mathit{prems}([\![P]\!]_{=p})$ such that $\tilde{t} = \tilde{x}\theta$. Then θ induces a substitution τ and a bijective renaming ρ for fresh, but not bound names (in Q) such that $P|_p\tau\rho = Q$ (see Definition 22).

From the form of the rule R, and since $Q = P|_p\tau\rho$, we can deduce that $Q = Q_1|Q_2$, for some processes $Q_1 = P|_{p\cdot 1}\tau\rho$ and $Q_2 = P|_{p\cdot 2}\tau\rho$.

We therefore chose the following transition:

$$\cdots \xrightarrow{F'_n} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\mathrm{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \rightarrow (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\mathrm{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $\mathcal{E}_{n'+1} = \mathcal{E}_{n'}$, $\mathcal{S}_{n'+1} = \mathcal{S}_{n'}$, $\mathcal{S}^{\mathrm{MS}}_{n'+1} = \mathcal{S}^{\mathrm{MS}}_{n'}$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^\# \{ Q_1 \mid Q_2 \}^\# \cup^\# \{ Q_1, Q_2 \}^\#$, $\sigma_{n'+1} = \sigma_{n'}$ and $\mathcal{L}_{n'+1} = \mathcal{L}_{n'}$.

We define f as on page 51. Therefore, Conditions 1 to 8 hold for $i < n - 1$. It is left to show that Conditions 1 to 8 hold for n.

By definition of $[\![P]\!]$ and $[\![P]\!]_{=p}$, we have that $Q_1 \leftrightarrow \mathsf{state}_{p\cdot 1}(\tilde{t})$ and $Q_2 \leftrightarrow \mathsf{state}_{p\cdot 2}(\tilde{t})$. Therefore, and since $Q \leftrightarrow \mathsf{state}_p(\tilde{t})$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^\# \{ Q_1 \mid Q_2 \}^\# \cup^\# \{ Q_1, Q_2 \}^\#$, and $S_n = S_{n-1} \setminus^\# \{ \mathsf{state}_p \tilde{t} \}^\# \cup^\# \{ \mathsf{state}_{p\cdot 1}(\tilde{t}), \mathsf{state}_{p\cdot 2}(\tilde{t}) \}^\#$, Condition 4 holds.

Conditions 1, 2, 3, 5 and 7 hold trivially.

Case: $ri = [!\mathsf{state}_p(\tilde{t})] -[]\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{t})]$ **(for some $p, \tilde{t}$).** By induction hypothesis, we have $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$, and thus, as previously established, $\mathcal{P}_{n'} \leftrightsquigarrow_P S_{n-1}$. Let $Q \in^\# \mathcal{P}_{n'}$ such that $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$. Let θ be a grounding substitution for $\mathsf{state}_p(\tilde{x}) \in \mathit{prems}([\![P]\!]_{=p})$ such that $\tilde{t} = \tilde{x}\theta$. Then θ induces a substitution τ and a bijective renaming ρ for fresh, but not bound names (in Q) such that $P|_p\tau\rho = Q$ (see Definition 22).

From the form of the rule R, and since $Q = P|_p\tau\rho$, we can deduce that $Q = !Q'$for a process $Q' = P|_{p\cdot 1}\tau\rho$..

We therefore chose the following transition:

$$\cdots \xrightarrow{F'_n} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\mathrm{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \rightarrow (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\mathrm{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $\mathcal{E}_{n'+1} = \mathcal{E}_{n'}$, $\mathcal{S}_{n'+1} = \mathcal{S}_{n'}$, $\mathcal{S}^{\mathrm{MS}}_{n'+1} = \mathcal{S}^{\mathrm{MS}}_{n'}$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \cup^\# \{ Q' \}^\#$, $\sigma_{n'+1} = \sigma_{n'}$ and $\mathcal{L}_{n'+1} = \mathcal{L}_{n'}$.

We define f as on page 51. Therefore, Conditions 1 to 8 hold for $i < n - 1$. It is left to show that Conditions 1 to 8 hold for n.

By definition of $[\![P]\!]$ and $[\![P]\!]_{=p}$, we have that $Q' \leftrightsquigarrow_P \mathsf{state}_{p\cdot 1}(\tilde{t})$. Therefore, and since $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \cup^\# \{ Q' \}^\#$, while $S_n = S_{n-1} \cup^\# \{ \mathsf{state}_{p\cdot 1}(\tilde{t}) \}^\#$, Condition 4 holds.

Conditions 1, 2, 3, 5 and 7 hold trivially.

Case: $ri = [\mathsf{state}_p(\tilde{t}), \mathsf{Fr}(a' \colon \mathit{fresh})] -[\mathit{ProtoNonce}(a' \colon \mathit{fresh})]\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{t}, a' \colon \mathit{fresh})]$ **(for some $p, \tilde{t}$ and $a' \in \mathit{FN}$).** By induction hypothesis, we have $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$, and thus, as previously established, $\mathcal{P}_{n'} \leftrightsquigarrow_P S_{n-1}$. Let $Q \in^\# \mathcal{P}_{n'}$ such that $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$. Let θ be a grounding substitution for $\mathsf{state}_p(\tilde{x}) \in \mathit{prems}([\![P]\!]_{=p})$ such that $\tilde{t} = \tilde{x}\theta$. Then θ induces a substitution τ and a bijective renaming ρ for fresh, but not bound names (in Q) such that $P|_p\tau\rho = Q$ (see Definition 22).

From the form of the rule R, and since $Q = P|_p\tau\rho$, we can deduce that $Q = \nu\, a;\, Q'$ for a name $a \in \mathit{FN}$ and a process $Q' = P|_{p\cdot 1}\tau\rho$.

By definition of $\mathit{exec}^{\mathit{msr}}$, the fact $\mathsf{Fr}(a')$ can only be produced once. Since this fact is linear it can only be consumed once. Every rule in $[\![P]\!]$ that produces a label $\mathit{ProtoNonce}(x)$ for some x consumes a fact $\mathsf{Fr}(x)$. Therefore,

$$a' \notin \{ a \in \mathit{FN} \mid \mathit{ProtoNonce}(a) \in_E \bigcup_{1 \leq j \leq n-1} E_j \}.$$

The induction hypothesis allows us to conclude that $a' \notin \mathcal{E}_{n'}$,i. e., a' is fresh. We therefore chose the following transition:

$$\cdots \xrightarrow{F'_n} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\mathrm{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \rightarrow (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\mathrm{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $\mathcal{E}_{n'+1} = \mathcal{E}_{n'} \cup a'$, $\mathcal{S}_{n'+1} = \mathcal{S}_{n'}$, $\mathcal{S}^{\mathrm{MS}}_{n'+1} = \mathcal{S}^{\mathrm{MS}}_{n'}$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{ \nu\ a; Q' \}^{\#} \cup^{\#} \{ Q'\{{}^{a}/_{a'}\} \}^{\#}$, $\sigma_{n'+1} = \sigma_{n'}$ and $\mathcal{L}_{n'+1} = \mathcal{L}_{n'}$.

We define f as on page 51. Therefore, Conditions 1 to 8 hold for $i < n-1$. It is left to show that Conditions 1 to 8 hold for n.

By definition of $[\![P]\!]$, $\mathit{state}_{p\cdot 1}(\tilde{x}, a) \in \mathit{prems}(R')$ for an $R' \in [\![P]\!]_{=p\cdot 1}$. We can choose $\theta' := \theta[n_a \mapsto a']$ and have $\mathsf{state}_{p\cdot 1}(\tilde{t}, a') = \mathsf{state}_{p\cdot 1}(\tilde{x}, a)\theta'$. Since $Q = P|_p\tau\rho$ for τ and ρ induced by θ, $Q'\{{}^{a'}/_{a}\} = P|_p\tau'\rho'$ for τ' and ρ' induced by θ', i. e., $\tau' = \tau$ and $\rho' = \rho[a \mapsto a']$. Therefore, $Q'\{{}^{a'}/_{a}\} \leftrightsquigarrow_P \mathsf{state}_{p\cdot 1}(\tilde{t}, a')$.

Condition 4 holds, since furthermore $\nu\ a'$; $\mathsf{Q'} \leftrightarrow \mathsf{state}_p(\tilde{t})$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{ \nu\ a';\ \mathsf{Q'} \}^{\#} \cup^{\#} \{ Q'\{{}^{a'}/_{a}\} \}^{\#}$, and $S_n = S_{n-1} \setminus^{\#} \{ \mathsf{Fr}(a), \mathsf{state}_p(\tilde{t}) \}^{\#} \cup^{\#} \mathsf{state}_{p\cdot 1}(\tilde{t}, a\colon \mathit{fresh})$.

Condition 1, holds since $\mathcal{E}_{n'+1} = \mathcal{E}_{n'} \cup a'$, and $E_n = \mathit{ProtoNonce}(a')$. Condition 7 holds since $\mathit{ProtoNonce}(a) \in \mathcal{F}_{\mathit{res}}$.

Conditions 2, 3 and 5 hold trivially.

Case: ri = $[\mathsf{state}_p(\tilde{t}), \mathsf{In}(t_1)] \ {-}[\mathrm{InEvent}(t_1)]{\rightarrow}\ [\mathsf{state}_{p\cdot 1}(\tilde{t}), \mathsf{Out}(t_2)]$ **(for some** $p, \tilde{t}$ **and** $t_1, t_2 \in \mathcal{M}$**).** Since the msr execution is normal, we have that $S_{n-2} \xrightarrow{K(t_1)}_{\mathrm{MDIN}} S_{n-1}$. Since $S_0 \xrightarrow{E_1}_{[\![P]\!]} \dots \xrightarrow{E_n}_{[\![P]\!]} S_n$ is normal, so is $S_0 \xrightarrow{E_1}_{[\![P]\!]} \dots \xrightarrow{E_{n-1}}_{[\![P]\!]} S_{n-1}$, and therefore $S_0 \xrightarrow{E_1}_{[\![P]\!]} \dots \xrightarrow{E_{n-2}}_{[\![P]\!]} S_{n-2}$. Hence there is an $m < n-2$ such $S_0 \xrightarrow{E_1}_{[\![P]\!]} \dots \xrightarrow{E_m}_{[\![P]\!]} S_m$ is a normal trace and $S_m \to^*_R S_{n-1}$ for $R = \{ \mathrm{MDO{\scriptsize UT}}, \mathrm{MDP{\scriptsize UB}}, \mathrm{MDF{\scriptsize RESH}}, \mathrm{MDA{\scriptsize PPL}}, \mathrm{F{\scriptsize RESH}} \}$.

By induction hypothesis, we have $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$, and thus, since $\{ \mathrm{MDO{\scriptsize UT}}, \mathrm{MDP{\scriptsize UB}}, \mathrm{MDF{\scriptsize RESH}}, \mathrm{MDA{\scriptsize PPL}} \}$and $\mathrm{F{\scriptsize RESH}}$ do not add or remove state-facts, $\mathcal{P}_{n'} \leftrightsquigarrow_P S_{n-2}$. Let $Q \in^{\#} \mathcal{P}_{n'}$ such that $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$. Let θ be a grounding substitution for $\mathit{state}(\tilde{x}) \in \mathit{prems}([\![P]\!]_{=p})$ such that $\tilde{t} = \tilde{x}\theta$. Then θ induces a substitution τ and a bijective renaming ρ for fresh, but not bound names (in Q) such that $P|_p\tau\rho = Q$ (see Definition 22).

From the form of the rule R, and since $Q = P|_p\tau\rho$, we can deduce that $Q = \mathrm{out}\ (t_1, t_2); Q'$ for a process $Q' = P|_{p\cdot 1}\tau\rho$.

From the induction hypothesis, and since $E_{m+1}, \dots, E_{n-2} = \emptyset$, we have that

$$\mathcal{E}_{n'} = \{ a \in \mathit{FN} \mid \mathit{ProtoNonce}(a) \in_E \bigcup_{1 \le j \le n-2} E_j \}.$$

From the induction hypothesis, and since no rule producing Out-facts is applied between step m and step $n-2$, we have that

$$\{ x\sigma_{n'} \mid x \in \mathbf{D}(\sigma_{n'}) \}^{\#} =_E \{ \mathsf{Out}(t) \in \cup_{k \le n-2} S_k \}^{\#}. \tag{3}$$

Let $\tilde{r} = \{ a : \mathit{fresh} \mid \mathit{RepNonce}(a) \in \bigcup_{1 \le j \le n-2} F_j \}$. Since $!\mathsf{K}(t_1) \in \mathit{prems}(\mathrm{MDI{\scriptsize N}}\sigma)$ for $\sigma(x) = t_1$, we have $!\mathsf{K}(t) \in_E S_{n-2}$. By Lemma 8 and Lemma 9, we have $\nu\mathcal{E}_{n'}, \tilde{r}.\sigma_{n'} \vdash t$. Therefore, $\nu\mathcal{E}_{n'}.\sigma_{n'} \vdash t$. We chose the following transition:

$$\dots \xrightarrow{F'_n} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\mathrm{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \xrightarrow{K(t_1)} (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\mathrm{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $\mathcal{E}_{n'+1} = \mathcal{E}_{n'}$, $\mathcal{S}_{n'+1} = \mathcal{S}_{n'}$, $\mathcal{S}^{\mathrm{MS}}_{n'+1} = \mathcal{S}^{\mathrm{MS}}_{n'}$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{ \mathrm{out}\ (t_1, t_2); Q' \}^{\#} \cup^{\#} \{ Q' \}^{\#}$, $\sigma_{n'+1} = \sigma_{n'} \cup \{ {}^{t_2}/_{x} \}$ and $\mathcal{L}_{n'+1} = \mathcal{L}_{n'}$ for a fresh x.

We define f as follows:

$$f(i) := \begin{cases} f_p(i) & \text{if } i \in \mathbb{N}_m \\ n' & \text{if } m < i < n-1 \\ n'+1 & \text{if } i = n \end{cases}$$

Therefore, Conditions 1 to 8 hold for $i < n-1$. It is left to show that Conditions 1 to 8 hold for n.

Condition 7 holds since $\mathit{hide}([E_1, \dots, E_m]) =_E [F_1, \dots, n']$, and $[E_{m+1}, \dots, E_{n-1}] =_E [F_{n'+1}]$, since $E_{n-1} = K(t_1)$.

Condition 5 holds since $\sigma_{n'+1} = \sigma_{n'} \cup \{ {}^{t_2}/_{x} \}$, and therefore:

$$\begin{aligned} \{ x\sigma_{n'+1} \mid x \in \mathbf{D}(\sigma_{n'+1}) \}^{\#} &= \{ x\sigma_{n'} \mid x \in \mathbf{D}(\sigma_{n'}) \}^{\#} \cup^{\#} \{ t_2 \}^{\#} \\ &=_E \{ \mathsf{Out}(t) \in \cup_{k \le n-2} S_k \}^{\#} \cup^{\#} \{ t_2 \}^{\#} && \text{(by (4))} \\ &= \{ \mathsf{Out}(t) \in \cup_{k \le n} S_k \}^{\#} \end{aligned}$$

By definition of $[\![P]\!]$ and $[\![P]\!]_{=p}$, we have that $Q' \leftrightsquigarrow_P \mathsf{state}_{p\cdot 1}(\tilde{t})$. Therefore, and since out $(t_1,t_2);Q' \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{\, \text{out } (t_1,t_2);Q' \,\}^{\#} \cup^{\#} \{\, Q' \,\}^{\#}$, and $S_n =_E S_{n-1} \setminus^{\#} \{\, \mathsf{In}(a), \mathsf{state}_p(\tilde{t}) \,\}^{\#} \cup^{\#} \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}), \mathsf{Out}(t_2) \,\}$, Condition 4 holds.

Conditions Condition 1, 2 and 3 hold trivially.

Case: $ri = [\mathsf{state}_p(\tilde{t}), \mathsf{In}(< t_1, t_2 >)] \;-\![\text{InEvent}(\langle t_1,t_2 \rangle)]\!\rightarrow\; [\mathsf{state}_{p\cdot 1}(\tilde{t}, \tilde{t}')]$ **(for some** $p, \tilde{t}, \tilde{t}'$ **and** $t_1, t_2 \in \mathcal{M}$**).** Since the msr execution is normal, we have that $S_{n-2} \xrightarrow{K(t_1)}_{\text{MDIN}} S_{n-1}$. Since $S_0 \xrightarrow{E_1}_{[\![P]\!]} \dots \xrightarrow{E_n}_{[\![P]\!]} S_n$ is normal, so is $S_0 \xrightarrow{E_1}_{[\![P]\!]} \dots \xrightarrow{E_{n-1}}_{[\![P]\!]} S_{n-1}$, and therefore $S_0 \xrightarrow{E_1}_{[\![P]\!]} \dots \xrightarrow{E_{n-2}}_{[\![P]\!]} S_{n-2}$. Hence there is an $m < n-2$ such $S_0 \xrightarrow{E_1}_{[\![P]\!]} \dots \xrightarrow{E_m}_{[\![P]\!]} S_m$ is a normal trace and $S_m \rightarrow^*_R S_{n-1}$ for $R = \{\, \text{MDOUT}, \text{MDPUB}, \text{MDFRESH}, \text{MDAPPL}, \text{FRESH} \,\}$.

By induction hypothesis, we have $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$. Since $\{\, \text{MDOUT}, \text{MDPUB}, \text{MDFRESH}, \text{MDAPPL} \,\}$, FRESH and MDIN do not add or remove state-facts, $\mathcal{P}_{n'} \leftrightsquigarrow_P S_{n-2}$. Let $Q \in^{\#} \mathcal{P}_{n'}$ such that $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$. Let θ be a grounding substitution for $\mathsf{state}_p(\tilde{x}) \in \mathit{prems}([\![P]\!]_{=p})$ such that $\tilde{t} =_E \tilde{x}\theta$. Then θ induces a substitution τ and a bijective renaming ρ for fresh, but not bound names (in Q) such that $P|_p\tau\rho = Q$ (see Definition 22). From the form of the rule R, and since $Q = P|_p\tau\rho$, we can deduce that $Q = \text{in } (t_1, N);Q'$, for N a term that is not necessarily ground, and a process $Q' = P|_{p\cdot 1}\tau\rho$. Since $ri \in_E \mathit{ginsts}(R)$, we have that there is a substitution τ' such that $N\tau' =_E t_2$.

From the induction hypothesis, and since $E_{m+1}, \dots, E_{n-2} = \emptyset$, we have that

$$\mathcal{E}_{n'} = \{\, a \mid \mathit{ProtoNonce}(a) \in \bigcup_{1 \le j \le n-2} E_j \,\}.$$

From the induction hypothesis, and since no rule producing Out-facts is applied between step m and step $n-2$, we have that

$$\{\, x\sigma_{n'} \mid x \in \mathbf{D}(\sigma_{n'}) \,\}^{\#} = \{\, \mathsf{Out}(t) \in \cup_{k \le n-2} S_k \,\}^{\#}. \tag{4}$$

Let $\tilde{r} = \{\, a : \mathit{fresh} \mid \mathit{RepNonce}(a) \in \bigcup_{1 \le j \le n-2} F_j \,\}$. Since $!\mathsf{K}(< t_1, t_2 >) \in \mathit{prems}(\text{MDIN}\sigma)$ for $\sigma(x) =< t_1, t_2 >$, we have $!\mathsf{K}(< t_1, t_2 >)_E \in S_{n-2}$. By Lemma 8 and Lemma 9, we have $\nu\mathcal{E}_{n'}, \tilde{r}.\sigma_{n'} \vdash < t_1, t_2 >$. Therefore, $\nu\mathcal{E}_{n'}.\sigma_{n'} \vdash < t_1, t2 >$. Using DEQ and DAPPL with the function symbols *fst* and *snd*, we have $\nu\mathcal{E}_{n'}.\sigma_{n'} \vdash t_1$ and $\nu\mathcal{E}_{n'}.\sigma_{n'} \vdash t2$. Therefore, we chose the following transition:

$$\dots \xrightarrow{F'_n} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\text{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \xrightarrow{K(t_1)} (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\text{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $\mathcal{E}_{n'+1} = \mathcal{E}_{n'}$, $\mathcal{S}_{n'+1} = \mathcal{S}_{n'}$, $\mathcal{S}^{\text{MS}}_{n'+1} = \mathcal{S}^{\text{MS}}_{n'}$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{\, \text{in } (t_1, N);Q' \,\}^{\#} \cup^{\#} \{\, Q'\tau' \,\}^{\#}$, $\sigma_{n'+1} = \sigma_{n'}$ and $\mathcal{L}_{n'+1} = \mathcal{L}_{n'}$.

We define f as follows:

$$f(i) := \begin{cases} f_p(i) & \text{if } i \in \mathbb{N}_m \\ n' & \text{if } m < i < n-1 \\ n'+1 & \text{if } i = n \end{cases}$$

Therefore, Conditions 1 to 8 hold for $i < n-1$. It is left to show that Conditions 1 to 8 hold for n.

Condition 7 holds since $\mathit{hide}([E_1, \dots, E_m]) = [F_1, \dots, n']$, and $[E_{m+1}, \dots, E_{n-1}] = [F_{n'+1}]$, since $E_{n-1} = K(t_1)$.

Let θ' such that $ri = \theta' R$. As established before, we have τ' such that $N\tau' =_E t_2$. By definition of $[\![P]\!]_{=p}$, we have that $\mathsf{state}_{p\cdot 1}(\tilde{t}, \tilde{t}') \in_E \mathit{ginsts}(P_{=p\cdot 1})$, and that $\theta' = \theta \cdot \tau'$. Since τ and ρ are induced by θ, θ' induces $\tau \cdot \tau'$ and the same ρ. We have that $Q'\tau' = (P|_{p\cdot 1}\tau\rho)\tau' = P|_p\tau\tau'\rho$ and therefore $Q'\tau \leftrightsquigarrow_P \mathsf{state}_{p\cdot 1}(\tilde{t}, \tilde{t}')$. Thus, and since in $(t_1, N);Q' \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{\, \text{in } (t_1, N);Q' \,\}^{\#} \cup^{\#} \{\, Q'\tau' \,\}^{\#}$ and $S_n = S_{n-1} \setminus^{\#} \{\, \mathsf{In}(< t_1, t_2 >), \mathsf{state}_p(\tilde{t}) \,\}^{\#} \cup^{\#} \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}, \tilde{t}') \,\}^{\#}$, Condition 4 holds.

Conditions Condition 1, 2, 3 and 5 hold trivially.

Case: $ri = [\mathsf{state}^{\mathsf{semi}}_p(\tilde{s}), \mathsf{Ack}(t_1, t_2)] \;-\![]\!\rightarrow\; [\mathsf{state}_{p\cdot 1}(\tilde{s})]$ **(for some** $p, \tilde{t}$ **and** $t_1, t_2 \in \mathcal{M}$**).** Since the msr execution is normal, we have that there p,q,$\tilde{x}, \tilde{y}, \tilde{y}'$ such that:

$$S_{n-3} \rightarrow_{R_1} S_{n-2} \rightarrow_{R_2} S_{n-1} \rightarrow_{R_3} S_n \quad \text{, where:}$$

- $R_1 = [\mathsf{state}_p(\tilde{x})] \to [\mathsf{Msg}(t_1, t_2), \mathsf{state}_p^{\mathsf{semi}}(\tilde{x})]$
- $R_2 = [\mathsf{state}_q(\tilde{y}), \mathsf{Msg}(t_1, t_2)] \to [\mathsf{state}_{q\cdot 1}(\tilde{y} \cup \tilde{y}'), \mathsf{Ack}(t_1, t_2)]$
- $R_3 = [\mathsf{state}_p^{\mathsf{semi}}(\tilde{x}), \mathsf{Ack}(t_1, t_2)] \to [\mathsf{state}_{p\cdot 1}(\tilde{x})]$

.

Since in this case, there is a fact with symbol Ack removed from S_{n-1} to S_n, we have to apply a different argument to apply the induction hypothesis.

Since $\emptyset \xrightarrow{E_1}_{[\![P]\!]} \cdots \xrightarrow{E_n}_{[\![P]\!]} S_n \in \mathit{exec}^{msr}([\![P]\!])$ is normal, $n \geq 2$, and $t_1, t_2 \in \mathcal{M}$, $\mathsf{Ack}(t_1, t_2) \in (S_{n-1} \setminus^{\#} S_n)$, there exists $m \leq n-3$ such that $S_m \to_R^* S_{n-3}$ for $R = \{\,\textsc{MDOut}, \textsc{MDPub}, \textsc{MDFresh}, \textsc{MDAppl}\,\} \cup \textsc{Fresh}$ and $\emptyset \xrightarrow{E_1}_{[\![P]\!]} \cdots \xrightarrow{E_m}_{[\![P]\!]} S_m \in \mathit{exec}^{msr}([\![P]\!])$ is normal. This allows us to apply the induction hypothesis on $\emptyset \xrightarrow{E_1}_{[\![P]\!]} \cdots \xrightarrow{E_m}_{[\![P]\!]} S_m \in \mathit{exec}^{msr}([\![P]\!])$. Hence there is a monotonically increasing function from $\mathbb{N}_m \to \mathbb{N}_{n'}$ and an execution such that Conditions 1 to 8 hold. Let f_p be this function and note that $n' = f_p(m)$.

In the following case distinction, we extend the previous execution by one step from $(\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\mathrm{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'})$ to $(\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\mathrm{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$, and prove that Conditions 1 to 7 hold for $n'+1$. By induction hypothesis, they hold for all $i \leq n'$. We define a function $f\colon \mathbb{N}_n \to \mathbb{N}_{n'+1}$ as follows:

$$f(i) := \begin{cases} f_p(i) & \text{if } i \in \mathbb{N}_m \\ n' & \text{if } m < i \leq n-3 \\ n'+1 & \text{if } i = n \end{cases}$$

Since, $S_m \to_R^* S_n$ for $R = \{\,\textsc{MDOut}, \textsc{MDPub}, \textsc{MDFresh}, \textsc{MDAppl}, \textsc{Fresh}\,\}$, only $S_n \setminus^{\#} S_m$ contains only Fr-facts and $\mathsf{!K}$-facts, and $S_m \setminus^{\#} S_n$ contains only Fr-facts and Out-facts. Therefore, Condition 3, 4 and 5 hold for all $i \leq n-3$. Since $E_{m+1}, \ldots, E_{n-1} = \emptyset$, Condition 1, 2, 6 and 7 hold for all $i \leq n-3$.

Fix a bijection such that $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$. We will abuse notation by writing $P \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$, if this bijection maps P to $\mathsf{state}_p(\tilde{t})$. Since $\{\,\textsc{MDOut}, \textsc{MDPub}, \textsc{MDFresh}, \textsc{MDAppl}\,\}$ and $\textsc{Fresh}$ do not add or remove state-facts, $\mathcal{P}_{n'} \leftrightsquigarrow_P S_{n-3}$. Let $P \in^{\#} \mathcal{P}_{n'}$ such that $P \leftrightsquigarrow_P \mathsf{state}_p(\tilde{s})$. Let $Q \in^{\#} \mathcal{P}_{n'}$ such that $Q \leftrightsquigarrow_P \mathsf{state}_q(\tilde{t})$.

Let θ' be a grounding substitution for $\mathsf{state}_q(\tilde{y}) \in \mathit{prems}([\![P]\!]_{=q})$ such that $\tilde{t} =_E \tilde{y}\theta'$. Then θ' induces a substitution τ' and a bijective renaming ρ' for fresh, but not bound names (in Q) such that $P|_q\tau'\rho' = Q$ (see Definition 22).

From the form of the rules R_1 and R_3, and since $P =_E P|_p\tau\rho$, for τ and ρ induced by the grounding substitution for $\mathsf{state}_p(\tilde{x})$, we can deduce that $P =_E \mathrm{out}\ t_1, t_2;\ P'$ for a process $P' = P|_{p\cdot 1}\tau\rho$. Similarly, from the form of R_2, we can deduce $Q =_E \mathrm{in}\ (t_1, N); Q'$, for N a term that is not necessarily ground, and a process $Q' = P|_{q\cdot 1}\tau'\rho'$. Since $S_{n-2} \to_{R_2} S_{n-1}$, we have that there is a substitution τ^* such that $N\tau'\rho'\tau^* =_E t_2$ and $((\tilde{y} \cup \mathit{vars}(N)) \setminus \tilde{y})\tau^* =_E \tilde{t}'$, where $\tilde{t}'$ such that $\mathsf{state}_{q\cdot 1}(\tilde{t}, \tilde{t}') \in S_{n-1} \setminus^{\#} S_{n-2}$.

Given that $Q =_E \mathrm{in}\ (t_1, N); Q'$ and $P =_E \mathrm{out}\ t_1, t_2;\ P'$, have that $\mathcal{P}_{n'} = \mathcal{P}' \cup^{\#} \{\mathrm{out}\ t_1, t_2; P', \mathrm{in}\ (t_1', N); Q'\,\}^{\#}$ with $t_1 =_E t_1'$ and $t_2 =_E N\tau^*$. Therefore, we chose the following transition:

$$\cdots \xrightarrow{F_n'} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\mathrm{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \xrightarrow{K(t_1)} (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\mathrm{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $\mathcal{E}_{n'+1} = \mathcal{E}_{n'}$, $\mathcal{S}_{n'+1} = \mathcal{S}_{n'}$, $\mathcal{S}^{\mathrm{MS}}_{n'+1} = \mathcal{S}^{\mathrm{MS}}_{n'}$, $\mathcal{P}_{n'+1} = \mathcal{P}' \cup^{\#} \{\,P', Q'\,\}^{\#}$, $\sigma_{n'+1} = \sigma_{n'}$ and $\mathcal{L}_{n'+1} = \mathcal{L}_{n'}$.

Conditions 1 to 8 hold for $i \leq n-3$. It is left to show that Conditions 1 to 8 hold for n.

As established before, we have τ^* such that $N\tau'\rho'\tau^* =_E t_2$. Let $\mathsf{state}_q(\tilde{t}, \tilde{t}')$ be the state variable added to S_{n-1}. Then, $((\tilde{y} \cup \mathit{vars}(N)) \setminus \tilde{y})\tau^* = \tilde{t}'$. By definition of $[\![P]\!]_{=q}$, we have that $\mathsf{state}_{q\cdot 1}(\tilde{t}, \tilde{t}') \in \mathit{prems}(\mathit{ginsts}(P_{=p\cdot 1}))$ for a grounding substitution $\theta_{q\cdot 1} = \theta' \cdot \tau^*$. Since τ' and ρ' are induced by θ', $\theta_{q\cdot 1}$ induces $\tau \cdot \tau'$ and the same ρ. We have that $Q'\tau' = (P|_{q\cdot 1}\tau'\rho')\tau* = P|_{q\cdot 1}\tau\tau'\rho$ and therefore $Q'\tau^* \leftrightsquigarrow_P \mathsf{state}_{q\cdot 1}(\tilde{t}, \tilde{t}')$. Similarly, we have $P' \leftrightsquigarrow_P \mathsf{state}_{q\cdot 1}(\tilde{s})$. We conclude that Condition 4 holds.

Conditions Condition 1, 2, 3, 5, 6 and 7 hold trivially.

Case: $ri = [\mathsf{state}_p(\tilde{t})]$ $-[\mathrm{Eq}(t_1,t_2)]\!\!\rightarrow$ $[\mathsf{state}_{p\cdot 1}(\tilde{t})]$ **(for some** $p, \tilde{t}$ **and** $t_1, t_2 \in \mathcal{M}$**).** By induction hypothesis, we have $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$, and thus, as previously established, $\mathcal{P}_{n'} \leftrightsquigarrow_P S_{n-1}$. Let $Q \in^{\#} \mathcal{P}_{n'}$ such that $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$. Let θ be a grounding substitution for $\mathsf{state}_p(\tilde{x}) \in \mathit{prems}([\![P]\!]_{=p})$ such that $\tilde{t} = \tilde{x}\theta$. Then θ induces a substitution τ and a bijective renaming ρ for fresh, but not bound names (in Q) such that $P|_p\tau\rho = Q$ (see Definition 22).

From the form of the rule R, and since $Q = P|_p\tau\rho$, we can deduce that $Q = \mathtt{if}\ t_1 = t_2\ \mathtt{then}\ Q_1\ \mathtt{else}\ Q_2$ for a process $Q' = P|_{p\cdot 1}\tau\rho$.

Since, $[E_1, \ldots, E_n \vDash \alpha$, and thus $[E_1, \ldots, E_m \vDash \alpha_{eq}$, we have that $t_1 =_E t_2$. We therefore chose the following transition:

$$\cdots \xrightarrow{F'_n} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\mathrm{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \rightarrow (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\mathrm{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $\mathcal{E}_{n'+1} = \mathcal{E}_{n'}$, $\mathcal{S}_{n'+1} = \mathcal{S}_{n'}$, $\mathcal{S}^{\mathrm{MS}}_{n'+1} = \mathcal{S}^{\mathrm{MS}}_{n'}$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{\, \mathtt{if}\ t_1 = t_2 \mathtt{then}\ Q_1\ \mathtt{else}\ Q_2 \,\}^{\#} \cup^{\#} \{\, Q_1 \,\}^{\#}$, $\sigma_{n'+1} = \sigma_{n'}$ and $\mathcal{L}_{n'+1} = \mathcal{L}_{n'}$.

We define f as on page 51. Therefore, Conditions 1 to 8 hold for $i < n-1$. It is left to show that Conditions 1 to 8 hold for n.

By definition of $[\![P]\!]$ and $[\![P]\!]_{=p}$, we have that $Q_1 \leftrightarrow \mathsf{state}_{p\cdot 1}(\tilde{t})$. Therefore, and since $\mathtt{if}\ t_1 = t_2\ \mathtt{then}\ Q_1\ \mathtt{else}\ Q_2 \leftrightarrow \mathsf{state}_p(\tilde{t})$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{\, \mathtt{if}\ t_1 = t_2\ \mathtt{then}\ Q_1\ \mathtt{else}\ Q_2 \,\}^{\#} \cup^{\#} \{\, Q_1 \,\}^{\#}$, and $S_n = S_{n-1} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t}) \,\}^{\#} \cup^{\#} \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}) \,\}^{\#}$, Condition 4 holds. Conditions 1, 2, 3, 5, 6 and 7 hold trivially.

Case: $ri = [\mathsf{state}_p(\tilde{t})]$ $-[\mathrm{NotEq}(t_1,t_2)]\!\!\rightarrow$ $[\mathsf{state}_{p\cdot 1}(\tilde{t})]$ **(for some** $p, \tilde{t}$ **and** $t_1, t_2 \in \mathcal{M}$**).** In this case, the proof is almost the same as in the previous case, except that α_{noteq} is the relevant axiom, Q_2 is chosen instead of Q_1 and and $S_n = S_{n-1} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t}) \,\}^{\#} \cup^{\#} \{\, \mathsf{state}_{p\cdot 2}(\tilde{t}) \,\}^{\#}$.

Case: $ri = [\mathsf{state}_p(\tilde{t})]$ $-[\mathrm{F}, \mathrm{Event}()]\!\!\rightarrow$ $[\mathsf{state}_{p\cdot 1}(\tilde{t})]$ **(for some** $p, \tilde{t}$**).** This is a special case of the case where $ri = [\mathsf{state}_p(\tilde{t}), l]$ $-[a]\!\!\rightarrow$ $[\mathsf{state}_{p\cdot 1}(\tilde{t}), r]$ for $l = r = \emptyset$ and $a = F$.

Case: $ri = [\mathsf{state}_p(\tilde{t})]$ $-[\mathit{Insert}(t_1,t_2)]\!\!\rightarrow$ $[\mathsf{state}_{p\cdot 1}(\tilde{t})]$ **(for some** $p, \tilde{t}$ **and** $t_1, t_2 \in \mathcal{M}$**).** By induction hypothesis, we have $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$, and thus, as previously established, $\mathcal{P}_{n'} \leftrightsquigarrow_P S_{n-1}$. Let $Q \in^{\#} \mathcal{P}_{n'}$ such that $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$. Let θ be a grounding substitution for $\mathsf{state}_p(\tilde{x}) \in \mathit{prems}([\![P]\!]_{=p})$ such that $\tilde{t} = \tilde{x}\theta$. Then θ induces a substitution τ and a bijective renaming ρ for fresh, but not bound names (in Q) such that $P|_p\tau\rho = Q$ (see Definition 22).

From the form of the rule R, and since $Q = P|_p\tau\rho$, we can deduce that $Q = \mathtt{insert}\ t_1, t_2; Q'$ for a process $Q' = P|_{p\cdot 1}\tau\rho$.

We therefore chose the following transition:

$$\cdots \xrightarrow{F'_n} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\mathrm{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \rightarrow (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\mathrm{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $\mathcal{E}_{n'+1} = \mathcal{E}_{n'}$, $\mathcal{S}_{n'+1} = \mathcal{S}_{n'}[t_1 \mapsto t_2]$, $\mathcal{S}^{\mathrm{MS}}_{n'+1} = \mathcal{S}^{\mathrm{MS}}_{n'}$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{\, \mathtt{insert}\ t_1, t_2; Q' \,\}^{\#} \cup^{\#} \{\, Q' \,\}^{\#}$, $\sigma_{n'+1} = \sigma_{n'}$ and $\mathcal{L}_{n'+1} = \mathcal{L}_{n'}$.

We define f as on page 51. Therefore, Conditions 1 to 8 hold for $i < n-1$. It is left to show that Conditions 1 to 8 hold for n.

By definition of $[\![P]\!]$ and $[\![P]\!]_{=p}$, we have that $Q' \leftrightarrow \mathsf{state}_{p\cdot 1}(\tilde{t})$. Therefore, and since $\mathtt{insert}\ t_1, t_2; Q' \leftrightarrow \mathsf{state}_p(\tilde{t})$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{\, \mathtt{insert}\ t_1, t_2; Q' \,\}^{\#} \cup^{\#} \{\, Q' \,\}^{\#}$, and $S_n = S_{n-1} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t}) \,\}^{\#} \cup^{\#} \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}) \,\}^{\#}$, Condition 4 holds.

Condition 2 holds, since $E_n = \mathit{Insert}(t_1, t_2)$ is the last element of the trace.

Conditions 1, 3, 5, 6 and 7 hold trivially.

Case: $ri = [\mathsf{state}_p(\tilde{t})]$ $-[\mathit{Delete}(t_1,t_2)]\!\!\rightarrow$ $[\mathsf{state}_{p\cdot 1}(\tilde{t})]$ **(for some** $p, \tilde{t}$ **and** $t_1, t_2 \in \mathcal{M}$**).** By induction hypothesis, we have $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$, and thus, as previously established, $\mathcal{P}_{n'} \leftrightsquigarrow_P S_{n-1}$. Let $Q \in^{\#} \mathcal{P}_{n'}$ such that $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$. Let θ be a grounding substitution for $\mathsf{state}_p(\tilde{x}) \in \mathit{prems}([\![P]\!]_{=p})$ such that $\tilde{t} = \tilde{x}\theta$. Then θ induces a substitution τ and a bijective renaming ρ for fresh, but not bound names (in Q) such that $P|_p\tau\rho = Q$ (see Definition 22).

From the form of the rule R, and since $Q = P|_p\tau\rho$, we can deduce that $Q = \texttt{delete}\ t_1; Q'$ for a process $Q' = P|_{p\cdot 1}\tau\rho$.

We therefore chose the following transition:

$$\dots \xrightarrow{F'_n} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\text{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \rightarrow (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\text{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $\mathcal{E}_{n'+1} = \mathcal{E}_{n'}$, $\mathcal{S}_{n'+1} = \mathcal{S}_{n'}[t_1 \mapsto t_2]$, $\mathcal{S}^{\text{MS}}_{n'+1} = \mathcal{S}^{\text{MS}}_{n'}$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^\# \{\, \texttt{delete}\ t_1; Q' \,\}^\# \cup^\# \{\, Q' \,\}^\#$, $\sigma_{n'+1} = \sigma_{n'}$ and $\mathcal{L}_{n'+1} = \mathcal{L}_{n'}$.

We define f as on page 51. Therefore, Conditions 1 to 8 hold for $i < n - 1$. It is left to show that Conditions 1 to 8 hold for n.

By definition of $[\![P]\!]$ and $[\![P]\!]_{=p}$, we have that $Q' \leftrightarrow \mathsf{state}_{p\cdot 1}(\tilde{t})$. Therefore, and since $\texttt{delete}\ t_1; Q' \leftrightarrow \mathsf{state}_p(\tilde{t})$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^\# \{\, \texttt{delete}\ t_1; Q' \,\}^\# \cup^\# \{\, Q' \,\}^\#$, and $S_n = S_{n-1} \setminus^\# \{\, \mathsf{state}_p(\tilde{t}) \,\}^\# \cup^\# \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}) \,\}^\#$, Condition 4 holds.

Condition 2 holds, since $E_n = \mathit{Delete}(t_1, t_2)$ is the last element of the trace.

Conditions 1, 3, 5, 6 and 7 hold trivially.

Case: $ri = [\mathsf{state}_p(\tilde{t})] \, -[\mathit{IsIn}(t_1, t_2)]\!\rightarrow \, [\mathsf{state}_{p\cdot 1}(\tilde{t}, t_2)]$ (for some $p, \tilde{t}$ and $t_1, t_2 \in \mathcal{M}$). By induction hypothesis, we have $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$, and thus, as previously established, $\mathcal{P}_{n'} \leftrightsquigarrow_P S_{n-1}$. Let $Q \in^\# \mathcal{P}_{n'}$ such that $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$. Let θ be a grounding substitution for $\mathsf{state}_p(\tilde{x}) \in \mathit{prems}([\![P]\!]_{=p})$ such that $\tilde{t} = \tilde{x}\theta$. Then θ induces a substitution τ and a bijective renaming ρ for fresh, but not bound names (in Q) such that $P|_p\tau\rho = Q$ (see Definition 22).

From the form of the rule R, and since $Q = P|_p\tau\rho$, we can deduce that $Q = \texttt{lookup}\ t_1\ \texttt{as}\ v\ \texttt{in}\ Q_1\ \texttt{else}\ Q_2$ for some variable V, and two processes $Q_1 = P|_{p\cdot 1}\tau\rho$ and $Q_2 = P|_{p\cdot 2}\tau\rho$.

Since $[E_1, \dots, E_n] \vDash \alpha_{in}$, there is an $i < n$ such that $\mathit{Insert}(t_1, t_2) \in_E E_i$ and there is no j such that $i < j < n$ and $\mathit{Delete}(t_1) \in_E E_j$ or and $\mathit{Insert}(t_1, t_2) \in_E \mathit{TE}_j$. Since $E_m, \dots, E_n = \emptyset$, we know that $i < m$. Hence, by induction hypothesis, $\mathcal{S}_{n'}(t_1) = t_2$. We therefore chose the following transition:

$$\dots \xrightarrow{F'_n} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\text{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \rightarrow (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\text{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $\mathcal{E}_{n'+1} = \mathcal{E}_{n'}$, $\mathcal{S}_{n'+1} = \mathcal{S}_{n'}$, $\mathcal{S}^{\text{MS}}_{n'+1} = \mathcal{S}^{\text{MS}}_{n'}$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^\# \{\, \texttt{lookup}\ t_1\ \texttt{as}\ v\ \texttt{in}\ Q_1\ \texttt{else}\ Q_2 \,\}^\# \cup^\# \{\, Q_1\{^{t_2}/_{v}\} \,\}^\#$, $\sigma_{n'+1} = \sigma_{n'}$ and $\mathcal{L}_{n'+1} = \mathcal{L}_{n'}$.

We define f as on page 51. Therefore, Conditions 1 to 8 hold for $i < n - 1$. It is left to show that Conditions 1 to 8 hold for n.

By definition of $[\![P]\!]$ and $[\![P]\!]_{=p}$, we have that $Q_1\{^{v}/_{t_2}\} \leftrightarrow \mathsf{state}_{p\cdot 1}(\tilde{t}, t_2)$ (for $\tau' = \tau[v \mapsto t_2]$ and $\rho' = \rho$). Therefore, and since $\texttt{lookup}\ t_1\ \texttt{as}\ v\ \texttt{in}\ Q_1\ \texttt{else}\ Q_2 \leftrightarrow \mathsf{state}_p(\tilde{t})$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^\# \{\, \texttt{lookup}\ t_1\ \texttt{as}\ v\ \texttt{in}\ Q_1\ \texttt{else}\ Q_2 \,\}^\# \cup^\# \{\, Q' \,\}^\#$, and $S_n = \S_{n-1} \setminus^\# \{\, \mathsf{state}_p(\tilde{t}) \,\}^\# \cup^\# \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}, t_2) \,\}^\#$, Condition 4 holds.

Conditions 1, 2, 3, 5, 6 and 7 hold trivially.

Case: $ri = [\mathsf{state}_p(\tilde{t})] \, -[\mathit{IsNotSet}(t_1)]\!\rightarrow \, [\mathsf{state}_{p\cdot 2}(\tilde{t})]$ (for some $p, \tilde{t}$ and $t_1 \in \mathcal{M}$). By induction hypothesis, we have $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$, and thus, as previously established, $\mathcal{P}_{n'} \leftrightsquigarrow_P S_{n-1}$. Let $Q \in^\# \mathcal{P}_{n'}$ such that $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$. Let θ be a grounding substitution for $\mathsf{state}_p(\tilde{x}) \in \mathit{prems}([\![P]\!]_{=p})$ such that $\tilde{t} = \tilde{x}\theta$. Then θ induces a substitution τ and a bijective renaming ρ for fresh, but not bound names (in Q) such that $P|_p\tau\rho = Q$ (see Definition 22).

From the form of the rule R, and since $Q = P|_p\tau\rho$, we can deduce that $Q = \texttt{lookup}\ t_1\ \texttt{as}\ v\ \texttt{in}\ Q_1\ \texttt{else}\ Q_2$ for a variable v and two processes $Q_1 = P|_{p\cdot 1}\tau\rho$ and $Q_2 = P|_{p\cdot 2}\tau\rho$.

Since $[E_1, \dots, E_n] \vDash \alpha_{notin}$, there is no $i < n$ such that $\mathit{Insert}(t_1, t_2) \in_E E_i$ and there is no j such that $i < j < n$ and $\mathit{Delete}(t_1) \in_E E_j$ or and $\mathit{Insert}(t_1, t_2) \in_E \mathit{TE}_j$. Since $E_m, \dots, E_n = \emptyset$, we know that holds $j < m$. Hence, by induction hypothesis, $\mathcal{S}_{n'}(t_1)$ is undefined. We therefore chose the following transition:

$$\dots \xrightarrow{F'_n} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\text{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \rightarrow (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\text{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $\mathcal{E}_{n'+1} = \mathcal{E}_{n'}$, $\mathcal{S}_{n'+1} = \mathcal{S}_{n'}$, $\mathcal{S}^{\text{MS}}_{n'+1} = \mathcal{S}^{\text{MS}}_{n'}$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^\# \{\, \texttt{lookup}\ t_1\ \texttt{as}\ v\ \texttt{in}\ Q_1\ \texttt{else}\ Q_2 \,\}^\# \cup^\# \{\, Q_2 \,\}^\#$, $\sigma_{n'+1} = \sigma_{n'}$ and $\mathcal{L}_{n'+1} = \mathcal{L}_{n'}$.

We define f as on page 51. Therefore, Conditions 1 to 8 hold for $i < n-1$. It is left to show that Conditions 1 to 8 hold for n.

By definition of $\llbracket P \rrbracket$ and $\llbracket P \rrbracket_{=p}$, we have that $Q_2 \leftrightarrow \mathsf{state}_{p\cdot 2}(\tilde{t})$. Therefore, and since $\texttt{lookup}$ t_1 $\texttt{as}$ v $\texttt{in}$ Q_1 $\texttt{else}$ $Q_2 \leftrightarrow \mathsf{state}_p(\tilde{t})$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{\, \texttt{lookup}\ t_1\ \texttt{as}\ v\ \texttt{in}\ Q_1\ \texttt{else}\ Q_2 \,\}^{\#} \cup^{\#} \{\, Q_2 \,\}^{\#}$, and $S_n = \S_{n-1} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t}) \,\}^{\#} \cup^{\#} \{\, \mathsf{state}_{p\cdot 2}(\tilde{t}) \,\}^{\#}$, Condition 4 holds.

Conditions 1, 2, 3, 5, 6 and 7 hold trivially.

Case: $ri = [\mathsf{state}_p(\tilde{t}), \mathsf{Fr}(lock_l)] \ {-}[Lock(lock_l, t)]{\rightarrow}\ [\mathsf{state}_{p\cdot 1}(\tilde{t}, lock_l)]$ **(for some** $p, \tilde{t}$**,** $lock_l \in FN$ **and** $t \in \mathcal{M}$**).** By induction hypothesis, we have $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$, and thus, as previously established, $\mathcal{P}_{n'} \leftrightsquigarrow_P S_{n-1}$. Let $Q \in^{\#} \mathcal{P}_{n'}$ such that $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$. Let θ be a grounding substitution for $\mathsf{state}_p(\tilde{x}) \in prems(\llbracket P \rrbracket_{=p})$ such that $\tilde{t} = \tilde{x}\theta$. Then θ induces a substitution τ and a bijective renaming ρ for fresh, but not bound names (in Q) such that $P|_p\tau\rho = Q$ (see Definition 22).

From the form of the rule R, and since $Q = P|_p\tau\rho$, we can deduce that $Q = \texttt{lock}^l\ t;\ Q'$ for $Q' = P|_{p\cdot 1}\tau\rho$.

Since $[E_1, \ldots, E_n] \vDash \alpha_{lock}$, for every $i < n$ such that $Lock(l_p, t) \in_E E_i$, there a j such that $i < j < n$ and $Unlock(l_p, t) \in_E E_j$, and in between i and j, there is no lock or unlock, i. e., for all k such that $i < k < j$, and all l_i, $Lock(l_i, t) \notin_E E_k$ and $Unlock(l_i, t) \notin_E E_k$.

Since $E_m, \ldots, E_n = \emptyset$, we know that this holds for $i < m$ and $j < m$ as well. By induction hypothesis, Condition 6, this implies that $t \notin_E \mathcal{L}_{n'}$. We therefore chose the following transition:

$$\cdots \xrightarrow{F'_n} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\mathrm{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \rightarrow (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\mathrm{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $\mathcal{E}_{n'+1} = \mathcal{E}_{n'}$, $\mathcal{S}_{n'+1} = \mathcal{S}_{n'}$, $\mathcal{S}^{\mathrm{MS}}_{n'+1} = \mathcal{S}^{\mathrm{MS}}_{n'}$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{\, \texttt{lock}^l\ t;\ Q' \,\}^{\#} \cup^{\#} \{\, Q' \,\}^{\#}$, $\sigma_{n'+1} = \sigma_{n'}$ and $\mathcal{L}_{n'+1} = \mathcal{L}_{n'} \cup \{\, t \,\}$.

We define f as on page 51. Therefore, Conditions 1 to 8 hold for $i < n-1$. It is left to show that Conditions 1 to 8 hold for n.

By definition of $\llbracket P \rrbracket$ and $\llbracket P \rrbracket_{=p}$, we have that $Q' \leftrightarrow \mathsf{state}_{p\cdot 1}(\tilde{t})$. Therefore, and since $\texttt{lock}^l\ t;\ Q' \leftrightarrow \mathsf{state}_p(\tilde{t})$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{\, \texttt{lock}^l\ t;\ Q' \,\}^{\#} \cup^{\#} \{\, Q' \,\}^{\#}$, and $S_n = \S_{n-1} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t}), \mathsf{Fr}(lock_l) \,\}^{\#} \cup^{\#} \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}, lock_l) \,\}^{\#}$, Condition 4 holds.

Condition 6 holds since $E_n = \{\, \mathrm{Lock}(lock_l, t) \,\}^{\#}$ is added to the end of the trace.

Conditions 1, 2, 3, 5 and 7 hold trivially.

Case: $ri = [\mathsf{state}_p(\tilde{t})] \ {-}[Unlock(n_l, t)]{\rightarrow}\ [\mathsf{state}_{p\cdot 1}(\tilde{t})]$ **(for some** $p, \tilde{t}$**,** $n_l \in FN$ **and** $t \in \mathcal{M}$**).** By induction hypothesis, we have $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$, and thus, as previously established, $\mathcal{P}_{n'} \leftrightsquigarrow_P S_{n-1}$. Let $Q \in^{\#} \mathcal{P}_{n'}$ such that $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$. Let θ be a grounding substitution for $\mathsf{state}_p(\tilde{x}) \in prems(\llbracket P \rrbracket_{=p})$ such that $\tilde{t} = \tilde{x}\theta$. Then θ induces a substitution τ and a bijective renaming ρ for fresh, but not bound names (in Q) such that $P|_p\tau\rho = Q$ (see Definition 22).

From the form of the rule R, and since $Q = P|_p\tau\rho$, we can deduce that $Q = \texttt{unlock}^l\ t;\ Q'$ for $Q' = P|_{p\cdot 1}\tau\rho$.

We therefore chose the following transition:

$$\cdots \xrightarrow{F'_n} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\mathrm{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \rightarrow (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\mathrm{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $\mathcal{E}_{n'+1} = \mathcal{E}_{n'}$, $\mathcal{S}_{n'+1} = \mathcal{S}_{n'}$, $\mathcal{S}^{\mathrm{MS}}_{n'+1} = \mathcal{S}^{\mathrm{MS}}_{n'}$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{\, \texttt{unlock}^l\ t;\ Q' \,\}^{\#} \cup^{\#} \{\, Q' \,\}^{\#}$, $\sigma_{n'+1} = \sigma_{n'}$ and $\mathcal{L}_{n'+1} = \mathcal{L}_{n'} \setminus \{\, t \,\}$.

We define f as on page 51. Therefore, Conditions 1 to 8 hold for $i < n-1$. It is left to show that Conditions 1 to 8 hold for n.

By definition of $\llbracket P \rrbracket$ and $\llbracket P \rrbracket_{=p}$, we have that $Q' \leftrightarrow \mathsf{state}_{p\cdot 1}(\tilde{t})$. Therefore, and since $\texttt{unlock}^l\ t;\ Q' \leftrightarrow \mathsf{state}_p(\tilde{t})$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{\, \texttt{unlock}^l\ t;\ Q' \,\}^{\#} \cup^{\#} \{\, Q' \,\}^{\#}$, and $S_n = \S_{n-1} \setminus^{\#} \{\, \mathsf{state}_p(\tilde{t}) \,\}^{\#} \cup^{\#} \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}) \,\}^{\#}$, Condition 4 holds.

We show that Condition 6 holds for $\mathcal{L}_{n'+1} = \mathcal{L}_{n'} \setminus \{\, t \,\}$: For all $t' \neq_E t$, $t' \in_E \mathcal{L}_{n'} \Leftrightarrow t' \in_E \mathcal{L}_{n'+1}$ by induction hypothesis. If $t \notin_E \mathcal{L}_{n'}$, then $\forall j \leq m, u.\mathrm{Lock}(u, t) \in_E E_j \rightarrow \exists j < k \leq n.\mathrm{Unlock}(u, t) \in_E E_k$. Since we have $E_m, \ldots, E_{n-1} = \emptyset$ and $E_n = \{\, \mathrm{Unlock}(n_l, t) \,\}^{\#}$, we can strengthen this to $\forall j \leq n, u.\mathrm{Lock}(u, t) \in_E E_j \rightarrow \exists j < k \leq n.\mathrm{Unlock}(u, t) \in_E E_k$, which means that the condition holds in this case. If $t \in_E \mathcal{L}_{n'}$, then $\exists j \leq n, u.\text{Lock}(u,t) \in_E E_j \wedge \forall j < k \leq n.\text{Unlock}(u,t) \notin_E E_k$ and since $E_m, \ldots, E_{n-1} = \emptyset$ and $E_n = \{\,\text{Unlock}(n_l,t)\,\}^{\#}$, a contradiction to Condition 6 would constitute of j and $u \neq_E n_l$ such that $\text{Lock}(u,t) \in_E E_j$ and $\forall j < k \leq n.\text{Unlock}(u,t) \notin_E E_k$.

We will show that this leads to a contradiction with $[E_1, \ldots, E_n] \vDash \alpha$. Fix j and u. By definition of $[\![P]\!]$ and well-formedness of P, there is a p_l that is a prefix of p such that $P|_{q_l} = \mathtt{lock}^l t; Q''$ for the same annotation l and parameter t. The form of the translation guarantees that if $\mathsf{state}_p(\tilde{t}) \in S_n$, then for some $\tilde{t}'$ there is $i \leq n$ such that $\mathsf{state}_{p'}(\tilde{t}') \in S_i$, if p' is a prefix of p. We therefore have that there is $i < n$ such that $E_i =_E \{\,\text{Lock}(n_l,t)\,\}^{\#}$. We proceed by case distinction:
<u>Case 1:</u> $j < i$ (see Figure 18). Since $\forall j < k \leq n.\text{Unlock}(u,t) \notin_E E_k$, $[E_1, \ldots, E_n] \not\vDash \alpha_{lock}$.

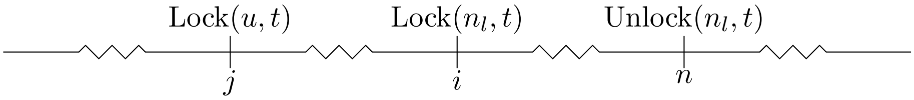

Figure 18: Visualisation of Case 1.

<u>Case 2:</u> $i < j$ (see Figure 19). By definition of $\overline{P}$, there is no parallel and no replication between p_l and p. Note that any rule in $[\![P]\!]$ that produces a state named state_q for a non-empty q is such that it requires a fact with name $\mathsf{state}_{q'}$ for $q = q' \cdot 1$ or $q = q' \cdot 2$ (in case of the translation of out, it might require $\mathsf{state}^{\mathsf{semi}}_{q'}$, which in turn requires $\mathsf{state}_{q'}$). Therefore, there cannot be a second $k \neq n$ such that $\text{Unlock}(n_l,t) \in_E E_k$ (since n_l was added in a Fr-fact in to S_i). This means in particular that there is not k such that $i < k < n$ and $\text{Unlock}(n_l,t) \in_E E_k$. Therefore, $[E_1, \ldots, E_n] \not\vDash \alpha_{lock}$.

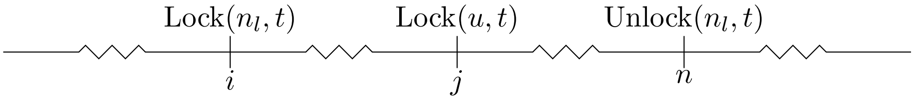

Figure 19: Visualisation of Case 2.

Conditions 1, 2, 3, 5 and 7 hold trivially.

Case: $ri = [\mathsf{state}_p(\tilde{t}), l'] -[a', \text{Event}()]\rightarrow [\mathsf{state}_{p\cdot 1}(\tilde{t}, \tilde{t}'), r']$ **(for some** $p, \tilde{t}, \tilde{t}'$ **and** $l', r', a' \in \mathcal{G}^*$**).** By induction hypothesis, we have $\mathcal{P}_{n'} \leftrightsquigarrow_P S_m$, and thus, as previously established, $\mathcal{P}_{n'} \leftrightsquigarrow_P S_{n-1}$. Let $Q \in^{\#} \mathcal{P}_{n'}$ such that $Q \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$. Let θ be a grounding substitution for $\mathsf{state}_p(\tilde{x}) \in \mathit{prems}([\![P]\!]_{=p})$ such that $\tilde{t} = \tilde{x}\theta$. Then θ induces a substitution τ and a bijective renaming ρ for fresh, but not bound names (in Q) such that $P|_p\tau\rho = Q$ (see Definition 22).

From the form of the rule R, and since $Q = P|_p\tau\rho$, we can deduce that $Q = l -[a]\rightarrow r; Q'$ for a process $Q' = P|_{p\cdot 1}\tau\rho$.

Since $ri \in_E \mathit{ginsts}(R)$, we have that there is a substitution τ^* such that $(l -[a]\rightarrow r)\tau^* = l' -[a]\rightarrow' r'$, $\mathit{lfacts}(l') \subset^{\#} S_{n-1}$, $\mathit{pfacts}(l') \subset_E S_{n-1}$ and, from the definition of $[\![P]\!]$ for embedded msr rules, $\mathit{vars}(l)\tau^* = \tilde{t}'$. Since P is well-formed, no fact in $\mathcal{F}_{res}$ appears in neither l nor r, so from Condition 3 in the induction hypothesis, we have that $\mathit{lfacts}(l') \subset^{\#} \mathcal{S}^{\text{MS}}_m = \mathcal{S}^{\text{MS}}_{n'}$ and $\mathit{pfacts}(l') \subset \mathcal{S}^{\text{MS}}_m = \mathcal{S}^{\text{MS}}_{n'}$. We therefore chose the following transition:

$$\cdots \xrightarrow{F'_n} (\mathcal{E}_{n'}, \mathcal{S}_{n'}, \mathcal{S}^{\text{MS}}_{n'}, \mathcal{P}_{n'}, \sigma_{n'}, \mathcal{L}_{n'}) \xrightarrow{a'} (\mathcal{E}_{n'+1}, \mathcal{S}_{n'+1}, \mathcal{S}^{\text{MS}}_{n'+1}, \mathcal{P}_{n'+1}, \sigma_{n'+1}, \mathcal{L}_{n'+1})$$

with $\mathcal{E}_{n'+1} = \mathcal{E}_{n'}$, $\mathcal{S}_{n'+1} = \mathcal{S}_{n'}$, $\mathcal{S}^{\text{MS}}_{n'+1} = \mathcal{S}^{\text{MS}}_{n'} \setminus \mathit{lfacts}(l') \cup^{\#} r'$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^{\#} \{\, l -[a]\rightarrow r; Q' \,\}^{\#} \cup^{\#} \{\, Q'\tau^* \,\}^{\#}$, $\sigma_{n'+1} = \sigma_{n'}$ and $\mathcal{L}_{n'+1} = \mathcal{L}_{n'}$.

We define f as on page 51. Therefore, Conditions 1 to 8 hold for $i < n - 1$. It is left to show that Conditions 1 to 8 hold for n.

Condition 7 holds since $\mathit{hide}([E_1, \ldots, E_m]) = [F_1, \ldots, n']$, and $E_{m+1}, \ldots, E_{n-1} = \emptyset$, while $E_n = F'_n \setminus \text{Event}() = a'$ (note that $\text{Event}() \in \mathcal{F}_{res}$).

As established before, we have τ^* such that $(l -[a]\rightarrow r)\tau^* =_E l -[a]\rightarrow r$. By the definition of $[\![P]\!]_{=p}$, we have that $\mathsf{state}_{p\cdot 1}(\tilde{t}, \tilde{t}') \in_E \mathit{ginsts}(P_{=p\cdot 1})$, and a $\theta' = \theta \cdot \tau^*$ that is grounding for $\mathsf{state}_{p\cdot 1}(\tilde{t}, \tilde{t}')$. Since τ and ρ are induced by θ, θ' induces $\tau \cdot \tau^*$ and the same ρ. We have that $Q'\tau^* = (P|_{p\cdot 1}\tau\rho)\tau^* = P|_p\tau\tau^*\rho$

and therefore $Q'\tau \leftrightsquigarrow_P \mathsf{state}_{p\cdot 1}(\tilde{t}, \tilde{t'})$. Thus, and since $l \ -[a]\!\rightarrow\ r; Q' \leftrightsquigarrow_P \mathsf{state}_p(\tilde{t})$, $\mathcal{P}_{n'+1} = \mathcal{P}_{n'} \setminus^\# \{\, l \ -[a]\!\rightarrow\ r; Q' \,\}^\# \cup^\# \{\, Q'\tau^* \,\}^\#$ and $S_n = S_{n-1} \setminus^\# \mathit{lfacts}(l') \cup^\# r' \setminus^\# \{\, \mathsf{state}_p(\tilde{t}) \,\}^\# \cup^\# \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}, \tilde{t'}) \,\}^\#$, Condition 4 holds.

Condition 3, holds since

$$\begin{aligned}
S_n \setminus^\# \mathcal{F}_{res} =& (S_{n-1} \setminus^\# \mathit{lfacts}(l') \cup^\# r' \setminus^\# \{\, \mathsf{state}_p(\tilde{t}) \,\}^\# \cup^\# \{\, \mathsf{state}_{p\cdot 1}(\tilde{t}, \tilde{t'}) \,\}^\#) \setminus^\# \mathcal{F}_{res}\\
=& (S_{n-1} \setminus^\# \mathit{lfacts}(l') \cup^\# r') \setminus^\# \mathcal{F}_{res}\\
=& S_{n-1} \setminus^\# \mathcal{F}_{res} \setminus^\# \mathit{lfacts}(l') \cup^\# r'\\
=& \mathcal{S}^{\mathrm{MS}}_{n'} \setminus^\# \mathit{lfacts}(l') \cup^\# r'\\
=& \mathcal{S}^{\mathrm{MS}}_{n'+1}
\end{aligned}$$

Conditions 1, 2, 5, 6 and 7 hold trivially.

□

Lemma 1. *Let P be a well-formed ground process. We have that*

$$\mathit{traces}^{pi}(P) = \mathit{hide}(\mathit{filter}(\mathit{traces}^{msr}([\![P]\!]))).$$

Proof. From Lemma 10, we can conclude that

$$\mathit{traces}^{pi}(P) \subseteq \{\, \mathit{hide}(t) | \mathit{tr} \in \mathit{traces}^{msr}([\![P]\!]) \text{ and } \mathit{tr} \vDash \alpha \,\} = \mathit{hide}(\mathit{filter}(\mathit{traces}^{msr}([\![P]\!]))).$$

From Lemma 11, we have that

$$\mathit{hide}(\mathit{filter}(\mathit{traces}^{msr}([\![P]\!]))) = \{\, \mathit{tr} \in \mathit{hide}(\mathit{filter}(\mathit{traces}^{msr}([\![P]\!]))) \mid \mathit{tr} \text{ is normal} \,\}.$$

From Lemma 12, we can conclude that

$$\{\, \mathit{tr} \in \mathit{hide}(\mathit{filter}(\mathit{traces}^{msr}([\![P]\!]))) \mid \mathit{tr} \text{ is normal} \,\} \subseteq \mathit{traces}^{pi}(P).$$

Hence

$$\mathit{hide}(\mathit{filter}(\mathit{traces}^{msr}([\![P]\!]))) \subseteq \mathit{traces}^{pi}(P).$$

□