65. Sailboat Racing. Referring to the figure, estimate to the nearest knot the speed of the sailboat sailing at the following angles to the wind: 30°, 75°, 135°, and 180°.

66. Sailboat Racing. Referring to the figure, estimate to the nearest knot the speed of the sailboat sailing at the following angles to the wind: 45°, 90°, 120°, and 150°.

67. Conic Sections. Using a graphing utility, graph the equation

$$r = \frac{8}{1 - e\cos\theta}$$

for the following values of e (called the **eccentricity** of the conic) and identify each curve as a hyperbola, an ellipse, or a parabola.

(A) $e = 0.4$ (B) $e = 1$ (C) $e = 1.6$

(It is instructive to explore the graph for other positive values of e.)

68. Conic Sections. Using a graphing utility, graph the equation

$$r = \frac{8}{1 - e\cos\theta}$$

for the following values of e and identify each curve as a hyperbola, an ellipse, or a parabola.

(A) $e = 0.6$ (B) $e = 1$ (C) $e = 2$

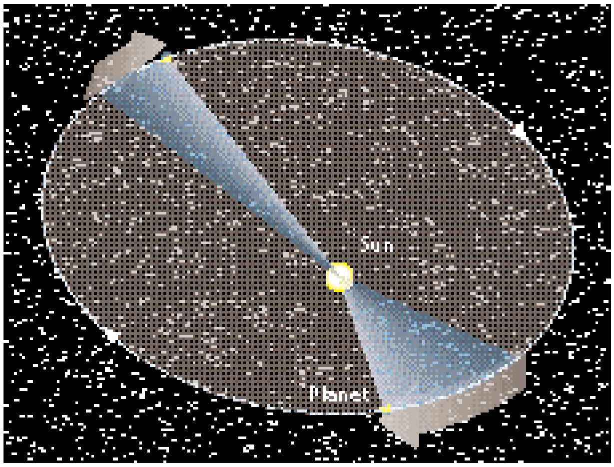
★ **69. Astronomy.**

(A) The planet Mercury travels around the sun in an elliptical orbit given approximately by

$$r = \frac{3.442 \times 10^7}{1 - 0.206\cos\theta}$$

where r is measured in miles and the sun is at the pole. Graph the orbit. Use TRACE to find the distance from Mercury to the sun at **aphelion** (greatest distance from the sun) and at **perihelion** (shortest distance from the sun).

(B) Johannes Kepler (1571–1630) showed that a line joining a planet to the sun sweeps out equal areas in space in equal intervals in time (see the figure). Use this information to determine whether a planet travels faster or slower at aphelion than at perihelion. Explain your answer.



Section 7-6 Complex Numbers in Rectangular and Polar Forms

- Rectangular Form
- Polar Form
- Multiplication and Division in Polar Form
- Historical Note

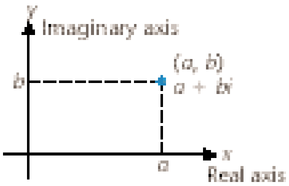
Utilizing polar concepts studied in the last two sections, we now show how complex numbers can be written in polar form, which can be very useful in many applications. A brief review of Section 2-4 on complex numbers should prove helpful before proceeding further.

Rectangular Form

Recall from Section 2-4 that a complex number is any number that can be written in the form

$$a + bi$$

FIGURE 1
Complex plane.



where a and b are real numbers and i is the imaginary unit. Thus, associated with each complex number $a + bi$ is a unique ordered pair of real numbers (a, b), and vice versa. For example,

$$3 - 5i \qquad \text{corresponds to} \qquad (3, -5)$$

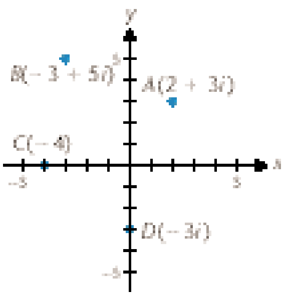
Associating these ordered pairs of real numbers with points in a rectangular coordinate system, we obtain a **complex plane** (see Fig. 1). When complex numbers are associated with points in a rectangular coordinate system, we refer to the x axis as the **real axis** and the y axis as the **imaginary axis.** The complex number $a + bi$ is said to be in **rectangular form.**

EXAMPLE 1 Plotting in the Complex Plane

Plot the following complex numbers in a complex plane:

$$A = 2 + 3i \qquad B = -3 + 5i \qquad C = -4 \qquad D = -3i$$

Solution



MATCHED PROBLEM 1

Plot the following complex numbers in a complex plane:

$$A = 4 + 2i \qquad B = 2 - 3i \qquad C = -5 \qquad D = 4i$$

Explore/Discuss 1

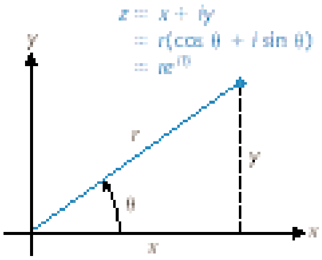
On a *real number line* there is a one-to-one correspondence between the set of real numbers and the set of points on the line: each real number is associated with exactly one point on the line and each point on the line is associated with exactly one real number. Does such a correspondence exist between the set of complex numbers and the set of points in an extended plane? Explain how a one-to-one correspondence can be established.

Polar Form

Complex numbers also can be written in **polar form.** Using the polar–rectangular relationships from Section 7-5,

$$x = r \cos \theta \qquad \text{and} \qquad y = r \sin \theta$$

FIGURE 2
Rectangular–polar relationship.



we can write the complex number $z = x + iy$ in polar form as follows:

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) \qquad (1)$$

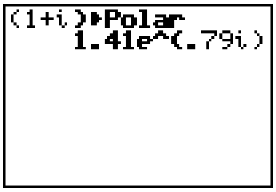
This rectangular–polar relationship is illustrated in Figure 2. In a more advanced treatment of the subject, the following famous equation is established:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad (2)$$

where $e^{i\theta}$ obeys all the basic laws of exponents. Thus, equation (1) takes on the form

$$z = x + yi = r(\cos\theta + i\sin\theta) = re^{i\theta} \qquad (3)$$

FIGURE 3
$(1 + i) = 1.41e^{0.79i}$.



We will freely use $re^{i\theta}$ as a polar form for a complex number. In fact, some graphing calculators display the polar form of $x + iy$ this way (see Fig. 3 where θ is in radians and numbers are displayed to two decimal places).

Since $\cos\theta$ and $\sin\theta$ are both periodic with period 2π, we have

$$\begin{aligned} \cos(\theta + 2k\pi) &= \cos\theta \\ \sin(\theta + 2k\pi) &= \sin\theta \end{aligned} \qquad k \text{ any integer}$$

Thus, we can write a more general polar form for a complex number $z = x + iy$, as given below, and observe that $re^{i\theta}$ is periodic with period $2k\pi$, k any integer.

DEFINITION 1

GENERAL POLAR FORM OF A COMPLEX NUMBER

For k any integer

$$z = x + iy = r[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]$$
$$z = re^{i(\theta + 2k\pi)}$$

The number r is called the **modulus,** or **absolute value,** of z and is denoted by **mod *z*** or $|z|$. The polar angle that the line joining z to the origin makes with the polar axis is called the **argument** of z and is denoted by **arg *z***. From Figure 2 we see the following relationships:

DEFINITION 2

MODULUS AND ARGUMENT FOR $z = x + iy$

$$\text{mod } z = r = \sqrt{x^2 + y^2} \qquad \text{Never negative}$$
$$\arg z = \theta + 2k\pi \qquad k \text{ any integer}$$

where $\sin\theta = y/r$ and $\cos\theta = x/r$. The argument θ is usually chosen so that $-180° < \theta \le 180°$ or $-\pi < \theta \le \pi$.

EXAMPLE 2

From Rectangular to Polar Form

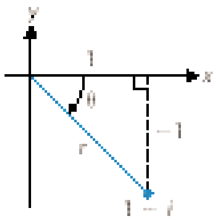
Write parts A–C in polar form, θ in radians, $-\pi < \theta \le \pi$. Compute the modulus and arguments for parts A and B exactly; compute the modulus and argument for part C to two decimal places.

(A) $z_1 = 1 - i$ (B) $z_2 = -\sqrt{3} + i$ (C) $z = -5 - 2i$

Solutions

Locate in a complex plane first; then if x and y are associated with special angles, r and θ can often be determined by inspection.

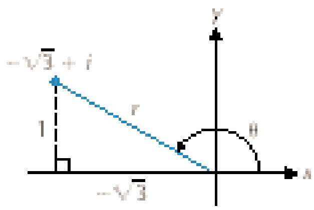
FIGURE 4



(A) A sketch shows that z_1 is associated with a special 45° triangle (Fig. 4). Thus, by inspection, $r = \sqrt{2}$, $\theta = -\pi/4$ (not $7\pi/4$), and

$$z_1 = \sqrt{2}[\cos(-\pi/4) + i\sin(-\pi/4)]$$
$$= \sqrt{2}\, e^{(-\pi/4)i}$$

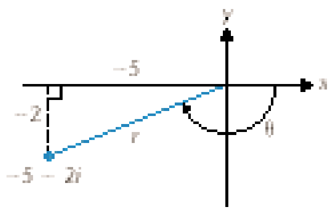
FIGURE 5



(B) A sketch shows that z_2 is associated with a special 30°–60° triangle (Fig. 5). Thus by inspection, $r = 2$, $\theta = 5\pi/6$, and

$$z_2 = 2(\cos 5\pi/6 + i \sin 5\pi/6)$$
$$= 2e^{(5\pi/6)i}$$

FIGURE 6



(C) A sketch shows that z_3 is not associated with a special triangle (Fig. 6). So, we proceed as follows:

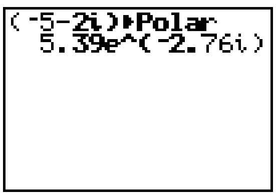
$$r = \sqrt{(-5)^2 + (-2)^2} = 5.39 \quad \text{To two decimal places}$$
$$\theta = -\pi + \tan^{-1}\left(\tfrac{2}{5}\right) = -2.76 \quad \text{To two decimal places}$$

Thus,

$$z_3 = 5.39[\cos(-2.76) + i\sin(-2.76)]$$
$$= 5.39e^{(-2.76)i} \quad \text{To two decimal places}$$

Figure 7 shows the same conversion done by a graphing calculator with a built-in conversion routine (with numbers displayed to two decimal places).

FIGURE 7
$(-5 - 2i) = 5.39e^{(-2.76)i}$.



MATCHED PROBLEM 2

Write parts A–C in polar form, θ in radians, $-\pi < \theta \le \pi$. Compute the modulus and arguments for parts A and B exactly; compute the modulus and argument for part C to two decimal places.

(A) $-1 + i$ (B) $1 + i\sqrt{3}$ (C) $-3 - 7i$

EXAMPLE 3

From Polar to Rectangular Form

Write parts A–C in rectangular form. Compute the exact values for parts A and B; for part C, compute a and b for $a + bi$ to two decimal places.

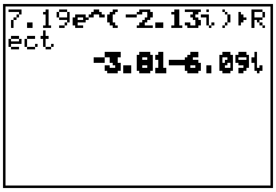
(A) $z_1 = 2e^{(5\pi/6)i}$ (B) $z_2 = 3e^{(-60°)i}$ (C) $z_3 = 7.19e^{(-2.13)i}$

Solutions

(A)
$$\begin{aligned} x + iy &= 2e^{(5\pi/6)i} \\ &= 2[\cos(5\pi/6) + i\sin(5\pi/6)] \\ &= 2\left(\frac{-\sqrt{3}}{2}\right) + i2\left(\frac{1}{2}\right) \\ &= -\sqrt{3} + i \end{aligned}$$

(B)
$$\begin{aligned} x + iy &= 3e^{(-60°)i} \\ &= 3[\cos(-60°) + i\sin(-60°)] \\ &= 3\left(\frac{1}{2}\right) + i3\left(\frac{-\sqrt{3}}{2}\right) \\ &= \frac{3}{2} - \frac{3\sqrt{3}}{2}i \end{aligned}$$

FIGURE 8
$7.19e^{(-2.13)i} = -3.81 - 6.09\,i$.



(C)
$$\begin{aligned} x + iy &= 7.19e^{(-2.13)i} \\ &= 7.19[\cos(-2.13) + i\sin(-2.13)] \\ &= -3.81 - 6.09\,i \end{aligned}$$

Figure 8 shows the same conversion done by a graphing calculator with a built-in conversion routine.

Explore/Discuss 2

If your calculator has a built-in polar-to-rectangular conversion routine, try it on $\sqrt{2}e^{45°i}$ and $\sqrt{2}e^{(\pi/4)i}$, then reverse the process to see if you get back where you started. (For complex numbers in exponential polar form, some calculators require θ to be in radian mode for calculations. Check your user's manual.)

MATCHED PROBLEM 3

Write parts A–C in rectangular form. Compute the exact values for parts A and B; for part C compute a and b for $a + bi$ to two decimal places.

(A) $z_1 = \sqrt{2}e^{(-\pi/2)i}$ (B) $z_2 = 3e^{120°i}$ (C) $z_3 = 6.49e^{(-2.08)i}$

Explore/Discuss 3

Let $z_1 = \sqrt{3} + i$ and $z_2 = 1 + i\sqrt{3}$.

(A) Find z_1z_2 and z_1/z_2 using the rectangular forms of z_1 and z_2.

(B) Find z_1z_2 and z_1/z_2 using the exponential polar forms of z_1 and z_2, θ in degrees. (Assume the product and quotient exponent laws hold for $e^{i\theta}$.)

(C) Convert the results from part B back to rectangular form and compare with the results in part A.

Multiplication and Division in Polar Form

There is a particular advantage in representing complex numbers in polar form: multiplication and division become very easy. Theorem 1 provides the reason. (The exponential polar form of a complex number obeys the product and quotient rules for exponents: $b^mb^n = b^{m+n}$ and $b^m/b^n = b^{m-n}$.)

THEOREM 1

PRODUCTS AND QUOTIENTS IN POLAR FORM

If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then

1. $z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$

2. $\dfrac{z_1}{z_2} = \dfrac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \dfrac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

We establish the multiplication property and leave the quotient property for Problem 32 in Exercise 7-6.

$$z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2}$$

Write in trigonometric form.

$$= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

Multiply.

$$= r_1r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$$

$$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$$

Use sum identities.

$$= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Write in exponential form.

$$= r_1r_2e^{i(\theta_1+\theta_2)}$$

EXAMPLE 4 **Products and Quotients**

If $z_1 = 8e^{45°i}$ and $z_2 = 2e^{30°i}$, find

(A) z_1z_2 (B) z_1/z_2

Solutions

(A) $z_1z_2 = 8e^{45°i} \cdot 2e^{30°i}$

$$\boxed{= 8 \cdot 2e^{i(45°+30°)}} = 16e^{75°i}$$

(B) $\dfrac{z_1}{z_2} = \dfrac{8e^{45°i}}{2e^{30°i}}$

$$\boxed{= \tfrac{8}{2}e^{i(45° - 30°)}} = 4e^{15°i}$$

MATCHED PROBLEM 4

If $z_1 = 9e^{165°i}$ and $z_2 = 3e^{55°i}$, find
(A) $z_1 z_2$ (B) z_1/z_2

Historical Note

There is hardly an area in mathematics that does not have some imprint of the famous Swiss mathematician Leonhard Euler (1707–1783), who spent most of his productive life at the New St. Petersburg Academy in Russia and the Prussian Academy in Berlin. One of the most prolific writers in the history of the subject, he is credited with making the following familiar notations standard:

$f(x)$ function notation
e natural logarithmic base
i imaginary unit, $\sqrt{-1}$

For our immediate interest, he is also responsible for the extraordinary relationship

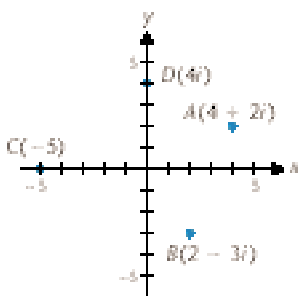
$$e^{i\theta} = \cos\theta + i\sin\theta$$

If we let $\theta = \pi$, we obtain an equation that relates five of the most important numbers in the history of mathematics:

$$e^{i\pi} + 1 = 0$$

Answers to Matched Problems

1.



2. (A) $\sqrt{2}[\cos(3\pi/4) + i\sin(3\pi/4)] = \sqrt{2}e^{(3\pi/4)i}$ (B) $2[\cos(\pi/3) + i\sin(\pi/3)] = 2e^{(\pi/3)i}$
(C) $5.83[\cos(-2.11) + i\sin(-2.11)] = 5.83e^{(-2.11)i}$

3. (A) $-i\sqrt{2}$ (B) $-\dfrac{3}{2} + \dfrac{3\sqrt{3}}{2}i$ (C) $-3.16 - 5.67i$

4. (A) $z_1 z_2 = 27e^{220°i}$ (B) $z_1/z_2 = 3e^{110°i}$

EXERCISE 7-6

A

In Problems 1–8, plot each set of complex numbers in a complex plane.

1. $A = 3 + 4i$, $B = -2 - i$, $C = 2i$
2. $A = 4 + i$, $B = -3 + 2i$, $C = -3i$
3. $A = 3 - 3i$, $B = 4$, $C = -2 + 3i$
4. $A = -3$, $B = -2 - i$, $C = 4 + 4i$
5. $A = 2e^{(\pi/3)i}$, $B = \sqrt{2}e^{(\pi/4)i}$, $C = 4e^{(\pi/2)i}$
6. $A = 2e^{(\pi/6)i}$, $B = 4e^{\pi i}$, $C = \sqrt{2}e^{(3\pi/4)i}$
7. $A = 4e^{(-150°)i}$, $B = 3e^{20°i}$, $C = 5e^{(-90°)i}$
8. $A = 2e^{150°i}$, $B = 3e^{(-50°)i}$, $C = 4e^{75°i}$

B

In Problems 9–12, change parts A–C to polar form. For Problems 9 and 10, choose θ in degrees, $-180° < \theta \le 180°$; for Problems 11 and 12 choose θ in radians, $-\pi < \theta \le \pi$. Compute the modulus and arguments for parts A and B exactly; compute the modulus and argument for part C to two decimal places.

9. (A) $\sqrt{3} + i$ (B) $-1 - i$ (C) $5 - 6i$
10. (A) $-1 + i\sqrt{3}$ (B) $-3i$ (C) $-7 - 4i$
11. (A) $-i\sqrt{3}$ (B) $-\sqrt{3} - i$ (C) $-8 + 5i$
12. (A) $\sqrt{3} - i$ (B) $-2 + 2i$ (C) $6 - 5i$

In Problems 13–16, change parts A–C to rectangular form. Compute the exact values for parts A and B; for part C compute a and b for a + bi to two decimal places.

13. (A) $2e^{(\pi/3)i}$ (B) $\sqrt{2}e^{(-45°)i}$ (C) $3.08e^{2.44i}$
14. (A) $2e^{30°i}$ (B) $\sqrt{2}e^{(-3\pi/4)i}$ (C) $5.71e^{(-0.48)i}$
15. (A) $6e^{(\pi/6)i}$ (B) $\sqrt{7}e^{(-90°)i}$ (C) $4.09e^{(-122.88°)i}$
16. (A) $\sqrt{3}e^{(-\pi/2)i}$ (B) $\sqrt{2}e^{135°i}$ (C) $6.83e^{(-108.82°)i}$

In Problems 17–22, find z_1z_2 and z_1/z_2.

17. $z_1 = 7e^{82°i}$, $z_2 = 2e^{31°i}$
18. $z_1 = 6e^{132°i}$, $z_2 = 3e^{93°i}$
19. $z_1 = 5e^{52°i}$, $z_2 = 2e^{83°i}$
20. $z_1 = 3e^{67°i}$, $z_2 = 2e^{97°i}$
21. $z_1 = 3.05e^{1.76i}$, $z_2 = 11.94e^{2.59i}$
22. $z_1 = 7.11e^{0.79i}$, $z_2 = 2.66e^{1.07i}$

Simplify Problems 23–26 directly and by using polar forms. Write answers in both rectangular and polar forms (θ is in degrees).

23. $(-1 + i)^2$
24. $(1 + i)^2$
25. $(-1 + i)(1 + i)$
26. $(1 + i\sqrt{3})(\sqrt{3} + i)$
27. $(1 - i)^3$
28. $(1 + i)^3$

C

29. Show that $r^{1/3}e^{(\theta/3)i}$ is a cube root of $re^{i\theta}$.
30. Show that $r^{1/2}e^{(\theta/2)i}$ is a square root of $re^{i\theta}$.
31. If $z = re^{i\theta}$, show that $z^2 = r^2e^{2\theta i}$ and $z^3 = r^3e^{3\theta i}$. What do you think z^n will be for n a natural number?
32. Prove

$$\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

APPLICATIONS

33. **Forces and Complex Numbers.** An object is located at the pole, and two forces **u** and **v** act on the object. Let the forces be vectors going from the pole to the complex numbers $20e^{0°i}$ and $10e^{60°i}$, respectively. Force **u** has a magnitude of 20 pounds in a direction of 0°. Force **v** has a magnitude of 10 pounds in a direction of 60°.
    (A) Convert the polar forms of these complex numbers to rectangular form and add.
    (B) Convert the sum from part A back to polar form.
    (C) The vector going from the pole to the complex number in part B is the resultant of the two original forces. What is its magnitude and direction?
34. **Forces and Complex Numbers.** Repeat Problem 33 with forces **u** and **v** associated with the complex numbers $8e^{0°i}$ and $6e^{30°i}$, respectively.