



Using the Real Power of Computer Algebra

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Who I Am, What I Do

Mathematics professor at ETS
(engineering school) since 1991:
single and multiple variable calculus,
differential equations, linear algebra,
complex analysis.

I started to use *Derive* in 1991 and I
continue to use it today.

Who I Am, What I Do

With the venue of the TI-92, the real possibilities of using computer algebra **IN THE CLASSROOM** were finally unified.

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Who I Am, What I Do

In an engineering school, it was natural to use an affordable CAS: since 1999, each undergraduate student at ETS has to buy a TI-92 Plus (now Voyage 200).

<http://www.seg.etsmtl.ca/ti/home.html>

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Who I Am, What I Do

In my engineering school, we have a great privilege: being able to use Computer Algebra in the classroom, when we want and during exams, all semester long.

We don't have to block internet access, Voyage 200 remains a "calculator".

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Who I Am, What I Do

Organizing TIME 2004 and ACA 2009 conferences would have been impossible without them:

Gilles Picard



Kathleen Pineau



Who I Am, What I Do

My (same and only) wife since **32** years:



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Who I Am, What I Do

But my main occupation for the past 2 years...



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Ideas and Themes of the Presentation

- Theme A: We can teach mathematics with a CAS.

Instead of removing items from the curriculum, we should revisit it with the CAS (example 1: V200, Nspire CAS).

With a CAS, you can (sometimes) solve the same problem, using 2 different approaches (examples 2 and 3: V200, *Derive*).

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Ideas and Themes of the Presentation

- Theme B: CAS are powerful but the teacher remains important.

If the CAS you are using does not have some built-in function, (sometimes) you can overcome this by using an appropriate approach (examples 4 and 5: *Derive*, V200).

Ideas and Themes of the Presentation

- Theme C: When computer algebra and theory become partners.

Some years ago, it was not easy to keep students' interest for concepts in analysis or linear algebra. Too much theory, not enough practice, no graphs (examples 6 and 7: Nspire CAS, *Derive*).

Comment about the examples to come

These selected examples are coming from my daily teaching/exchanges with colleagues at ETS and David Jeffrey from UWO.

This means that the level of mathematics involved is the one at university level, for engineering studies. But, one can easily apply it to any level of teaching.

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Comment about the examples to come

Some examples will be (now have been) performed live: if someone would like to obtain a file showing the details, please email michel.beaudin@etsmtl.ca.

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Theme A

We can teach with CAS

- Example 1** At ETS, some colleagues don't teach partial fractions anymore. Some never say a word about integrals of trig expressions. Why? You can do so much with a CAS. Consider these expressions

$$\frac{x}{3x^4 + 5x^3 + 25x^2 + 45x - 18}, \quad \sin^6 x.$$

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The image shows four sequential screenshots of a CAS interface:

- Top Left:** The function $f(x) = \frac{x}{3x^4 + 5x^3 + 25x^2 + 45x - 18}$ is defined. The command `∫(f,x)` is entered.
- Top Right:** The integral result is displayed: $\frac{-29 \cdot \ln(x^2 + 9)}{2132} + \frac{3 \cdot \ln(3 \cdot x - 1)}{574} + \frac{2 \cdot \ln(x + 2)}{91}$.
- Bottom Left:** The command `getDenom(f)` is used to find the denominator, and `factor(3*x^4 + 5*x^3 + 25*x^2 + 45*x - 18, x)` is used to factor it into $(x+2)(3x-1)(x^2+9)$.
- Bottom Right:** The command `propFrac(ans(1))` is used to perform partial fraction decomposition, resulting in $\frac{x}{45(x+2)(3x-1)(x^2+9)} = \frac{1}{1066(x^2+9)} + \frac{1}{574(3x-1)} + \frac{2}{91(x+2)}$.

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F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up F6

$(\sin(x))^6 \rightarrow g$
 $\int g dx \rightarrow ig$
 $\frac{5 \cdot x}{16} - \frac{\sin(x) \cdot (8 \cdot (\sin(x))^4 + 10 \cdot (\sin(x))^2 + 15) \cdot \cos(x)}{48}$
 $\int(g, x) \rightarrow ig$

MAIN RAD AUTO PAR 2/30

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up F6

$\int g dx \rightarrow ig$
 $\frac{5 \cdot x}{16} - \frac{\sin(x) \cdot (8 \cdot (\sin(x))^4 + 10 \cdot (\sin(x))^2 + 15) \cdot \cos(x)}{48}$
 $\frac{d}{dx}(ig)$
 $\left(\frac{-5 \cdot (\sin(x))^4}{6} - \frac{5 \cdot (\sin(x))^2}{8} - \frac{5}{16} \right) \cdot (\cos(x))$
 $\frac{d}{dx}(ig, x)$

MAIN RAD AUTO PAR 3/30

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up F6

$\left(\frac{-5 \cdot (\sin(x))^4}{6} - \frac{5 \cdot (\sin(x))^2}{8} - \frac{5}{16} \right) \cdot (\cos(x))$
 $tExpand \left(\left(\frac{-5 \cdot (\sin(x))^4}{6} - \frac{5 \cdot (\sin(x))^2}{8} - \frac{5}{16} \right) \cdot (\cos(x))^2 + \right.$
 $\left. -40 \cdot (\sin(x))^4 \cdot (\cos(x))^2 - 30 \cdot (\sin(x))^2 \cdot (\cos(x))^2 \right)$
 $tExpand(ans(1))$

MAIN RAD AUTO PAR 4/30

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up F6

$\left(\frac{-5 \cdot (\sin(x))^4}{6} - \frac{5 \cdot (\sin(x))^2}{8} - \frac{5}{16} \right) \cdot (\cos(x))$
 $tExpand$
 $-40 \cdot (\cos(x))^2$
 $ans(1) \rightarrow sin$

MAIN RAD AUTO PAR 4/30

1.2 1.3 1.4 1.5 RAD AUTO RECT

$\frac{5 \cdot x}{16} - \frac{\sin(x) \cdot (8 \cdot (\sin(x))^4 + 10 \cdot (\sin(x))^2 + 15) \cdot \cos(x)}{48}$
 $\frac{d}{dx}(ig)$
 $\left(\frac{-5 \cdot (\sin(x))^4}{6} - \frac{5 \cdot (\sin(x))^2}{8} - \frac{5}{16} \right) \cdot (\cos(x))$

1.2 1.3 1.4 1.5 RAD AUTO RECT

$\left(\frac{-5 \cdot (\sin(x))^4}{6} - \frac{5 \cdot (\sin(x))^2}{8} - \frac{5}{16} \right) \cdot (\cos(x))^2 +$
 $\frac{\sin(x)^6}{6} + \frac{5 \cdot (\sin(x))^4}{24} + \frac{5 \cdot (\sin(x))^2}{16} + \frac{5}{16} \rightarrow sin$
 $(\sin(x))^6$

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- **Example 2:** In the same course, we should try to link different subjects and show the students how to use their handheld.

For example, let's solve, using power series, the following problem:

$$(2x-11)y'' - y' - 3xy = 0, \quad y(0) = 10, \quad y'(0) = 4.$$

Then, we will use it to estimate $y(3)$.

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Series solution: $y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$

(with $c_0 = 10$ and $c_1 = 4$).

By hand, the students will find the following recurrence formula:

$$c_n = \frac{(2n^2 - 7n + 5)c_{n-1} - 3c_{n-3}}{11n(n-1)}$$

Then $c_2 = -2/11$, $c_3 = -167/363$, $c_4 = -651/5324, \dots$

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If we use only the first 5 terms, we will conclude that $y(3) = -1.96$.

But the correct answer is $y(3) = 2.04$. Students can obtain this answer with the help of their device.

Here is how.

Voyage 200 has a SEQ graphic mode. Let's use it:

Calculator screen showing the definition of a sequence u_1 in SEQ mode. The screen displays the formula $u_1 = \frac{(2 \cdot n^2 - 7 \cdot n + 5) \cdot u_1(n-1) - 3 \cdot u_1(n-3)}{11 \cdot n \cdot (n-1)}$ and the initial value $u_1(0) = -2/11$. Below the formula, the sequence values $u_1(1)$ through $u_1(4)$ are listed.

n	u1
0.	10.
1.	4.
2.	-.181818
3.	-.460055
4.	-.122276
5.	-.008637
6.	.003266
7.	.001176

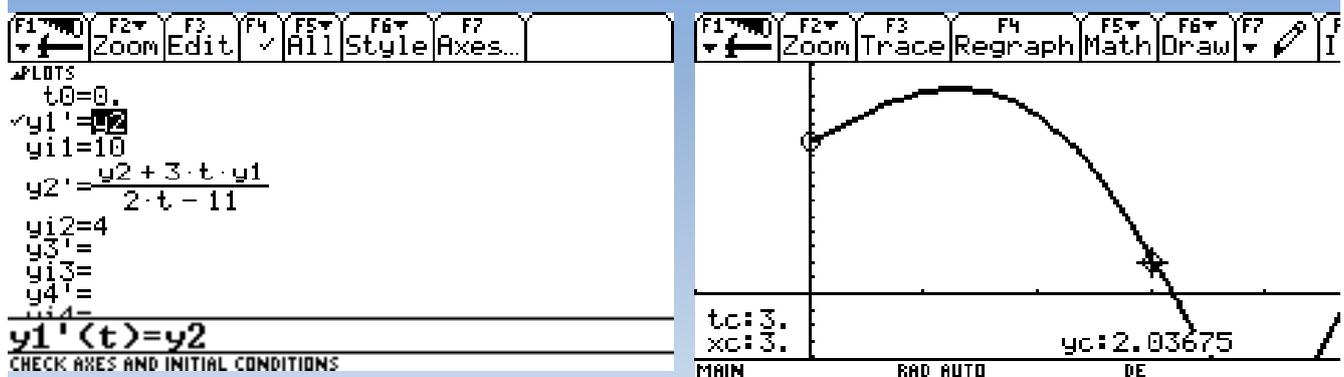
Calculator screen showing the summation of the sequence $u_1(n) \cdot 3^n$ in SEQ mode. The screen displays the formula $\sum_{n=0}^p (u_1(n) \cdot 3^n) \rightarrow s(p)$ and the values of $s(4)$ through $s(30)$.

Voyage 200 also has a DIFF EQUATIONS graphic mode. We have to convert (by hand) the second order ODE into a first order system:

$$\begin{cases} y' = z \\ z' = \frac{z + 3xy}{2x - 11} \end{cases} \quad y(0) = 10, z(0) = 4$$

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We type the system into the Y Editor, plot the graph of y_1 (the solution) and observe the value of $y(3)$ (RK method is being used, here):



So the value $y(3) = 2.04$ is correct!

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- **Example 3** Why not use the power of the CAS? Especially for *Derive*, its ability to integrate piecewise continuous functions. Find the steady-state solution of the following problem:

$$y'' + 2y' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0,$$

$$f(t) = \begin{cases} \sin(t) & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases} \quad f(t + 2\pi) = f(t).$$

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One can show, using the undetermined coefficients method and Fourier series, that the (particular) steady-state solution looks like this:

$$\frac{1}{\pi} - \frac{1}{4} \cdot \cos(t) - \frac{8}{75 \cdot \pi} \cdot \sin(2 \cdot t) + \frac{2}{25 \cdot \pi} \cdot \cos(2 \cdot t)$$

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But, according to Laplace transform theory, the **whole** solution is given by a convolution integral:

If $y'' + 2y' + y = f(t)$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = h(t) * f(t) = \int_0^t h(\tau) f(t - \tau) d\tau, \text{ where } h(t) = te^{-t}.$$

(This is because the inverse Laplace transform
of $1/(s^2 + 2s + 1)$ is te^{-t} .)

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No one will try to compute the convolution integral by hand (and how can we define easily a non trivial periodic input?).

We can use *Derive* in order to compute the convolution of f and h . Let us recall that

$$f(t) = \begin{cases} \sin(t) & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases} \quad f(t + 2\pi) = f(t).$$

$$h(t) = te^{-t}.$$

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Theme B

My CAS can't do this... What can I do?

- **Example 4** There is no “LambertW” function implemented into *Derive*. And no built-in Dirac Delta “function”. How can you deal with this?

Let us show that these absent functions can be turned into a better understanding of some mathematical concepts.

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Let's find all real solutions and one complex solution for the equation

$$2^x = x^{10}.$$

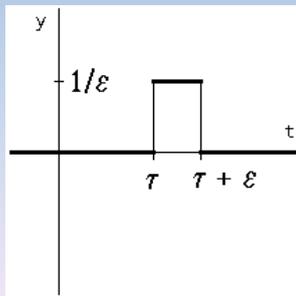
Note: some systems (*Maple* for example) can solve this equation because they have defined a special function (Lambert W) which is the inverse function of $f(x) = x \exp(x)$.

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Now let's solve the differential equation

$$y'' + 4y = 50\delta(t - \pi), \quad y(0) = 10, \quad y'(0) = 5.$$

Note: the “Dirac delta” function $\delta(t - \tau)$ can be seen as a limit of indicator function over a very small interval of time with area 1.



$1/\epsilon$ CHI($\tau, t, \tau + \epsilon$) in *Derive*.

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“`dsolve2_iv(p, q, r, t, to, yo, vo)`” is the command that solves the second order linear differential equation in *Derive*.

$$y'' + p(t)y' + q(t)y = r(t), \quad y(t_0) = y_0, \quad y'(t_0) = v_0.$$

$$y'' + 4y = 50\delta(t - \pi), \quad y(0) = 10, \quad y'(0) = 5.$$

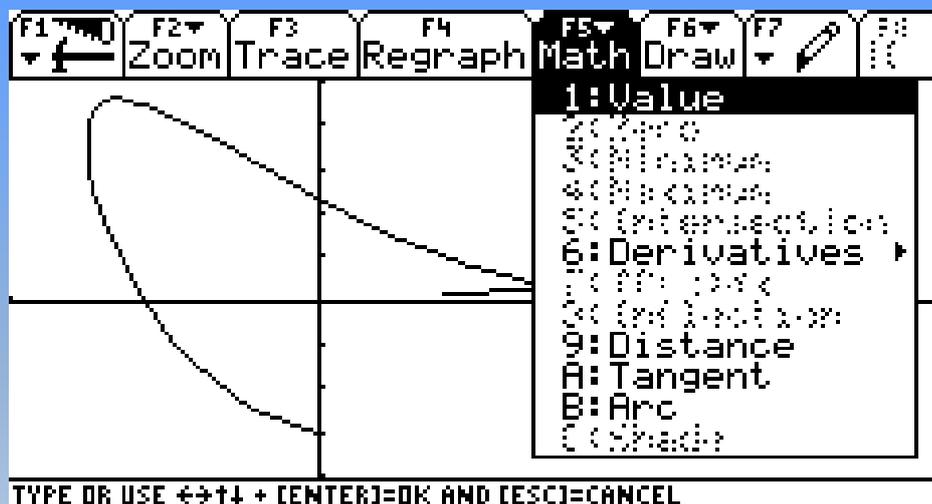
Derive is using the method of variation of parameters, which involves the computation of integrals ... And *Derive* has no problem to integrate piecewise functions!

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Example 5 Using Voyage 200, how can I find the coordinates of the point of intersection of two parametric 2D defined curves?

The F5 Math menu shows the item “Intersection” in Function graphic mode only...

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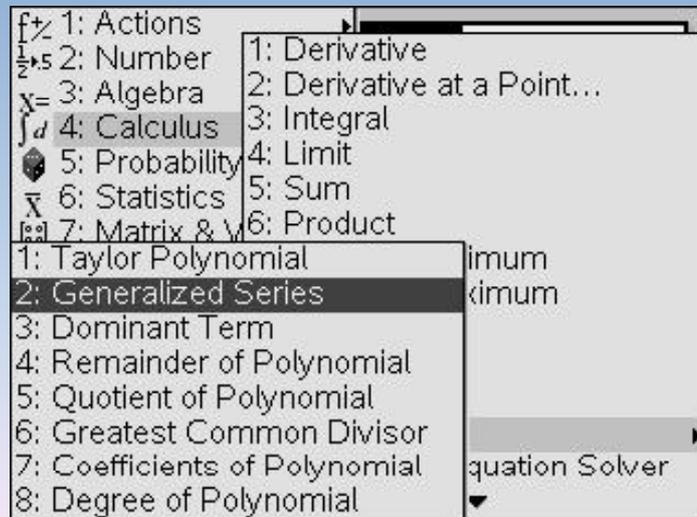
Let's see this, using Voyage 200 :

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Theme C

Theory and CAS

(Since OS 1.7) Nspire CAS has a “Generalized Series” built-in function:



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- **Example 6** Question from complex analysis. Compute the following line integral:

$$I \equiv \oint_C f(z) dz, \quad C \text{ the boundary of the square}$$

with vertices $\pm 3 \pm 3i$,

$$\text{and } f(z) = \frac{\cos(z) + \sinh(z)}{(z^2 - 1)^3}.$$

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We usually compute this kind of integral using the residue integration method.

If f has an isolated singularity at z_0 and Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \quad \text{for } 0 < |z-z_0| < R,$$

then the coefficient c_{-1} is called the residue of f at z_0 .

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By the residue theorem, we have $I = 2\pi i (B_1 + B_2)$ where

B_1 = residue at the triple pole $z = 1$ and

B_2 = residue at the triple pole $z = -1$.

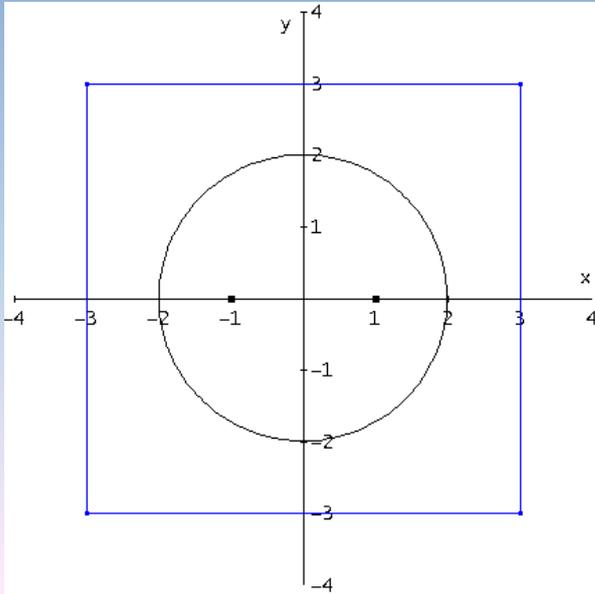
$$f(z) = \frac{\cos(z) + \sinh(z)}{(z^2 - 1)^3}$$

Nspire CAS will show that $B_1 + B_2 = \frac{e - 7e^{-1}}{16}$.

$$\text{So, } I = \left(\frac{e - 7e^{-1}}{16} \right) 2\pi i \approx 0.056i.$$

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But we can also use the definition of a line integral. Of course, it would be long to integrate over each side of the square... Using the principle of deformation of paths, we can impose a continuous deformation of the square into the circle of radius 2, located at the origin:



$$f(z) = \frac{\cos(z) + \sinh(z)}{(z^2 - 1)^3}$$

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This means that we can compute the value of the integral by using the definition.

$$\int_r f(z) dz = \int_a^b f(r(t)) r'(t) dt \quad \text{where } r: [a, b] \rightarrow \square$$

is a smooth curve in the plane.

Let's use $r(t) = 2\exp(it)$ for the circle of radius 2, centered at 0.

Now take a look at the power of Nspire CAS!

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- **Example 7** The importance to see it.

In *Derive*, $\text{ROTATE_X}(\theta)$ simplifies to a matrix \mathbf{A} such that $\mathbf{A}\cdot\mathbf{v}$ rotates the 3D coordinate vector \mathbf{v} through an angle of θ radians about the x -axis, counterclockwise when viewed from the positive x -axis toward the origin.

Can we see an example?

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Conclusion

What are the main benefits of using technology in the classroom?

Some (personal) answers.

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Computer algebra allows me to continue to teach (almost) the same courses with a taste of new: “make new from old” (Josef Böhm).

This situation has many advantages, namely:

- Retirement can wait.
- I can revisit some mathematical results.
- When I have to prepare new material, CAS explorations are helping me.

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But using technology has also some disadvantages:

- Weaker students can succeed thanks to the CAS.
- Is this really a problem?
- In some cases, less time CAN be spent on proving results because more time MUST be spent on learning how to use the CAS.

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This is why we need a “good mathematical assistant”. As far as I am concerned, *Derive*, Voyage 200 and Nspire CAS are in this category.

And don't forget this (adapted from the *Derive* user manual).

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“Making mathematics more exciting and enjoyable should be the driving force behind the development of a CAS program.”

“It gives you the freedom to explore different approaches to problems ... Something you probably would never consider if you had to do the calculations by hand”.

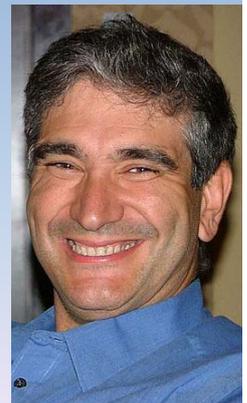
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But I have to tell you something: why do I like so much to attend TIME conferences?

Or why Computer Algebra is so powerful according to me?

It gives me the chance to meet nice people (or, at least, think of them).

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I have the opportunity to travel very far and have beer with friends:



I can see (almost yearly) very **SERIOUS** colleagues:



Thank You!